Theoretical Astrophysics

Heidelberg University
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Problem Sheet 9

Discussion in the tutorial groups in the week of December 18, 2023

1. **Comprehension Question**.

   (i) Summarise the derivation of the dielectric response function, and explain why it acquires an imaginary part.

   (ii) Explain the relation between dissipation and an imaginary part of a response function.

   (iii) How are dispersion relations for longitudinal and transverse electromagnetic waves determined?

2. **Landau damping**. Assume that a longitudinal electromagnetic wave with amplitude $\hat{E}$, wave number k, and frequency ω propagates through a thermal plasma. Assuming that the wave travels in x-direction, the *mean energy dissipation rate* due to Landau damping is given by

$$\langle Q\rangle = -|\hat{E}|^2 \frac{\pi m_\mathrm{e} e^2 \omega}{k^2} \left.\frac{\mathrm{d}\bar{f}}{\mathrm{d}p_x}\right|_{p_x = m_\mathrm{e}\omega/k},$$

   where $\bar{f}$ is the one-particle phase-space distribution that is already integrated over the momentum components p_y and p_z.

   (a) Determine $\bar{f}(p_x)$ for a spatially homogeneous thermal plasma with mean electron density n_e.

   (b) Assume that the wave's frequency is approximately equal to the *plasma frequency* $\omega_\mathrm{p} = \sqrt{4\pi n_\mathrm{e} e^2/m_\mathrm{e}}$. Determine $\langle Q\rangle$ as a function of the amplitude $\hat{E}$ and the wave number $k(\omega_\mathrm{p})$, which has to fulfill the dispersion relation for longitudinal waves. *Hint:* You do not have to calculate $k(\omega_\mathrm{p})$ explicitly.

   (c) Determine the characteristic damping time scale τ for waves with wave number k using that the energy density per Fourier mode is given by $|\hat{E}|^2/(4\pi)$.

3. **Time delays in a thermal plasma**. Electromagnetic waves of frequency ω that propagate through a thermal plasma in a deep gravitational well experience two kinds of time delay. The first is the delay

$$\Delta t_\omega = \frac{2\pi e^2}{m_\mathrm{e} c \omega^2} \int \mathrm{d}l\, n_\mathrm{e}$$

   due to the electromagnetic dispersion, and the second is the so-called *Shapiro delay* due to relativistic time dilatation,

$$\Delta t_\mathrm{Shap} = -\frac{2}{c^3} \int \mathrm{d}l\, \Phi,$$

where Φ is the Newtonian gravitational potential. Both integrals have to be evaluated along the wave's trajectory. Assume that the electromagnetic wave travels through a spherical cloud of fully ionised hydrogen of radius R and that both the total mass density ρ and the electron number density n_e are constant throughout the sphere, hence

$$\rho(r) = \begin{cases} \bar{\rho} & \text{for } r \le R \\ 0 & \text{for } r > R \end{cases}$$

and

$$n_\mathrm{e}(r) = \begin{cases} \bar{n}_\mathrm{e} & \text{for } r \le R \\ 0 & \text{for } r > R. \end{cases}$$

(a) Determine both time delays Δt_ω and Δt_Shap that the wave experiences between entering and leaving the plasma. Assume that the wave's frequency is equal to the plasma frequency ω_p and that it travels through the sphere's centre. The gravitational potential inside a homogeneous sphere of density ρ and radius R is given by

$$\Phi(r) = -2\pi G\rho\left(R^2 - \frac{r^2}{3}\right)$$

if the potential $\Phi(r)$ is chosen to vanish for $r \to \infty$.

(b) What is the relative magnitude of both time delays? *Hint:* The Schwarzschild radius for an object of mass M is given by $R_\mathrm{S} = 2GM/c^2$.

4. **Induction equation**. Assume that a plasma with conductivity $\sigma = \text{const.}$ and a velocity profile $\vec{v}$ is embedded into a magnetic field whose field lines are parallel at $t = 0$. Does the magnetic field change its direction for $t > 0$ if at $t = 0$

   (a) $\vec{v} \parallel \vec{B}$,

   (b) $\vec{v} \perp \vec{B}$?

5. **Faddeeva function (classroom assignment)**. The Faddeeva function is given by

$$w(z) = \frac{\mathrm{i}}{\pi}\int_{-\infty}^{+\infty} \mathrm{d}t \, \frac{\mathrm{e}^{-t^2}}{z - t}.$$

For $z = 0$, it follows that $w(z = 0) = w(0) = 1$.

   (a) Show that

$$w'(z) = \frac{2\mathrm{i}}{\sqrt{\pi}} - 2zw(z).$$

   Hint: $\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi}$.

   (b) Use this relation to Taylor approximate the function $yZ(y) = y\,\mathrm{i}\sqrt{\pi}w(y)$ for small y.