Standard Model of Particle Physics

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Problem Sheet 00

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Problem 1. Index gymnastics

These small exercises are intended for you to review some basic ideas about tensors and get some practice in doing index manipulations. In the Standard Model, you will come across a lot of different indices (many of which will sometimes be omitted), so it's a good idea that you become familiar in dealing with them.

On this sheet, Greek (space-time) indices will range from 0 to $d-1=4-1=3$ (the Standard Model is constructed in 4 space-time dimensions), whereas Latin indices from the middle of the alphabet starting with i will be spatial indices, and so will run from 1 to $d-1=3$. Keep in mind that in the lectures you will find indices that are not space-time or spatial ones (for example, gauge or family indices), and those might follow different conventions and run over a different set of values!

We will use the Einstein summation convention, which means that repeated indices (usually appearing as one lower and one upper index) are summed over all the possible values of the index.

1. Let $g^{\mu\nu}$ denote the inverse of the metric $g_{\mu\nu}$, i.e. the tensor for which $g^{\mu\nu}g_{\nu\sigma}=\delta^{\mu}_{\sigma}$. Evaluate $g^{\mu\nu}g_{\nu\mu}$.
2. Evaluate $g^{ij}g_{ij}$.
3. In these lectures, $g^{\mu\nu}$ always denotes the flat-space metric. However, in general this can be a function of space-time. Prove that, in these cases, $\partial_\mu g^{\nu\rho}=-g^{\nu\kappa}g^{\rho\lambda}\partial_\mu g_{\lambda\kappa}$.
4. We denote symmetrization of a tensor $T_{\alpha\beta\ldots\gamma}$ as $T_{(\alpha\beta\ldots\gamma)}$, and antisymmetrization as $T_{[\alpha\beta\ldots\gamma]}$. Evaluate $T^{(\alpha\beta)}\omega_{[\alpha\beta]}$.
5. Evaluate $\omega_{[\alpha\beta]}U^\alpha U^\beta$. Hence, show that $\omega_{\alpha\beta}U^\alpha U^\beta=\omega_{(\alpha\beta)}U^\alpha U^\beta$.

Problem 2. Dimensional Analysis

This problem is intended as a summary of some basic concepts in collider physics, and it is for you to gain better control on the quantities that you will be working with during the course.

a) At a particle collider, the number of events of a certain scattering process is given by

$$N=\mathcal{L}\,\sigma \tag{1}$$

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where $\mathcal{L}$ is called *luminosity* and it's a property of the collider, and σ is the *cross-section* and is a property of the process.

Let's assume that the collider operates with two beams of particles traveling in opposite directions, that therefore collide head-on. In a time interval Δt, each beam delivers N particles, that are contained in cylinder with effective area A and height $c\Delta t$ (if the particles move at the speed of light). In this case the integrated luminosity during Δt is

$$\mathcal{L} = \frac{N^2}{A} \tag{2}$$

It's more common to define a *instantaneous luminosity*, differentiating over time

$$\frac{d\mathcal{L}}{dt} = \frac{N^2}{A\Delta t} = \frac{N^2 f}{A} \tag{3}$$

and $f = 1/\Delta t$ is the frequency with which bunches of N particles cross. In this way $R = \sigma d\mathcal{L}/dt$ is the rate at which the process takes place.

What dimensions do $\mathcal{L}$ and σ have? And the instantaneous luminosity?

b) In particle physics is very common to use natural units, ie. to set $\hbar = 1 = c$. What are the dimensions in this case?

Hint: remember $\hbar$ has dimensions of an action and c of a speed: $[\hbar] = [ET]$, $[c] = [LT^{-1}]$.

c) When doing a measurement, one often measures the cross section as a function of a kinematic quantity, for instance the invariant mass of the collision s or the angle between two scattering products.

What are the dimensions of the differential cross section $\frac{d\sigma}{ds}$? and of $\frac{d\sigma}{d\cos\theta}$?

d) Imagine that you are observing 1000 collision events of a given process. For each of them you have a measurement of $\sqrt{s}$, with values that span between 0 and 500 GeV. You divide this range in 20 bins of equal size and you make a histogram of $\sqrt{s}$ values.

What is the quantity you are representing in the y axis? What units would it have?

e) Unstable particles can decay into other states, which gives them a finite lifetime τ.

The lifetime is related to the decay width of the particle Γ by $\tau = \hbar/\Gamma$.

What are the dimensions of the decay width? And in natural units?

