Particle Physics 1+2
Theory Chapters

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Abstract

These lecture notes cover the six theory modules in master-level Particle Physics 1 and 2. They assume a solid theory background from the theory bachelor courses, for instance analytical mechanics, electromagnetism, and quantum mechanics. Aspects of quantum field theory are sketched and then skipped, we refer to the specialized lectures for more details. The goal of the lecture is to understand the theory background of modern particle physics, from low to high energies, and be able to do simple calculations.

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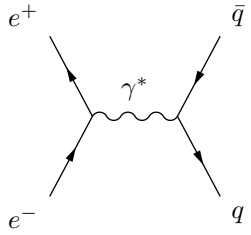
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1 A simple scattering process

When we compute transition amplitudes for collider like LEP or LHC, we usually combine building blocks defined by Feynman rules in a way which does not make it obvious that we are dealing with a quantum field theory. One of the easiest processes we can look at is

$$e^+e^- \to \gamma^* \to q\bar{q}\,, \tag{1.1}$$

only through a photon, where the star indicates that the photon is not a physical particle on its mass shell. We can graphically represent this process with the scattering or time proceeding from the left to the right,



We will see how we can compute this scattering process using so-called Feynman rules. We first need to describe four external fermions, an electron and the positron as its antiparticle and a quark-antiquark pair. We also need to describe the fermion coupling to a photon, and the propagation of this boson from the e^+e^- annihilation to the point where is splits into a $\bar{q}$ pair.

From theoretical mechanics we remember that there are several ways to describe a physical system and calculate its time evolution. Assuming one degree of freedom, corresponding to a real scalar field ϕ, we can for instance start with the action

$$S = \int d^4x\, \mathscr{L}(\phi, \partial_\mu\phi) \qquad \text{with} \qquad x = \begin{pmatrix} x_0 \\ \vec{x} \end{pmatrix}\,. \tag{1.2}$$

The position x is given by a space-time 4-vector, a space and metric defined by special relativity. Under the integral there is a Lagrange density, which works exactly like the Lagrange function in theoretical mechanics. The object ϕ is now a quantum field, and in particle physics we use space-time with the Minkoswki metric (+ - - -). The equation of motion or time dependence of the scalar field is provided by the invariance of the action under a variation $\delta S = 0$. From theoretical mechanics we know that we can translate this condition on the acton into the Euler-Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)} \right) = \frac{\partial \mathscr{L}}{\partial \phi} \qquad \text{with} \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}\,. \tag{1.3}$$

A second field which allows us to switch from a Lagrangian to a Hamiltonian is the (conjugate) momentum,

$$\pi(x) = \frac{\partial \mathscr{L}}{\partial(\partial_0\phi)} = \dot{\phi}\,. \tag{1.4}$$

With these two fields we define the third object which we can use to describe the dynamics of a system, the Hamiltonian or energy functional

$$\mathcal{H}(t) = \int d^3x\, \left(\pi\dot{\phi} - \mathscr{L} \right)\,. \tag{1.5}$$

While for example in quantum mechanics this Hamiltonian is the basis of most calculations, in quantum field theory we usually start from the Lagrangian.

1.1 Boson field

We already know that for our scattering process we need to compute a transition amplitude between two kinds of matter particles, incoming (anti-) electrons and outgoing (anti-) quarks. They interact via their electric charges. The interaction is classically described by the electromagnetic Lagrangian based on the abelian $U(1)$ field theory,

$$\mathscr{L}_{\text{photon}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \text{with} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \; , \tag{1.6}$$

in terms of a photon 4-vector field A_μ. An abelian group is defined by commutating field operators,

$$[A_\mu, A_\nu] = 0 \; . \tag{1.7}$$

We know this interaction from classical electrodynamics, written in a covariant way. The Maxwell equations

$$0 = \partial^\mu F_{\mu\nu} = \partial^\mu \partial_\mu A_\nu - \partial^\mu \partial_\nu A_\mu = \Box A_\nu \qquad \text{with} \qquad \Box = \partial_\mu \partial^\mu \tag{1.8}$$

are the equations of motion for this photon field. In the last step we assume the Lorenz gauge condition $\partial_\mu A^\mu = 0$ and find the so-called d'Alembert equation for the vector potential A_μ.

