Theoretical Astrophysics

Heidelberg University
Winter Term 2023/24

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Problem Sheet 6

Discussion in the tutorial groups in the week of November 27, 2023

1. **Comprehension Question**.

   (i) Describe the concepts of the adiabatic and the isothermal sound speed.

   (ii) What is the Jeans length?

   (iii) What is a polytropic stratification?

2. **Sound speed in gases**. It was shown in the lecture that the sound speed at constant entropy is

$$c_s^2 = \left(\frac{\partial P}{\partial \rho}\right)_S = \gamma \frac{P}{\rho \alpha T}$$

with the thermal isobaric volume expansion coefficient

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P .$$

   (a) Convince yourself that $\alpha = T^{-1}$ for an *ideal* gas.

   (b) Use the same steps as in the lecture to prove that the *isothermal* sound speed is

$$c_s^2 = \frac{P}{\rho \alpha T}.$$

   (c) Consider the equation of state of a van-der-Waals gas,

$$\left(P + \frac{a\nu^2}{V^2}\right)(V - \nu b) = \nu R T.$$

   Calculate the isentropic and isothermal sound speeds for a van-der-Waals gas.

   (d) Estimate the correction term for air, CO_2, and H_2 under normal conditions; that is, $V = \nu V_0$ with $V_0 = 22.4\,\mathrm{l\,mol^{-1}} = 22.4 \times 10^3\,\mathrm{cm^3\,mol^{-1}}$.

	a	b
air	1358.0	36.4
CO_2	3637.0	42.7
H_2	247.0	15.5

   The units of the parameters are $[a]$ =kbar cm^6 mol^{-2} and $[b]$ =cm^3 mol^{-1}. *Hint:* Why is it appropriate to eliminate RT using the ideal gas law?

3. **Fermi pressure of a white dwarf**. White dwarfs are the main leftovers of main-sequence stars with low or intermediate mass, which have already lost their envelopes. They are stabilised by the Fermi pressure P_F of the electrons against further gravitational collapse and can be treated non-relativistically. The corresponding phase-space density is given by

$$f(\vec{p}) = \frac{n}{V_F}\,\Theta(p_F - |\vec{p}|),$$

where n is the average electron number density, p_F the Fermi momentum, $V_F = 4\pi p_F^3/3$ the corresponding volume in momentum space, and $\Theta(x)$ the Heaviside step function.

   (a) Show that

$$p_F = \hbar\left(3\pi^2 n\right)^{1/3}$$

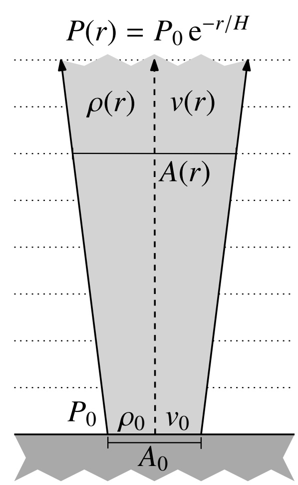
   by calculating the number of available phase-space cells.

   (b) Determine the Fermi pressure P_F as a function of n from the energy-momentum tensor

$$T^{\mu\nu} \equiv c^2 \int \frac{d^3p}{E} p^\mu p^\nu f(\vec{p})$$

   (c) A polytropic equation of state relates the pressure P to the mass density ρ by $P = P_0\,(\rho/\rho_0)^\alpha$. Determine the polytropic index α in the case of a white dwarf.

4. **Properties of a jet**.
Jets are large directed outflows of material and a common astrophysical phenomenon. They can be observed under various circumstances, e.g. together with young T Tauri stars and the accretion onto black holes in the centres of galaxies, so-called active galactic nuclei.
Assume that a stationary jet has its origin on the surface of a spherical star with mass M and radius R and has initially the velocity v_0 and the cross-sectional area A_0. The outflowing material satisfies the polytropic relation $P = P_0\,(\rho/\rho_0)^\gamma$, where γ is the adiabatic index. The atmosphere of the star is also adiabatic, i.e. the pressure drops exponentially with the distance r from the surface, $P(r) = P_0 e^{-r/H}$, where H is the scale height and P_0 the pressure on the surface. Since the jet's extent is very small compared to the star, any curvature effects can be safely neglected.



   (a) Determine the specific enthalpy $\tilde{h}$ as a function of P and ρ.

   (b) Determine the velocity $v(r)$.

   (c) Determine the cross-sectional area $A(r)$.

5. **Rotation of an incompressible fluid (classroom assignment)**. A cylinder that contains an incompressible fluid rotates with constant angular velocity $\vec{\omega} = \omega\,\vec{e}_z$ in the gravitational field of the Earth, characterised by the gravitational acceleration $\vec{g} = -g\,\vec{e}_z$.

   (a) Use the Navier-Stokes equation

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\,\vec{v}\right] + \vec{\nabla}P = \eta\vec{\nabla}^2\vec{v} + \left(\zeta + \frac{\eta}{3}\right)\vec{\nabla}\,(\vec{\nabla}\cdot\vec{v}) + \vec{f},$$

with $\vec{f} = -\rho\,\vec{\nabla}\Phi$ and the potential Φ, to derive differential equations for the components of the pressure gradient that contain ω, ρ, and g. Why does the continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

not yield any additional information?

(b) Determine a function $P(x, y, z)$ from these differential equations. What does the surface of the rotating fluid look like?