Theoretical Astrophysics

Heidelberg University
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Problem Sheet 5

Discussion in the tutorial groups in the week of November 20, 2023

1. **Comprehension Question**.
   (i) What is a curve of growth for a spectral line? What does it look like schematically?
   (ii) Name in your own words the foundational concepts of hydrodynamics.
   (iii) What is ideal hydrodynamics, and which additional processes occur in the non-ideal case?

2. **Energy-momentum tensor of a single particle**. The energy-momentum tensor is defined as

$$T^{\mu\nu} = c^2 \int \frac{\mathrm{d}^3 p}{E(p)} p^\mu p^\nu f(\vec{x}, \vec{p}, t)$$

   where $(p^\mu) = (E/c, \vec{p})^T$ is the four-momentum, E is the energy, and $f(\vec{x}, \vec{p}, t)$ the one-particle phase-space density distribution. While the energy density is $\epsilon = T^{00}$, the pressure is given by one third of the stress-energy tensor's trace, hence $P = (1/3) \sum_{i=1}^{3} T^{ii}$.

   (a) Determine $T^{\mu\nu}$ for a single particle of mass m with trajectory $\vec{x}_0(t)$ and momentum $\vec{p}_0(t)$.
   (b) Determine $T^{\mu\nu}$ for a photon of frequency ω with trajectory $\vec{x}_0(t)$.
   (c) How is the energy density related to the pressure in the two cases of (a) and (b)?

3. **Isothermal fluids**. The hydrodynamical equations describing mass conservation, momentum conservation, and energy conservation for an ideal fluid are

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \tag{1}$$

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla P}{\rho} \tag{2}$$

$$\partial_t \epsilon + \nabla \cdot (\epsilon \vec{v}) = -P \nabla \cdot \vec{v}, \tag{3}$$

   respectively, with the mass density ρ, the pressure P, and the energy density ϵ.

   (a) Show using equation (3) that an isothermal ideal fluid, i.e. a fluid with constant temperature $T(x, t) = T_0$ and non-vanishing pressure $P \neq 0$, is also incompressible, that is $\nabla \cdot \vec{v} = 0$.
   (b) Show that for a spherically symmetric and isothermal flow of an ideal gas, equations (1) through (3) simplify to

$$\partial_t \rho + v \partial_r \rho = 0$$

$$\partial_t v - \frac{2v^2}{r} = -c_s^2 \partial_r \ln \rho,$$

   where $c_s := \sqrt{k_B T_0 / m}$ is a characteristic thermal speed and $v = |\vec{v}|$.

4. **Keplerian disk**. Young stars often form in the centre of a thin accretion disk whose height is much smaller than its radius. If the mass of the central object M is much larger than the disk's mass, the gas particles move on approximately Keplerian orbits which are almost circular.

   (a) What is the velocity v of a gas particle as a function of the radius r? Determine also the divergence of the velocity field.

   (b) Calculate the components of the velocity tensor

   $$v_{ij} = \frac{1}{2}\left(\partial_j v_i - \partial_i v_j\right)$$

   for the Keplerian disk. Use explicitly Cartesian coordinates.

5. **Fourier transform (classroom assignment)**. Calculate by hand the Fourier transform of

   $$f_n(r) = r^{-n}\mathrm{e}^{-ar}$$

   with $a = \text{const.}$ and $r = |\vec{x}|$ for $n = 0$ and $n = 1$ in 3 dimensions.