Theoretical Astrophysics

Heidelberg University
Winter Term 2023/24

Lecturer: Prof. Dr. Matthias Bartelmann
Head tutor: Dr. Michael Zacharias

Problem Sheet 3

Discussion in the tutorial groups in the week of November 6th, 2023

1. **Comprehension Question**.

   (i) Summarise the essential properties of synchrotron radiation and thermal bremsstrahlung.

   (ii) Explain the main difference between Thomson and Compton scattering.

   (iii) What is Born's approximation?

2. **Klein-Nishina scattering**. The differential cross section for photons with energy $\hbar\omega$ that are scattered off free electrons is given by the Klein-Nishina formula

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{r_e^2}{2} F^2(\omega,\Omega)\left[F(\omega,\Omega) + \frac{1}{F(\omega,\Omega)} - 1 + \cos^2\theta\right]$$

   with the classical electron radius r_e, and

$$F(\omega,\Omega) = \left[1 + \frac{\hbar\omega}{m_e c^2}(1-\cos\theta)\right]^{-1}.$$

   (a) What is the ratio $\hbar\omega/m_e c^2$ for visible light? How does the Klein-Nishina formula simplify in this case? Is the solution familiar to you?

   (b) Assume that an electron is hit by a photon with energy $\hbar\omega = m_e c^2$. Calculate the total cross section

$$\sigma_{\mathrm{KN}} = \int \mathrm{d}\Omega\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$$

   and compare it to the classical Thomson cross section $\sigma_T = 8\pi r_e^2/3$.

3. **Lorentz boost for photons**. Consider a photon with frequency ω scattered by a resting electron under the angle θ. By the scattering process, its frequency changes to $\omega' < \omega$. One can transform into the barycenter system, defined by $\vec{p}_{\mathrm{tot}} = \vec{0}$ before and after the scattering, by applying a proper Lorentz boost

$$(\Lambda^\mu_\nu) = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

   to the four-momentum $(p^\mu) = (E/c, \vec{p})^T$, assuming that the incoming photon moves along the negative z-direction.

(a) Calculate the energies and momenta of both the electron and the photon in the barycenter system as a function of β.

(b) Determine the velocity β as a function of ω and the electron mass m_e.

(c) Express the scattering angle θ^* in the barycenter system as a function of the scattering angle θ in the rest frame of the electron, ω and m_e.

4. **Fourier transformation of the acceleration employing Born's approximation**. Relate the acceleration of a particle with mass m in Born's approximation to the Fourier transform of the accelerating potential. In order to so, follow these steps:

   (a) Write the acceleration $\vec{a}$ as a function of the potential energy and Fourier transform $V(\vec{x}) \to \tilde{V}(k)$ to find

   $$\vec{a}(\vec{x}) = -\frac{\mathrm{i}}{m} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \vec{k}\tilde{V}(k)\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}}.$$

   (b) Introduce the particle orbit in Born's approximation.

   (c) Now, perform the Fourier transformation from time to frequency space to arrive at

   $$\tilde{\vec{a}}(\omega) = -\frac{\mathrm{i}}{m} \int \frac{\mathrm{d}^3k}{(2\pi)^2} \vec{k}\tilde{V}(k)\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}_0}\delta_D(\vec{k}\cdot\vec{v} - \omega),$$

   where $\delta_D(u)$ is Dirac's delta-distribution.

   (d) Finally, assume $\vec{x}_0 = b\hat{e}_y$ and $\vec{v} = v\hat{e}_x$ and solve one of the three integrals.

5. **Energy loss time scale of electrons (classroom assignment)**. Relativistic electrons passing through a (thermal) photon field with energy density U lose energy through inverse-Compton scattering on the characteristic time scale

   $$\tau = \frac{3m_e c}{4U\sigma_T}.$$

   Derive the cooling time scale for an electron passing through the cosmic microwave background (CMB) today and when the CMB was emitted.
   Hints: The energy density of a thermal photon field of temperature T is

   $$U = \frac{\pi^2(k_B T)^4}{15(\hbar c)^3}$$

   with $k_B = 8.617\times10^{-5}$ eV K^{-1}, $\hbar = 6.582\times10^{-16}$ eV s, $c = 3\times10^{10}$ cm s^{-1}, and $\sigma_T = 6.65\times10^{-25}$ cm^2. Furthermore, the temperature of the CMB scales as $T(z) = (1+z)T_{\mathrm{now}}$ with the cosmological redshift z and the present-day ($z = 0$) temperature $T_{\mathrm{now}} = 2.72$ K. The Universe was opaque at temperatures above 3000 K. Which redshift does this correspond to?