Theoretical Astrophysics

Heidelberg University
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Lecturer: Prof. Dr. Matthias Bartelmann
Head tutor: Dr. Michael Zacharias

Problem Sheet 7

Discussion in the tutorial groups in the week of December 4, 2023

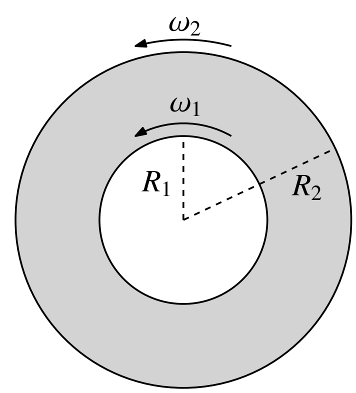
1. **Comprehension Question**.

   (i) Summarize the essential steps from the Euler equation to the Bernoulli equation.

   (ii) Describe the main steps from the Bernoulli equation to Bondi accretion.

   (iii) Which principles lead to the shock jump conditions?

2. **The Navier-Stokes equation in curvilinear coordinates**.
   Assume that two coaxial and infinitely long cylinders with radii R_1 and $R_2 > R_1$ rotate around their axes with the angular speeds ω_1 and ω_2, respectively, in a freely falling reference frame. The space between them is filled with a viscous and incompressible fluid.
   The three components of the Navier-Stokes equation in cylindrical coordinates then read



$$\frac{\partial v_r}{\partial t} + (\vec{v}\cdot\vec{\nabla})\,v_r - \frac{v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{\eta}{\rho}\left(\Delta v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\varphi}{\partial \varphi}\right),$$
$$\frac{\partial v_\varphi}{\partial t} + (\vec{v}\cdot\vec{\nabla})\,v_\varphi + \frac{v_r v_\varphi}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial \varphi} + \frac{\eta}{\rho}\left(\Delta v_\varphi - \frac{v_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \varphi}\right),$$
$$\frac{\partial v_z}{\partial t} + (\vec{v}\cdot\vec{\nabla})\,v_z = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{\eta}{\rho}\Delta v_z.$$

   (a) Show that the three components of the Navier-Stokes equation simplify to the two equations

$$\frac{\mathrm{d}^2 v_\varphi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}v_\varphi}{\mathrm{d}r} - \frac{v_\varphi}{r^2} = 0,$$
$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\rho v_\varphi^2}{r}.$$

   (b) Solve the differential equation for v_φ using the ansatz $v_\varphi(r) \propto r^n$ and setting appropriate boundary conditions.

   (c) Determine the pressure profile $P(r)$ with the boundary condition $P(R_1) = P_0$.

3. **The method of characteristics**. In astrophysics the hydrodynamical equations often simplify a lot due to spherical symmetry, e.g. when a young star accretes mass. In this case $\vec{v} = v(r)\,\vec{e}_r$ and the continuity and the Euler equations simplify to

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\rho v)}{\partial r} = 0,$$
$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{c_\mathrm{s}^2}{\rho}\frac{\partial \rho}{\partial r},$$

respectively, where it was used that $\vec{\nabla} P = c_s^2 \vec{\nabla}\rho$. Let us for simplicity further assume that c_s is constant everywhere. In order to use the method of characteristics, the equations have to be brought into the form

$$T_{ij}\frac{\partial u_j}{\partial t} + R_{ij}\frac{\partial u_j}{\partial r} = Z_i, \tag{1}$$

where one has to sum over j. The matrix elements T_{ij} and R_{ij} are given by the coefficients in front of the partial derivatives, while $\vec{Z}$ is given by the inhomogeneities.

(a) Bring the continuity and the Euler equations into the form of Eq. (1) and identify T, R, $\vec{u}$, and $\vec{Z}$.

(b) Determine a relation between the differentials $\mathrm{d}r$ and $\mathrm{d}t$ from the condition $\det(\mathsf{T}\,\mathrm{d}r - \mathsf{R}\,\mathrm{d}t) = 0$. What does the result mean physically?

4. **Continuity conditions in relativistic hydrodynamics**. Assume that a comoving coordinate system is chosen such that the yz-plane always coincides with the shock front. The gas flows from side 1 to side 2. With the energy-momentum tensor

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u^\mu u^\nu + \eta^{\mu\nu} P$$

and the 4-velocity $(u^\mu) = \gamma\,(c, \vec{v})$, where $\gamma = (1 - v^2/c^2)^{-1/2}$, the relativistic generalisations of the continuity conditions for the densities of the particle current, the momentum current and the energy current are

$$\begin{aligned} n_1 u_1^x &= n_2 u_2^x, \\ T_1^{xx} &= T_2^{xx}, \\ cT_1^{0x} &= cT_2^{0x}, \end{aligned}$$

respectively.

(a) Express the continuity conditions as functions of the velocities $v_i \equiv v_i^x$, the pressures P_i, the enthalpies per volume $w_i = \varepsilon_i + P_i$, where $\varepsilon = \rho c^2$, and the number densities n_i (here and in the following, $i = 1, 2$).

(b) Determine the velocities on both sides of the discontinuity as a function of P_i and ε_i.

5. **The Chandrasekhar mass (classroom assignment)**. Consider a solution to the Lane-Emden equation for $n = 3$, $\gamma = 4/3$, that is for monoatomic, ideal, realtivistic gas. As can be shown (e.g. numerically with the Jupyter notebook on the lecture's website), it is in this case

$$C = \int_0^{x_{\max}} x^2 \Theta^n(x)\mathrm{d}x \approx 2.$$

Insert the Fermi pressure into the expression for the scale radius of the Lane-Emden solution. Write down an expression for the total mass M and identify the Planck mass

$$m_{\mathrm{Pl}} = \sqrt{\frac{\hbar c}{G}}.$$

Set the mean mass of an atom per electron to $\bar{m} = 2m_p$ (why?) – m_p is the proton mass – and verify that the total mass is the Chandrasekhar mass

$$M = \frac{\sqrt{3\pi}}{4}\frac{m_{\mathrm{Pl}}^3}{m_p^2} = 1.44 M_\odot.$$