To not have to write out the vector index of the photon field, let us instead use the real scalar ϕ from Eq.(1.2) to illustrate interaction or bosonic fields. Allowing for a mass for this real scalar field we can write down its equation of motion which is the same for a spin-zero scalar boson as for the spin-one vector boson of Eq.(1.8)

$$\left(\Box + m^2\right) \, \phi(x) = 0 \; . \tag{1.9}$$

This scalar Klein–Gordon equation corresponds to the d'Alembert equation for the electromagnetic vector potential in Lorentz gauge. It corresponds to a Lagrangian

$$\mathscr{L}_{\text{scalar}} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2} \phi^2 \; , \tag{1.10}$$

which we can confirm using the Euler-Lagrange equation Eq.(1.3). If we want to compute the scattering amplitude in momentum space, we need to Fourier-transform the scalar field into momentum space and then quantize it, *i.e.* define commutation properties for the field in position and momentum space.

For our scattering process in Eq.(1.1) we need an object that describes the propagation of the (virtual) photon from its production from an e^+e^- pair to its splitting into a $q\bar{q}$ pair. Until now, we never used the fact that for instance $\phi(x)$ is an operator, not just a field. The so-called propagator in position space is defined as a time–ordered product of two field operators sandwiched between vacuum states. We can think of it as describing a photon starting from its birth out of an e^+e^- pair at space-time x' to its death as a $q\bar{q}$ pair at x,

$$\Delta(x - x') \equiv i \, \langle 0 | T \left(\phi(x) \phi(x') \right) | 0 \rangle \; . \tag{1.11}$$

The time–ordered product of two operators ensures that the photon is first generated and annihilated later,

$$T \left(A(x) B(x') \right) = \begin{cases} A(x) B(x') & x_0 > x_0' \\ B(x') A(x) & x_0' > x_0 \end{cases} \; . \tag{1.12}$$

We know that this propagator is the Green function for the Klein–Gordon equation Eq.(1.9). We can transform it into Fourier space and find

$$\Delta(x - x') = - \int \frac{d^4 k}{(2\pi)^4} \, e^{-ik(x-x')} \, \frac{1}{k^2 - m^2 + i\varepsilon} \; . \tag{1.13}$$

The $+i\varepsilon$ in the denominator ensures causality and will, for real particles, often be replaced by a resummation of finite width contributions. We can explicitly confirm that this propagator is the Green function for the Klein–Gordon

equation Eq.(1.9),

$$\begin{aligned}\left(\Box + m^2\right)\Delta(x-x') &= -\int \frac{d^4k}{(2\pi)^4}\ \left(\Box + m^2\right) e^{-ik\cdot(x-x')}\ \frac{1}{k^2-m^2} \\ &= \int \frac{d^4k}{(2\pi)^4}\ \left((-ik)^2 + m^2\right) e^{-ik\cdot(x-x')}\ \frac{(-1)}{k^2-m^2} \\ &= \int \frac{d^4k}{(2\pi)^4}\ e^{-ik\cdot(x-x')} \\ &= \delta^4(x-x')\ . \end{aligned} \tag{1.14}$$

All these properties we will later use for the photon field A^μ, a vector field, where each component obeys the Klein–Gordon equation. The propagator and quantization aspects like commutation relations for the field operators will not change. In an ideally-chosen (Feynman) gauge, boson propagators only get dressed by factors $g_{\mu\nu}$ where appropriate.

1.2 Fermion field

Next, we need to describe (external) fermion fields or matter particles. Fermions, like leptons or quarks, have a different equation of motion and a different contribution to the Lagrangian. A field describing a fermionic particle has to include two spin states of this particles. Moreover, in quantum field theory every fermion $\chi^\dagger$ has an antiparticle with the same mass. The altogether four degrees of freedom naturally combine to one equation with the same mass and the particle and the antiparticle described by one field ψ. Our task is two write a fermion Lagrangian that respects the observed Lorentz invariance of relativistic physics.