Can you relate $\tau = \hbar/\Gamma$ to the uncertainty principle?

f) In general, an unstable particle p can decay to different final states: $p \to f_i$. The particle width is then the sum over the partial widths into each channel:

$$\Gamma_p = \sum_i \Gamma(p \to f_i) \tag{4}$$

In this case it is convenient to define quantities called *branching ratios*, that indicate the probability of each decay compared to the total rate:

$$\mathrm{Br}(p \to f_i) = \frac{\Gamma(p \to f_i)}{\sum_j \Gamma(p \to f_j)} \tag{5}$$

What are the dimensions of a branching ratio?

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Problem 3. Symmetries

a) Suppose we have a theory of two independent $U(1)$ gauge groups with field strength tensors: $F^{\mu\nu}$ and $B^{\mu\nu}$ respectively. Does the following Lagrangian density respect the basic symmetries (Lorentz and the $U(1) \times U(1)$ gauge symmetry)?

$$\mathcal{L} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{4} B^{\mu\nu} B_{\mu\nu} + \frac{1}{4} F^{\mu\nu} B_{\mu\nu} \tag{6}$$

b) We repeat the same exercise but this time with two $SU(2)$ gauge groups with field strength tensors $F^{I\mu\nu}$ and $B^{I\mu\nu}$ where I is the $SU(2)$ generator index. Does the following Lagrangian density respect the basic symmetries (Lorentz and the $SU(2) \times SU(2)$ gauge symmetry)?

$$\mathcal{L} = \frac{1}{4} F^{I\mu\nu} F^{I}_{\mu\nu} + \frac{1}{4} B^{I\mu\nu} B^{I}_{\mu\nu} + \frac{1}{4} F^{I\mu\nu} B^{I}_{\mu\nu} \tag{7}$$

Problem 4. Lagrangian and field redefinitions

In this exercise we will construct the most general Lagrangian for a fermion field, compatible with Lorentz invariance.

Consider a theory that contains N non-interacting fermion states ψ_m. The Lagrangian is a quadratic function in the fields, with dimensions $[\mathcal{L}] = [E^4]$. The most general set of Lorentz-invariant terms that we can write is then

$$\mathcal{L} = A_{mn} i \bar{\psi}_m \gamma^\mu \partial_\mu \psi_n + B_{mn} i \bar{\psi}_m (i\gamma_5) \gamma^\mu \partial_\mu \psi_n - C_{mn} \bar{\psi}_m \psi_n - D_{mn} \bar{\psi}_m i\gamma_5 \psi_n + E \tag{8}$$

a) The Lagrangian must always be a Hermitian function. What does this imply for the quantities $A_{mn}, B_{mn}, C_{mn}, D_{mn}, E$?

b) Rewrite the Lagrangian above separating the left-handed and right-handed components of the spinors:

$$\psi_L = P_L \psi, \qquad \psi_R = P_R \psi \tag{9}$$

Hint: remember the projectors properties $P_L + P_R = \mathbb{1}$, *and the definition* $P_L = (1 - \gamma_5)/2$, $P_R = (1 + \gamma_5)/2$.

c) The Lagrangian given above is *redundant*, ie. some of the terms can be removed performing field redefinitions. Consider the redefinition

$$\psi_{L,m} \mapsto \mathcal{V}_{mn} \psi'_{Ln} \,, \tag{10}$$

$$\psi_{R,m} \mapsto \mathcal{V}^*_{mn} \psi'_{Rn} \,. \tag{11}$$

Show that this is equivalent to

$$\psi_m \mapsto V_{mn} \psi'_n - U_{mn} i\gamma_5 \psi'_n \,. \tag{12}$$

for $\mathcal{V} = V + iU$.

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d) Define the complex matrices $\mathcal{A} = A + iB$, $\mathcal{C} = C + iD$. What properties do they satisfy, given those of A, B, C, D?

What properties does $\mathcal{A}$ have to fulfill for the theory to be bounded from below?

Apply the transformation in Eqs (10), (11) to the Lagrangian written in terms of chiral fields.

e) We can still choose the form of $\mathcal{V}$. Define $\mathcal{V} = (\mathcal{A}^*)^{-\frac{1}{2}} \mathcal{M}^*$ where $\mathcal{M}$ is the unitary matrix that satisfies the following property:

$$\mathcal{M}^T \mathcal{C}' \mathcal{M} = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_N \end{pmatrix}, \tag{13}$$

where $\mathcal{C}'$ is the symmetric matrix defined by: $\mathcal{C}' = \left[(\mathcal{A}^*)^{-\frac{1}{2}} \mathcal{C} \mathcal{A}^{-\frac{1}{2}}\right]$. One can show that such a matrix $\mathcal{M}$ always exists and that we can chose the $c_1 \ldots c_N$ to be real and non-negative.

Apply this transformation to the Lagrangian. What is the final form of the Lagrangian?