Tutorials for the Lecture
Introduction to Astronomy and Astrophysics
(Summer Term 2022)

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Day 3

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9 Points

1. **M stars, red giants and the pressure broadening of spectral lines** [3 pt]
There are stars that have the same spectral type (hence the same temperature) but very different absolute luminosity L_ν. For example: an M dwarf and a red giant. When we observe a star of spectral type M, we do not know whether this is a M dwarf near to us or a red giant far from us.

We assume an M dwarf to be a star with $T_{\text{eff},*} = 3000\text{K}$, $R_* = 0.1\, R_\odot$, and $M_* = 0.1\, M_\odot$. We also assume a red giant to be a star with $T_{\text{eff},*} = 3000\text{K}$, $R_* = 30\, R_\odot$, and $M_* = 1\, M_\odot$.

(a) How much farther away should the red giant be from us with respect to the M dwarf in order to have the same flux as the M dwarf?

Unfortunately, it is difficult to measure the distance of most stars via their parallaxes, and thus it will be difficult to distinguish between a M dwarf and a red giant.

Luckily enough, though, one can estimate the strength of the gravitational constant g at the stellar surface from the spectrum of a star, and then assess the difference between a M dwarf and a red giant.

(b) Determine g in cm/s^2 for both the M dwarf and the red giant using their properties we assume above.

Now, how do we derive g from the spectrum? First, we assume that the gas pressure in the stellar photosphere P_ν is directly proportional to g (i.e. P_ν increases with increasing g). We then define P_ν as the pressure at the height in the photosphere where the optical depth τ_ν (here ν is frequency, the unit of our spectrum) is 2/3. At $\tau_\nu = 2/3$ the Eddington-Barbier relation holds:

$$I_\nu^{\text{obs}} \simeq B_\nu(T(z(\tau_\nu = 2/3))) \tag{1}$$

where $z(\tau_\nu = 2/3)$ is the value of z and $\tau_\nu(z) = 2/3$. If we initially assume an opacity κ_ν that does not depend on ρ and T, we can express the continuum optical depth from a height z in the atmosphere to the observer as:

$$\tau_\nu(z) = \kappa_\nu \int_z^\infty \rho(z')dz' \tag{2}$$

where the line of sight is considered to be perpendicular to the atmosphere. If we also at first suppose that the temperature of the atmosphere is constant with z, the structure of atmosphere can be described with a simple exponential function, as we have seen in the lecture.

(c) Demonstrate with this assumption that

$$P_\nu = \frac{2}{3}\frac{g}{\kappa_\nu} \tag{3}$$

We now waive the assumption that T = constant, since an atmosphere without a temperature gradient will not give rise to spectral lines. We now use the proportionality between the line width $\Delta\nu/\nu$ and the gas pressure:

$$\frac{\Delta\nu}{\nu} \propto \frac{P}{\sqrt{T}} \tag{4}$$

(d) Argue that the lines in the spectrum of our M dwarf are wider than those in the spectrum of the red giant.

2. **Polytropic stars** [6 pt]
We consider a star that is made of ideal gas and is perfectly polytropic. This means that the pressure and density follow, everywhere in the star, the equation of state of an ideal gas:

$$P(r) \equiv \frac{k_B}{\mu m_p}\rho(r)T(r) = K\rho(r)^\gamma \tag{5}$$

where r is the radial coordinate, γ the adiabatic index, k_B the Boltzmann constant, μm_p the mass of one gas molecule, ρ the density, P the pressure, T the temperature and K a constant that depends on the type of gas. Often, γ is replaced by the polytropic index n:

$$n = \frac{1}{\gamma - 1} \tag{6}$$

We have seen in the lecture that the solution of the equation of hydrostatic equilibrium can be written in the following dimensionless form:

$$r = \alpha x, \qquad \rho(r) = \rho_c\Theta(x)^n, \qquad \alpha^2 = \frac{K(n+1)}{4\pi G}\rho_c^{(1/n)-1} \tag{7}$$

where G is the gravitational constant, ρ_c the density at the centre of the star, x one dimensionless coordinate, and $\Theta(x)$ one solution of the Lane-Emden equation. For a given n there is only one $\Theta(x)$ solution to the Lane-Emden equation.

(a) Demonstrate that one can write K as:

$$K = \frac{k_B T}{\mu m_p}\frac{1}{\rho^{1/n}} = \frac{k_B T_c}{\mu m_p}\frac{1}{\rho_c^{1/n}} \tag{8}$$

where T_c is the temperature at the centre of the star.

(b) Show that the radius of the star depends on T_c and ρ_c as follows:

$$R_* = A\sqrt{\frac{T_c}{\rho_c}} \tag{9}$$

where A is a constant, which depends on universal constants as well as on n and μ (but not on ρ_c or T_c).

(c) Demonstrate that the mass of the star depends on T_c and ρ_c as follows:

$$M_* = B \frac{T_c^{3/2}}{\rho_c^{1/2}} \tag{10}$$

where B is a constant, which depends on universal constants as well as on n und μ (but not on ρ_c or T_c). Important: unfortunately no general analytical expression exists for B (except for very specific values of n) because B contains an integral that cannot be solved analytically. Do not be put off by it.

(d) Demonstrate that the escape velocity at the stellar surface $v_{\text{esc}} = \sqrt{2GM_*/R_*}$ depends on universal constants as well as on the central temperature T_c and not on the stellar mass or the central density ρ_c.

(e) Show that

$$R_* \propto \frac{M_*}{T_c} \tag{11}$$

We now consider what happens when a star radiates away part of its energy. We assume that no nuclear reaction is taking place in the stellar core. This means that K changes with time (but n remains constant), and so do R_*, ρ_c, T_c and $E_{\text{tot}} = E_{\text{therm}} + E_{\text{pot}}$ also. We wish to find out how they vary (will they increase or decrease with time?). One can demonstrate with this polytropic model that

$$E_{\text{pot}} \equiv -4\pi \int_0^{R_*} r^2 \frac{G\rho(r)}{r} M(r) dr \propto -\frac{M_*^2}{R_*} \tag{12}$$

which is also physically justified. From the virial theorem it follows that the total energy scales in this way:

$$E_{\text{tot}} \propto -\frac{1}{2} \frac{M_*^2}{R_*} \tag{13}$$

which one can also derive by directly integrating the polytropic model (but this is not part of the exercise).

(f) Show that, when the star loses energy, the central temperature increases. In other words: that the star has a negative heat capacity.

(g) The energy production of a star via nuclear fusion is a strongly increasing function of temperature, for example $\epsilon \propto \rho T^5$ for the so-called "pp chain". Discuss why the negative heat capacity of the star keeps the star stable (main-sequence star), or, on the contrary, if the star had a positive heat capacity (like every "normal" system has), it would explosively convert all its fuel into heat like a giant bomb.