Quantum Field Theory 1 – Problem set 3

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Suggested reading before solving these problems: Chapter 3 in the script and/or Chapter 4.2 of *Peskin & Schroeder*.

Problem 1: Unitary evolution and T-product

Consider the decomposition of a Hamiltonian operator H in free and interaction parts, $H = H_0 + H_{\text{int}}$. In the interaction picture, operators evolve in time with the free Hamiltonian H_0, while states $|f\rangle$ evolve with the interaction Hamiltonian,

$$i\,\partial_t\,|f\rangle \,=\, H_{\text{int}}(t)\,|f\rangle\,.$$

Show that this implies that we can write $|f(t)\rangle = U(t,\,t_0)\,|f(t_0)\rangle$, where the unitary operator $U(t,t_0)$ satisfies the differential equation (Schrödinger equation)

$$i\,\partial_t\,U(t,\,t_0) \,=\, H_{\text{int}}(t)\,U(t,\,t_0)\,. \tag{1}$$

Show that the solution of equation (1) can be expressed as a power series, in which each term is an operator,

$$U(t,\,t_0) \,=\, \mathbf{1} - i\int_{t_0}^{t} dt_1 H_{\text{int}}(t_1) \,+\, (-i)^2\int_{t_0}^{t}\int_{t_0}^{t_1} dt_1 dt_2\, H_{\text{int}}(t_1)H_{\text{int}}(t_2) + \ldots \tag{2}$$

Convince yourself that this series can be re-expressed as

$$U(t,\,t_0) \,=\, T\left\{\exp\left[-i\int_{t_0}^{t} dt' H_{\text{int}}(t')\right]\right\}$$

where the T-product acts as $T\,A(t)B(t') \,=\, A(t)B(t')\,\Theta(t-t') + B(t')A(t)\,\Theta(t'-t)$. In particular, show that the expansion of the exponential up to second term provides Eq. (2), and try to generalise your argument for the higher order terms.

Problem 2: 2-to-2 scattering

In the lecture course you have learned how to describe the scattering of two particles in the interaction picture. Assume that the particles are characterised by momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ in the initial state, and by momenta $\boldsymbol{p}_1^{'}$ and $\boldsymbol{p}_2^{'}$ in the final state. The interaction Hamiltonian is $H_I = \frac{\lambda}{4!}\,\phi^4$, where $\phi(t,\boldsymbol{x})$ is the time dependent operator associated with a real scalar field.

In particular, you have learned that the amplitude controlling the process can be obtained from the following quantity

$$i\,T_{fi} \simeq -i\,\langle \boldsymbol{p}_1^{'}\,\boldsymbol{p}_2^{'}|\left[\frac{\lambda}{4!}\int d^4x\,\phi^4(x)\right]|\boldsymbol{p}_1\,\boldsymbol{p}_2\rangle \tag{3}$$

by isolating the contributions proportional to $\delta^4\left(p_1+p_2-p_1'-p_2'\right)$, and defining the scattering amplitude $\mathcal{M}_{fi}$ as

$$i\,T_{fi} \equiv i\,\mathcal{M}_{fi}\,(2\pi)^4\,\delta^4\left(p_1+p_2-p_1'-p_2'\right)$$

The amplitude can be extracted by expanding the scalar field $\phi(t,\,\boldsymbol{x})$ in terms of ladder operators $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$, and by plugging this expansion in eq. (3). Then, using the commutation relations of a and $a^\dagger$, one extracts the terms that are proportional to $\delta^{(4)}\left(p_1+p_2-p_1'-p_2'\right)$.

- Identify the relevant terms in the expansion! How many are there? Do they all give the same contribution?
- Show that the final result is $\mathcal{M}_{fi} = -4!\,\frac{\lambda}{4!} = -\lambda$. Is there a connection between the coefficient in this result, and the number of terms in the expansion of eq. (3) that contribute to the scattering amplitude?
- Try to generalise the previous results to the $n-n$ scattering of a theory with interaction Hamiltonian $H_I = \frac{\lambda}{(2n)!}\,\phi^{2n}$, with n being a natural number.
- By inserting the scalar field expansion into eq. (3), do you also find terms that are *not* proportional to $\delta^{(4)}\left(p_1+p_2-p_1'-p_2'\right)$? If so, what is their physical interpretation?

Problem 3: Charge of a complex scalar field

Consider the Lagrangian density for a free complex scalar field

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi.$$

and define the associated conjugate momenta π and π^*. The Noether theorem leads to a conserved charge, given in terms of 0-component of the Noether current,

$$Q \equiv \int d^3x\,j^0\,.$$

For a complex scalar field, the four-vector associated with the current j reads

$$j^\mu = i\,\left[(\partial^\mu\phi)^*\,\phi - \phi^*\,(\partial^\mu\phi)\right],$$

from which an expression for the corresponding charge Q can be easily obtained.

The theory is quantised by promoting ϕ, ϕ^* and their conjugate momenta to operators. To that end it is convenient to introduce the creation and annihilation operators

$$\phi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left\{a(\boldsymbol{p})\, e^{i\boldsymbol{px}} + b^\dagger(\boldsymbol{p})\, e^{-i\boldsymbol{px}}\right\}, \tag{4}$$

$$\pi(\boldsymbol{x}) = -\, i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\boldsymbol{p}}}{2}} \left\{b(\boldsymbol{p})\, e^{i\boldsymbol{px}} - a^\dagger(\boldsymbol{p})\, e^{-i\boldsymbol{px}}\right\}. \tag{5}$$

with commutation relations

$$\begin{aligned}
[a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})] &= [b(\boldsymbol{p}), b^\dagger(\boldsymbol{q})] = (2\pi)^3\, \delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \\
[a(\boldsymbol{p}), a(\boldsymbol{q})] &= [b(\boldsymbol{p}), b(\boldsymbol{q})] = 0, \\
[a(\boldsymbol{p}), b(\boldsymbol{q})] &= [a(\boldsymbol{p}), b^\dagger(\boldsymbol{q})] = 0\ .
\end{aligned}$$

Express the charge Q in terms of the operators a, $a^\dagger$ and b, $b^\dagger$, carrying out all the details of the calculation.