The form of the fermion field is given by the transformation property under the Lorentz transformation. We remind ourselves that a scalar field $\phi(x)$ transforms under a Lorentz boost introduced in electrodynamics as

$$x^\mu\ \to\ \Lambda^\mu{}_\nu x^\nu \qquad \text{and} \qquad \frac{\partial}{\partial^\mu} \equiv \partial_\mu\ \to\ \Lambda_\mu{}^\nu \partial_\nu \tag{1.15}$$

via a unitary operator $U(\Lambda)$ as

$$U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x)\ . \tag{1.16}$$

The fermion field has to live in a different, the so-called spinor representation. It describes on-shell particles and antiparticles including spin and transforms under the Lorentz transformation as

$$U(\Lambda)^{-1}\psi(x)U(\Lambda) = \Lambda_{1/2}\,\psi(\Lambda^{-1}x)\ , \tag{1.17}$$

where $\Lambda_{1/2}$ is this special representation of the Lorentz transformation. We can define it using the four Dirac matrices γ^μ with their anti–commutator Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\ \mathbb{1}\ . \tag{1.18}$$

The unit matrix has the same size as the γ matrices. That we usually write them as 4×4 matrices has nothing to do with the number of — also four — γ matrices. The explicit form of the γ_μ matrices is not relevant because it never appears in actual calculations; all we will ever need is a few trace relations arising from the (anti-) commutators. In terms of the Dirac matrices we can also write a representation of the Lorentz transformation

$$\Lambda_{1/2} = \exp\left(\frac{\omega_{\mu\nu}}{8}\ [\gamma^\mu, \gamma^\nu]\right)\ . \tag{1.19}$$

We only quote the result that this representaion gives us the transformation rule for the Dirac matrices

$$\Lambda_{1/2}^{-1}\ \gamma^\mu\ \Lambda_{1/2} = \Lambda^\mu{}_\nu \gamma^\nu\ . \tag{1.20}$$

We now postulate an equation of motion for the fermions, the Dirac equation, which describes fermions in Nature perfectly. One way to motivate this form is by taking some kind of square root of the Klein-Gordon equation. Another way is to write 2-spinors separately, for the two-spin states of the particle or of the antiparticle. Because of the spin, each of these two Dirac equations is then written with the help of the Pauli matrices as generators of the spin group $SU(2)$. Both derivations motivate the same equation of motion for the full Dirac spinor

$$(i\gamma^\mu \partial_\mu - m\mathbb{1})\ \psi(x) \equiv (i\partial\!\!\!/ - m\mathbb{1})\ \psi(x) = 0\ . \tag{1.21}$$

The unit matrix in the mass term is a 4-by-four matrix, just like the Dirac matrices. Note that the size of the γ-matrix and the number of Lorentz indices μ are only coincidentally identical.

We want to check that Eq.(1.21) as the fermionic equation of motion is invariant under Lorentz transformations, keeping in mind that $\Lambda_{1/2}$ commutes with everything except for the Dirac matrices

$$\begin{aligned}
(i\gamma^\mu \partial_\mu - m\mathbb{1})\ \psi(x) &\to (i\gamma^\mu \Lambda_\mu{}^\nu \partial_\nu - m\mathbb{1})\ \Lambda_{1/2}\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\Lambda_{1/2}^{-1}\,(i\gamma^\mu \Lambda_\mu{}^\nu \partial_\nu - m\mathbb{1})\ \Lambda_{1/2}\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\left(i\Lambda_{1/2}^{-1}\gamma^\mu \Lambda_{1/2}\ \Lambda_\mu{}^\nu \partial_\nu - m\mathbb{1}\right)\ \psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\left(i\Lambda^\mu{}_\rho \gamma^\rho (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m\mathbb{1}\right)\ \psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\left(ig^\nu{}_\rho \gamma^\rho \partial_\nu - m\mathbb{1}\right)\ \psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\,(i\gamma^\nu \partial_\nu - m\mathbb{1})\ \psi(\Lambda^{-1}x) = 0\ .
\end{aligned} \tag{1.22}$$

