Cosmic Structure Formation

Heidelberg University
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Problem Sheet 2

Discussion in the tutorial group on Nov. 10th, 2022

1. **Cosmic flow with vorticity**

   We have argued in the lecture that a curl component of the peculiar-velocity field decays in linear order and thus can often be neglected.

   a) Look up Helmholtz' decomposition theorem.

   b) Take the linearised Euler equation for the peculiar-velocity field,

   $$\partial_t \vec{u} + 2H\vec{u} = -\frac{1}{a^2}\vec{\nabla}\phi \ ,$$

   and consider a curl component of the velocity field, $\vec{u} = \vec{\nabla} \times \vec{w}$. Assuming without loss of generality that $\vec{\nabla} \cdot \vec{w} = 0$, show that $\vec{w}$ satisfies

   $$\frac{\vec{\nabla}^2 \dot{w}_i}{\vec{\nabla}^2 w_i} = -2H$$

   for all components w_i of $\vec{w}$.

   c) Solve this equation and use the definition of the Hubble function H to show that the amplitude of w_i decreases with increasing scale factor a as a^{-2}.

2. **Perturbative solution of Hamilton's equations**

   Consider the motion of a test particle of constant mass m with Hamiltonian

   $$H = \frac{p^2}{2m} + m\varphi$$

   in a static space-time.

   a) Show that the particle's phase-space trajectory $x(t)$ is given by

   $$x(t) = G(t,0)x^{(\mathrm{i})} - m\int_0^t G(t,t')\begin{pmatrix} 0 \\ \nabla\varphi \end{pmatrix} \tag{I}$$

   with

   $$G(t,t') = \begin{pmatrix} 1 & (t-t')/m \\ 0 & 1 \end{pmatrix} .$$

   b) Split Eq. (I) into its position and momentum parts.

   c) Construct a first-order, perturbative solution in the following way: find the unperturbed solution by setting $\varphi = 0$ first, then insert this solution into the full equations for position and momentum. Solve them adopting the potential

   $$\varphi = -\frac{\alpha}{r}$$

   with a positive constant α.