Theoretical Astrophysics

Heidelberg University
Winter Term 2023/24

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Problem Sheet 2

Discussion in the tutorial groups in the week of October 30th, 2023

1. **Comprehension Question**.

   (i) Summarise in your own words the essential steps for calculating the spectrum of an accelerated point charge.

   (ii) Why is the synchrotron spectrum broad despite the sharp angular frequency of an electron in a magnetic field?

   (iii) Discuss when and why it is permitted to set $\beta = 1$ for an ultrarelativistic charge, but not $1 - \beta = 0$.

2. **Spectrum of an electron**. Consider an electron whose one-dimensional trajectory $x(t)$ satisfies the differential equation of a damped harmonic oscillator,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0.$$

   (a) What is the oscillator frequency ω if ω_0 is the system's eigenfrequency? *Hint:* Use the ansatz $x(t) \propto \mathrm{e}^{\mathrm{i}\omega t}$. What does a complex frequency mean physically?

   (b) Show that the solution of the differential equation is given by

$$x(t) = \frac{v_0}{\bar{\omega}}\mathrm{e}^{-\gamma t}\sin\bar{\omega}t \quad \text{with} \quad \bar{\omega} := \sqrt{\omega_0^2 - \gamma^2}$$

   if $\omega_0 > \gamma$ and the initial conditions are $x(t = 0) = 0$ and $\dot{x}(t = 0) = v_0$.

   (c) Calculate the Fourier transform $\hat{x}(\omega)$. Assume that $x(t) = 0$ for $t < 0$.

   (d) Calculate the spectrum $\mathrm{d}E/\mathrm{d}\omega$ of the moving electron.

   (e) What does the spectrum look like if both $\omega \gg \omega_0$ and $\omega \gg \gamma$?

3. **Spectrum of a thermal electron distribution**. Due to the Lorentz force, a non-relativistic electron moving with a velocity $\vec{v}$ through the magnetic field $\vec{B}$ experiences the acceleration

$$\ddot{x} = -\frac{e}{m_e c}\left(\vec{v} \times \vec{B}\right),$$

   with the elementary charge e, the electron mass m_e and the speed of light c.

   (a) What is the average amount of energy per unit time and volume, $\left\langle \mathrm{d}^2E/(\mathrm{d}t\mathrm{d}V)\right\rangle$, radiated away by an isotropic electron distribution with number density n_e?

(b) Assume now further that the electrons are in thermal equilibrium. In this case, the probability for an electron to have the velocity $v = |\vec{v}|$ is given by the Maxwell-Boltzmann distribution

$$p(v)\mathrm{d}v = \sqrt{\frac{2}{\pi}} \left(\frac{m_e}{k_B T}\right)^{3/2} v^2 \exp\left(-\frac{m_e v^2}{2k_B T}\right),$$

where T is the temperature of the electron gas and k_B is Boltzmann's constant. Calculate $\mathrm{d}^2E/(\mathrm{d}t\mathrm{d}V)$ as a function of the electron gas temperature T and the magnetic field $\vec{B}$. *Hint:* You may (or not) need

$$\int_0^\infty \mathrm{d}x\, x^4 \mathrm{e}^{-ax^2} = \frac{3\sqrt{\pi}}{8} a^{-5/2}.$$

4. **Synchrotron spectrum of a nonthermal electron distribution (classroom assignment)**. The synchrotron spectrum in the orbital plane of a single electron with Larmor frequency ω_L is

$$\frac{\mathrm{d}^2E}{\mathrm{d}\omega\mathrm{d}\Omega} = \frac{3e^2\gamma^2}{\pi c} \left(\frac{\omega}{\omega_c}\right)^2 K_{2/3}^2\left(\frac{\omega}{\omega_c}\right)$$

where $\omega_c = 3\omega_L\gamma^3 = (3eB/mc)\gamma^2$, $K_{2/3}(x)$ is the modified Bessel function of order 2/3 of the second kind, m is the electron mass, e is the elementary charge and c the speed of light.

(a) In stochastic particle-acceleration processes, the accelerated electrons typically follow an energy distribution of the power-law form

$$\frac{\mathrm{d}N}{\mathrm{d}\epsilon}\mathrm{d}\epsilon = A\epsilon^{-\alpha}\mathrm{d}\epsilon$$

where A is a normalisation constant, and $\alpha > 1$. Calculate the spectrum for such a population of electrons. *Hint:* Express the electron energy ϵ by γ and use

$$\int_0^\infty \mathrm{d}x\, x^a K_{2/3}^2\left(bx^2\right) = b^{-(a+1)/2} \frac{\sqrt{\pi}\Gamma\left(\frac{3a-5}{12}\right)\Gamma\left(\frac{3a+11}{12}\right)\Gamma\left(\frac{a+1}{4}\right)}{8\Gamma\left(\frac{a+3}{4}\right)}$$

valid for $a > 5/3$.

(b) Draw the expected spectrum schematically in a double-logarithmic plot.