We also see that we can multiply the Dirac equation with $-(i\gamma^\mu \partial_\mu + m\mathbb{1})$ and obtain

$$\begin{aligned}
-\,(i\gamma^\mu \partial_\mu + m\mathbb{1})\,(i\gamma^\nu \partial_\nu - m\mathbb{1})\ \psi(x) &= \left(\gamma^\mu\gamma^\nu \partial_\mu\partial_\nu + m^2\mathbb{1}\right)\ \psi(x) \\
&= \left(\frac{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu}{2}\partial_\mu\partial_\nu + m^2\mathbb{1}\right)\ \psi(x) \\
&= \left(\partial^2 + m^2\mathbb{1}\right)\ \psi(x)\ ,
\end{aligned} \tag{1.23}$$

which means that every fermion field that obeys the Dirac equation also fulfills a Klein–Gordon equation. We will see this similarity when we construct the fermion propagator.

To define a mass term in the Lagrangian we need to form Lorentz scalars or invariants out of the fermion fields ψ. Naively, $(\psi^\dagger\psi)$ would work if the Lorentz transformations in $(\psi^\dagger\Lambda_{1/2}^\dagger\Lambda_{1/2}\psi)$ cancelled. Unfortunately $\Lambda_{1/2}$ is not a unitary transformation, so they do not cancel. Instead, one can show that the Dirac adjoint has the correct transformation property, namely

$$\overline{\psi} = \psi^\dagger\gamma^0 \qquad \text{with} \qquad \overline{\psi}\psi \to \overline{\psi}\psi \tag{1.24}$$

This allows us to write down the Lagrangian which corresponds to the Dirac equation for a fermion field

$$\mathscr{L}_{\text{fermion}} = \overline{\psi}(i\partial\!\!\!/ - m\mathbb{1})\psi\ . \tag{1.25}$$

Because we will later need the fermion–photon interaction in the Lagrangian, we introduce the convenient form of the covariant derivative

$$\begin{aligned}
\mathscr{L}_{\text{fermion-photon}} &= \overline{\psi}\,(i\not{D} - m\mathbb{1})\,\psi \\
&\equiv \overline{\psi}\,(i(\partial\!\!\!/ + ieqA\!\!\!/) - m\mathbb{1})\,\psi \\
&= \overline{\psi}\,(i\partial\!\!\!/ - m\mathbb{1})\,\psi - eqA_\mu\,\overline{\psi}\gamma^\mu\psi
\end{aligned} \tag{1.26}$$

The last term describes the coupling of a vector photon field A_μ to a vector-like expression $\overline{\psi}\gamma^\mu\psi$ which we call a vector current of a spinor field.

Just like the bosonic propagator, we now Fourier-transform the Dirac field operators. Now taking care of the Lorentz-invariant Fourier transformation, this gives us four degrees of freedom, (anti-) particles with two spins, explicitly

$$\begin{aligned}\psi(x) &= \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{\text{spin } s\,=\,\pm 1/2} \left(e^{ikx} v_s(k) b_s^\dagger(\vec{k}) + e^{-ikx} u_s(k) a_s(\vec{k}) \right) \\ \overline{\psi}(x) &= \int \frac{d^3k}{(2\pi)^2 2k_0} \sum_{\text{spin } s\,=\,\pm 1/2} \left(e^{ikx} \bar{u}_s(k) a_s^\dagger(\vec{k}) + e^{-ikx} \bar{v}_s(k) b_s(\vec{k}) \right) \ . \end{aligned} \tag{1.27}$$

The 4-dimensional spinors u and v in Fourier space. Acting on the vacuum a and b are annihilation operators and $a^\dagger$ and $b^\dagger$ are creation operators. These operators only depend on the momentum three-vector, the energy component follows from the dispersion relation of the on–shell particles. The operators are listed in Tab. 1. As for the boson propagator, we skip the quantization steps and directly quote the resulting spin sums for the spinors u and v and their Dirac adjoints

$$\begin{aligned}\sum_{\text{spin } s\,=\,\pm 1/2} u_s(k) \bar{u}_s(k) &= \not{k} + m\mathbb{1} \\ \sum_{\text{spin } s\,=\,\pm 1/2} v_s(k) \bar{v}_s(k) &= \not{k} - m\mathbb{1} \ . \end{aligned} \tag{1.28}$$

These spin sums do not combine spinors to scalars but to Dirac matrices. For many applications in LHC physics, the masses are negligible, and since they really complicate formulas, we evaluate the spin sums without masses whenever we can.

