Cosmic Structure Formation

Heidelberg University
Winter term 2022/23

Lecturer: Prof. Dr. Matthias Bartelmann
Head tutor: Selin Üstündağ

Problem Sheet 3

Discussion in the tutorial group on Nov. 17th, 2022

1. **Burgers' equation**

   In the Zel'dovich approximation, the peculiar velocity of a particle remains constant,

   $$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = \partial_t\vec{u} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} = 0\ .$$

   In the so-called adhesion approximation, this equation is augmented by a viscosity term on the right-hand side,

   $$\partial_t\vec{u} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} = \nu\vec{\nabla}^2\vec{u}$$

   with constant viscosity ν. This is Burgers' equation.

   a) Introduce a velocity potential ψ such that $\vec{u} = \vec{\nabla}\psi$, and with it the exponential potential

   $$U(\vec{q}, t) = \exp\left(-\frac{\psi(\vec{q}, t)}{2\nu}\right)$$

   Show that this Hopf-Cole transformation turns Burgers' equation into the diffusion equation

   $$\partial_t U - \nu\vec{\nabla}^2 U = 0\ .$$

   b) Show that this equation is solved by the convolution

   $$U(\vec{q}, t) = \left(\frac{1}{4\pi\nu t}\right)^{3/2}\int \mathrm{d}^3y\, U^{(\mathrm{i})}(\vec{y})\,\exp\left(-\frac{(\vec{q}-\vec{y})^2}{4t\nu}\right)$$

   of the initial exponential potential $U^{(\mathrm{i})}$ with a Gaussian kernel of increasing width.

   c) Extract the peculiar velocity from this result and interpret the result.

2. **Power spectra**

   Let the initial velocity potential ψ be a Gaussian random field with power spectrum P_ψ.

   a) Explain why the power spectrum of the density contrast is related to it by

   $$P_\delta = k^4 P_\psi\ .$$

   b) Find the power spectrum of the peculiar-velocity field in terms of P_ψ and P_δ.

   c) Determine a suitable rotational invariant of this power spectrum.