Theoretical Astrophysics

Heidelberg University
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Lecturer: Prof. Dr. Matthias Bartelmann
Head tutor: Dr. Michael Zacharias

Problem Sheet 1

Discussion in the tutorial groups in the week of Oct. 23rd, 2023

1. **Comprehension questions**.
   - Summarize how Maxwell's equations lead to a wave equation for the electrodynamic potentials. Which gauge is typically being used here?
   - How can this wave equation generally be solved?
   - What is the energy current density of an electromagnetic field?

2. **Liénard-Wiechert Potentials**. Maxwell's equations can be brought into the form of the wave equation

$$\Box A^\mu = -\frac{4\pi}{c} j^\mu \ ,$$

which is solved by means of the retarded Green's function

$$G\left(\vec{x}, \vec{x}', t, t'\right) = \frac{1}{|\vec{x} - \vec{x}'|} \delta_\mathrm{D} \left(t - t' - \frac{|\vec{x} - \vec{x}'|}{c} \right) \ .$$

   (a) Find an expression for the current-density four-vector j^μ of a point charge q moving along a trajectory $\vec{x}_0(t)$.

   (b) Show that the potentials of a point charge q moving along a trajectory $\vec{x}_0(t)$ are given by

$$A^\mu\left(\vec{x}, t\right) = \begin{pmatrix} 1 \\ \vec{\beta} \end{pmatrix} \frac{q}{R\left(1 - \hat{e} \cdot \vec{\beta}\right)} \ ,$$

   where $\vec{R} = \vec{x} - \vec{x}_0(t')$, $R = |\vec{R}|$, $\hat{e} = \vec{R}/R$, and $\vec{\beta}(t') = \vec{v}(t')/c$.

3. **Derivatives of the retarded time and distance**. With the definition of the retarded time,

$$t' = t - \frac{R(t')}{c} \ ,$$

show that

   (a) the derivative of the retarded time t' with respect to the time t is

$$\partial_t t' = \frac{1}{1 - \hat{e} \cdot \vec{\beta}}$$

   (b) and that its gradient with respect to $\vec{x}$ is

$$\vec{\nabla} t' = -\frac{\hat{e}}{c\left(1 - \hat{e} \cdot \vec{\beta}\right)} \ .$$

4. **Far-field electric field (classroom assignment)**. In terms of the potentials A^μ, the electric field is given by

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\partial_t \vec{A} \, .$$

(a) Take the gradient of $\Phi = A^0$, sort the terms by how steeply they depend on R, and show that the one term falling of like R^{-1} is

$$\vec{\nabla}\Phi = -\frac{q\hat{e}\left(\hat{e}\cdot\dot{\vec{\beta}}\right)}{cR\left(1-\hat{e}\cdot\vec{\beta}\right)^3} \, .$$

(b) Take the time derivative of $\vec{A} = (A^1, A^2, A^3)^\top$ and show that its contribution falling off like R^{-1} reads

$$\partial_t \vec{A} = \frac{q}{R\left(1-\hat{e}\cdot\vec{\beta}\right)^3}\left[\left(1-\hat{e}\cdot\vec{\beta}\right)\dot{\vec{\beta}} + \vec{\beta}\left(\hat{e}\cdot\dot{\vec{\beta}}\right)\right]$$

(c) Combine the two results and use the identity $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ twice to bring the electric field far from its source into the form

$$\vec{E} = \frac{q}{cR\left(1-\hat{e}\cdot\vec{\beta}\right)^3}\,\hat{e}\times\left[\left(\hat{e}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right] \, .$$