1.3 Scattering

Now we have everything we need to compute a transition amplitude for our scattering process

$$e^-(k_1, s_1) + e^+(k_2, s_2) \to q(k_3, s_3) + \bar{q}(k_4, s_4) \ , \tag{1.29}$$

where k_j are the 4-momenta and s_j the spins of the external fermions. We neglect the electron and quark masses, and energy-momentum conservation reads $k_1 + k_2 + k_3 + k_4 = 0$. Asymptotically, the initial state will evolve into a final state via a scattering operator $\mathcal{S}$,

$$\lim_{t \to -\infty} |t\rangle \equiv |i\rangle \quad \xrightarrow{\mathcal{S}} \quad \lim_{t \to \infty} |t\rangle = \mathcal{S}|i\rangle \tag{1.30}$$

To describe this scattering from an initial state $|i\rangle$ into a final state $|f\rangle$ we need to compute the transition amplitude

$$S \equiv \langle f|\mathcal{S}|i\rangle = \langle q_3 \bar{q}_4 |\mathcal{S}| e_1^+ e_2^- \rangle = \langle 0| a_3 b_4 \, \mathcal{S} \, a_1^\dagger b_2^\dagger |0\rangle \ . \tag{1.31}$$

We use a single index to indicate the momenta and spins of the external particles. This transition amplitude is not a vacuum expectation value, but the operator $\mathcal{S}$ sandwiched between physically measurable states made from the vacuum using the generation and annihilation field operators $a, a^\dagger, b, b^\dagger$ defined in Eq.(1.27).

symbol	meaning	operator	acting on
$u_s(k)$	incoming fermion (e^-, q) with momentum k and spin s	a_s	$a_s^\dagger\|0\rangle$
$\bar{v}_s(k)$	incoming anti–fermion $(e^+, \bar{q})$	b_s	$b_s^\dagger\|0\rangle$
$\bar{u}_s(k)$	outgoing fermion (e^-, q)	$a_s^\dagger$	$\langle 0\|a_s$
$v_s(k)$	outgoing anti–fermion $(e^+, \bar{q})$	$b_s^\dagger$	$\langle 0\|b_s$

Table 1: Assignment of generation and annihilating operators for particle and antiparticle spinors in momentum space, as given in Eq.(1.27), including the combination of operator and vaccuum they act on.

We will not show that, but the transition operator can be computed from the time evolution of the initial state $i\partial_t|t\rangle = \mathcal{H}(t)|t\rangle$ in the interaction picture with a time-dependent Hamilton operator,

$$\mathcal{S} = T\left(e^{-i\int dt\,\mathcal{H}(t)}\right) \;, \tag{1.32}$$

again with time ordering T. This form ensures a unitary transformation. For our computation, we will be fine using the interaction-Hamiltonian part of Eq.(1.5). The Lagrangian in Eq.(1.26) gives us the interaction of two incoming and two outgoing fermions, each pair involving a different particle species j with charge q_j,

$$\mathcal{H}_{\text{int}}(t) = -\int d^3x\,\mathscr{L}_{\text{int}}(x) \supset \sum_j -eq_j \int d^3x\,A_\mu\,\overline{\psi}_j\gamma^\mu\psi_j \;, \tag{1.33}$$

in terms of the 4-vector x including its first entry $t = x_0$. We skip the corresponding calculation and how is automatically Fourier-transforms us into momentum space and just give the result for the numbering of the incoming and outgoing particles defined in Eq.(1.29)

