Theoretical Astrophysics

Heidelberg University
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Problem Sheet 4

Discussion in the tutorial groups in the week of November 13, 2023

1. **Comprehension Question**.
   (i) Summarise in your own words the steps and approximations leading to the common cross section for radiative quantum transitions.
   (ii) What is a "forbidden line", and how could it be allowed?
   (iii) Describe broadening mechanisms of spectral lines and what shape of the line profile they produce.

2. **Hydrogen ionisation cross section**. Consider the bound-free transition between the hydrogen ground state $(n, l, m) = (1, 0, 0)$ and the free state of the electron. The momentum of the free electron follows the de-Broglie relation $\vec{p} = \hbar\vec{k}$.
   (a) Calculate the dipole matrix element $\vec{d}_{nm}$ for this transition using the wave functions

$$\langle x|m\rangle = \frac{1}{\sqrt{V}}\mathrm{e}^{i\vec{k}\cdot\vec{x}}$$
$$\langle x|n\rangle = \frac{1}{\sqrt{\pi a_B}}\mathrm{e}^{-r/a_B}$$

   and the Fourier transform

$$\mathcal{F}[\mathrm{e}^{-r/a_B}](k) = \frac{8\pi a_B^3}{(1 + a_B^2 k^2)^2}$$

   of the exponential. We set the Bohr radius $a_B := \hbar/(\alpha m_e c)$ with the finestructure constant $\alpha = e^2/\hbar c$.
   (b) Explain why the cross section

$$\sigma_\omega = 4\pi^2 r_e c f_{nm} \delta_D(\omega_{nm} - \omega)$$

   needs to be multiplied by the number of free electron states

$$\frac{m_e V}{2\pi^2\hbar} k\mathrm{d}\omega$$

   so that the frequency-integrated cross section becomes

$$\sigma = \frac{2r_e m_e c}{\hbar} V k f_{nm}.$$

   (c) Calculate the oscillator strength f_{nm} and express ω_{nm} by k, a_B, and the Rydberg energy $\mathrm{Ry} = \alpha^2 m_e c^2/2$. Plug it in to obtain the cross section σ.

3. **Excited state**. The cross section for a transition between an initial state $|n\rangle$ and a final state $|m\rangle$ was derived as

$$\sigma_{mn} = \frac{4\pi}{3c\hbar}\omega_{mn}\left|\vec{d}_{mn}\right|^2 \delta_D(\omega_{mn} - \omega),$$

where $\vec{d}_{mn} = \langle m|e\hat{x}|n\rangle$ is the dipole matrix element and $\omega_{mn} = (E_m - E_n)/\hbar$ is the frequency corresponding to the energy difference between the states $|m\rangle$ and $|n\rangle$. The delta distribution $\delta_D(x)$ assures that only those photons contribute to the cross section that have the correct frequency.

   (a) Consider the one-dimensional harmonic oscillator with energy levels $E_n = \hbar\omega(n + 1/2)$ and corresponding wave functions

$$\Psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right) \exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

   with the Hermite polynomials

$$H_n(x) = (-1)^n \mathrm{e}^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^{-x^2}.$$

   Calculate the cross section σ_{10} for the transition from the ground state ($n = 0$) to the first excited state ($n = 1$). *Hint:* It may be helpful to use

$$\int_{-\infty}^{\infty} \mathrm{d}x\, x^2 \mathrm{e}^{-\alpha x^2} = -\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\partial}{\partial\alpha}\mathrm{e}^{-\alpha x^2}.$$

   (b) Consider now an infinitely deep potential well of length L with energy levels

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

   with $n \in \mathbb{N}$ and wave functions

$$\Psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi}{L}x\right) & \text{if } n \text{ is odd} \\ \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right) & \text{if } n \text{ is even} \end{cases},$$

   with $x \in [-L/2, L/2]$. What is the cross section σ_{21} for the transition from the ground state ($n = 1$) to the first excited state ($n = 2$)?
   Compare the factor in front of the delta distribution with that for the harmonic oscillator.

4. **Title (classroom assignment)**. Besides their natural line width, emission lines with transition frequency ω_0 are broadened due to collisions of the emitting atoms and their thermal velocities. The collisional broadening leads to a line shape that is described by a Lorentz profile

$$\Phi_{\Gamma_c}(\omega - \omega_{12}) = \frac{1}{\pi}\frac{\Gamma_c/2}{(\omega - \omega_{12})^2 + \Gamma_c^2/4},$$

where $\Gamma_c = \sigma\langle nv\rangle$ is the collision rate, σ is the cross section for collisions, n is the number density of atoms, v their velocity and $\langle\cdot\rangle$ indicates the thermal average. The Doppler broadening leads to the Gaussian profile function

$$\Phi_D = \frac{c}{\sqrt{2\pi}\omega_0\sigma_v}\exp\left[-\frac{c^2}{2\sigma_v^2}\left(\frac{\omega - \omega_0}{\omega_0}\right)^2\right],$$

where σ_v is the velocity dispersion.

(a) Estimate the line width for Doppler broadening from the full width at half maximum (FWHM) $\Delta\omega_D$ from the Gaussian profile, defined by $\Phi_D(\omega_0 \pm \Delta\omega_D/2) = \Phi_D(\omega_0)/2$, as a function of temperature T.

(b) Estimate the line width $\Delta\omega_c$ due to collisions from the FWHM of the Lorentz profile $\Phi_{\Gamma_c}(\omega)$ as a function of T. Assume that σ is set by the Bohr radius $a_B = \hbar/(\alpha mc)$ and that the density does not depend on temperature. *Hint:* What is the relation between the line width $\Delta\omega_c$ and the collision rate Γ_c?

(c) How can the results from (a) and (b) be combined to determine the density of an emitting medium?

(d) Calculate the ratio $\Delta\omega_c/\Delta\omega_D$ for the Hα line (6563Å) emitted from a cloud of atomic hydrogen with $n = 16\,\mathrm{cm}^{-3}$.