Theoretical Astrophysics

Heidelberg University
Winter Term 2023/24

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Problem Sheet 8

Discussion in the tutorial groups in the week of December 11, 2023

1. **Comprehension Question**.
   (i) What is the meaning of the Debye length?
   (ii) What is the plasma parameter, and in which parameter range of this value is a plasma considered ideal?
   (iii) Under which condition can electromagnetic waves in media have longitudinal components?

2. **Stability in an ideal gas**. It was shown in the lecture that stability against convection requires

$$\frac{\mathrm{d}\ln P}{\mathrm{d}\ln\rho} < \gamma.$$

   (a) Show that for an ideal gas this implies

$$\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} < \frac{\gamma-1}{\gamma}.$$

   (b) Combine this result with hydrostatics in a constant gravitational field to show that

$$-\frac{\mathrm{d}T}{\mathrm{d}r} < \frac{g}{c_p}$$

   where c_p is the heat capacity per mass.

3. **Surface gravity waves**. If the surface of a free fluid in the gravitational field of the Earth is disturbed from its equilibrium state, the occurring flow propagates in form of *surface gravity waves*. These waves propagate mainly on the fluid's surface and become quickly damped for deeper layers. We will consider waves whose amplitude A is much smaller than its wavelength λ, hence $A \ll \lambda$. In this case the term $(\vec{v}\cdot\vec{\nabla})\,\vec{v}$ in Euler's equation can be neglected since the velocity of the moving fluid particles is small, and the fluid can be described by a *potential flow* with $\vec{v} = \vec{\nabla}\psi$, where ψ is the corresponding potential. If the fluid is additionally incompressible and the coordinate system is chosen such that the xy-plane is parallel to the unperturbed surface with $z = 0$ at the surface, it was demonstrated in the lecture that the potential ψ satisfies the Laplace equation and a boundary equation,

$$\begin{aligned} 0 &= \Delta\psi, \\ 0 &= \left.\left(\frac{\partial\psi}{\partial z} + \frac{1}{g}\frac{\partial^2\psi}{\partial t^2}\right)\right|_{z=0}, \end{aligned}$$

   respectively, where g is the gravitational acceleration.

(a) Consider waves that propagate in x-direction and that are homogeneous in y-direction. Use the ansatz $\psi(x, z, t) = f(z) \cos(kx - \omega t)$, where $k = 2\pi/\lambda$ is the wave number and ω the frequency, to find a differential equation for $f(z)$. Determine the most general solution for $f(z)$.

(b) Assume that the fluid has a finite depth h. What is the resulting boundary condition for $\partial\psi/\partial z$? Determine $f(z)$ using this boundary condition.

(c) Determine the dispersion relation $\omega(k)$ and the propagation velocity $u = \partial\omega/\partial k$ of the waves.

(d) Determine the velocity field $\vec{v}^{\top} = (v_x, v_y, v_z)$ of the particles.

4. **Electromagnetic waves in conducting media**. In isotropic media the electric and magnetic fields $\vec{E}$ and $\vec{B}$ in Maxwell's inhomogeneous equations need to be replaced by $\vec{D} = \epsilon\vec{E}$ and $\vec{H} = \vec{B}/\mu$, respectively, while Maxwell's homogeneous equations remain unchanged.

   (a) Derive the *telegraph equation*

$$\Delta\vec{E} - \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = \frac{4\pi\sigma\mu}{c^2}\frac{\partial}{\partial t}\vec{E}$$

   from Maxwell's equations in media and Ohm's law $\vec{j} = \sigma\vec{E}$. Assume plane-wave solutions for $\vec{E}$ and derive the dispersion relation for waves in media.

   (b) From the equation of motion

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = -\frac{e}{m_\mathrm{e}}\vec{E} - \frac{\vec{v}}{\tau}$$

   for an electron, containing a damping term with a characteristic collision time τ, derive an equation for the current density $\vec{j}$. Assume again plane-wave solutions for $\vec{E}$ and $\vec{j}$ and identify the conductivity

$$\sigma = \frac{n_\mathrm{e}e^2}{m_\mathrm{e}}\frac{\tau}{1 - \mathrm{i}\omega\tau},$$

   where n_e is the number density of electrons.

   (c) Combine the results from a) and b) for the dispersion relation, assume $\mu = 1$ and $\omega\tau \gg 1$ and identify the plasma frequency ω_p. What does the limit $\omega\tau \gg 1$ mean?

5. **Plasma oscillations (classroom assignment)**. Consider a plasma that is electrically neutral on large scales. When a non-equilibrium distribution of charges is produced in the electron gas with mean number density $\bar{n}_\mathrm{e}$, the electrons start moving in such a way as to screen the static electric field originating from the ions. Since the electrons possess a finite mass, they cannot stop at the exact state of equilibrium but overshoot and produce another non-equilibrium state. After they have stopped, they start moving in the reverse direction. Ignoring dissipation, this process repeats indefinitely, leading to *plasma oscillations*.

   (a) Determine the electric field E if the electron gas is diplaced by the distance x with respect to the ions.

   (b) Determine the restoring force originating from this electric field on a single electron.

   (c) Determine the equation of motion for the electron and identify the oscillation frequency. Does it look familiar to you?