$$\begin{aligned} S &= i(2\pi)^4 \sum_{\text{spins}} \underbrace{\delta^4(k_1+k_2+k_3+k_4)}_{\text{energy-momentum conservation}} \quad \underbrace{e^2 q_e q_q}_{\text{charges}} \quad \underbrace{(\bar{u}_3\gamma_\mu v_4)\,\frac{1}{(k_1+k_2)^2}\,(\bar{v}_2\gamma^\mu u_1)}_{\text{quantum field theory}} \\ \Rightarrow \qquad \mathcal{M} &= e^2 q_e q_q\,(\bar{u}_3\gamma_\mu v_4)\,\frac{1}{(k_1+k_2)^2}\,(\bar{v}_2\gamma^\mu u_1) \;. \end{aligned} \tag{1.34}$$

In the second step we strip off some prefactors to define the transition matrix element for quark–antiquark production in QED. Just like in quantum mechanics we square this transition matrix element to compute the transition probability,

$$\begin{aligned} |\mathcal{M}|^2 &= \sum_{\text{spin, color}} e^4 q_e^2 q_q^2\,\frac{1}{(k_1+k_2)^4}\,(\bar{v}_4\gamma_\nu u_3)(\bar{u}_1\gamma^\nu v_2)\,(\bar{u}_3\gamma_\mu v_4)\,(\bar{v}_2\gamma^\mu u_1) \\ &= e^4 q_e^2 q_q^2 N_c\,\frac{1}{(k_1+k_2)^4}\,\sum_{\text{spin}}(\bar{v}_4\gamma_\nu u_3)(\bar{u}_1\gamma^\nu v_2)\,(\bar{u}_3\gamma_\mu v_4)\,(\bar{v}_2\gamma^\mu u_1) \;. \end{aligned} \tag{1.35}$$

The color factor N_c is the number of outgoing color singlet states we can form out of a quark and an antiquark with opposite color charges. Because color only appears in the final state, we sum over all possible color states or multiply by N_c.

In the next step, we can observe how the crucial structure of transition amplitudes with external fermions magically forms, namely traces of chains of Dirac matrices. Part of the squaring is the sum over all spins which uses the spin sums Eq.(1.28) to replace the spinors by Dirac matrices. We then use trace rules to remove all Dirac matrices. Neither for the spinors nor for the Dirac matrices do we need to know their explicit form to find

$$\begin{aligned} |\mathcal{M}|^2 =& e^4 q_e^2 q_q^2 N_c\,\frac{1}{(k_1+k_2)^4}\,\sum_{\text{spin}}(\bar{v}_4)_i(\gamma_\nu)_{ij}(u_3)_j\,(\bar{u}_3)_k(\gamma_\mu)_{kl}(v_4)_l\cdots && \text{for one trace} \\ =& e^4 q_e^2 q_q^2 N_c\,\frac{1}{(k_1+k_2)^4}\,\left(\sum_{\text{spin}}(v_4)_l(\bar{v}_4)_i\right)\,\left(\sum_{\text{spin}}(u_3)_j(\bar{u}_3)_k\right)\,(\gamma_\nu)_{ij}(\gamma_\mu)_{kl}\cdots && \\ =& e^4 q_e^2 q_q^2 N_c\,\frac{1}{(k_1+k_2)^4}\,(\not{k}_4)_{li}\,(\not{k}_3)_{jk}\,(\gamma_\nu)_{ij}(\gamma_\mu)_{kl}\cdots && \text{using Eq.(1.28), no masses} \\ \equiv& e^4 q_e^2 q_q^2 N_c\,\frac{1}{(k_1+k_2)^4}\,\text{Tr}\,(\not{k}_4\gamma_\nu\not{k}_3\gamma_\mu)\;\text{Tr}\,(\not{k}_1\gamma^\nu\not{k}_2\gamma^\mu) && \text{written as traces} \end{aligned} \tag{1.36}$$

In the final step, we need a standard expression for the Dirac trace. Longer traces become very complicated very fast,

and we evaluate them using symbolic manipulation on the computer. We find

$$\begin{aligned}
|\mathcal{M}|^2 &= e^4 q_e^2 q_q^2 N_c \; \frac{1}{(k_1+k_2)^4} \; 4\,(k_{3\nu}k_{4\mu} + k_{3\mu}k_{4\nu} - g_{\mu\nu}(k_3k_4)) \; 4\,(k_1^\nu k_2^\mu + k_1^\mu k_2^\nu - g_{\mu\nu}(k_1k_2)) \\
&= 16 e^4 q_e^2 q_q^2 N_c \; \frac{1}{(k_1+k_2)^4} \; [2(k_1k_3)(k_2k_4) + 2(k_1k_4)(k_2k_3) + (-2-2+g_{\mu\nu}g^{\mu\nu})(k_3k_4)(k_1k_2)] \\
&= 32 e^4 q_e^2 q_q^2 N_c \; \frac{1}{(k_1+k_2)^4} \; [(k_1k_3)(k_2k_4) + (k_1k_4)(k_2k_3)] \; . \qquad (1.37)
\end{aligned}$$

To evaluate this matrix element we first introduce Mandelstam Lorentz-invariants as squares of sums of 4-vectors,

$$s = (k_1+k_2)^2 \approx 2(k_1k_2) > 0 \qquad t = (k_1+k_3)^2 \approx 2(k_1k_3) < 0 \qquad u = (k_1+k_4)^2 \approx 2(k_1k_4) \; . \qquad (1.38)$$

In this sign convention all momenta are incoming, $k_1+k_2+k_3+k_4 = 0$, and in the second step we neglect the masses. The second Mandelstam variable can be expressed through the polar or scattering angle

$$t = \frac{s}{2}(-1+\cos\theta) \in [-s, 0] \; . \qquad (1.39)$$

Allowing the external particles to have a finite mass, we can use this 4-momentum conservation to show

$$s+t+u = k_1^2 + k_2^2 + k_3^2 + k_4^2 \equiv m_1^2 + m_2^2 + m_3^3 + m_4^2 \; . \qquad (1.40)$$

We assume that all masses be zero and we find the compact form

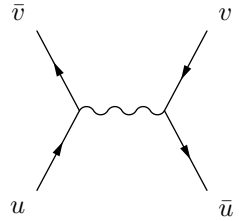
$$\begin{aligned}
|\mathcal{M}|^2 &= 32 e^4 \; q_e^2 q_q^2 N_c \; \frac{1}{s^2} \; \left[\frac{t^2}{4} + \frac{u^2}{4}\right] \\
&= 8 e^4 \; q_e^2 q_q^2 N_c \; \frac{1}{s^2} \; \left[s^2 + 2st + 2t^2\right] \\
&= 8 e^4 \; q_e^2 q_q^2 N_c \; \left[1 + 2\frac{t}{s} + 2\frac{t^2}{s^2}\right] \; . \qquad (1.41)
\end{aligned}$$

We can briefly check if this number is indeed positive, using the definition of the Mandelstam variable t for massless external particles in terms of the polar angle: the upper phase space boundary $t = 0$ inserted into the brackets in Eq.(1.41) gives $[\cdots] = 1$, just as the lower boundary $t = -s$ with $[\cdots] = 1 - 2 + 2 = 1$. For the central value $t = -s/2$ the minimum value of the brackets is $[\cdots] = 1 - 1 + 0.5 = 0.5$.

The azimuthal angle ϕ does not play a role at colliders, unless we want to compute gravitational effects on Higgs production at ATLAS and CMS. Any LHC Monte Carlo will either random-generate a reference angle ϕ for the partonic process or pick one and keep it fixed.

1.4 Feynman rules

Feynman rules are calculation rules that we can extract easily from the Lagrangian and which allow us to derive Eq.(1.34) without going through the time evolution and the interaction Hamiltonian. We start by drawing Feynman diagrams for all ways we can link the given initial and final states through interaction vertices and internal propagators. For our scattering process, we first conform that there is only one Feynman diagram with a photon in the s-channel,



It consists of four external fermions labeled according to Tab. 1, one internal photon, and two interaction vertices, for which we need to reproduce $\mathcal{M}$ in momentum space, as given in Eq.(1.34)

1. From Eq.(1.27) and Tab. 1 we know how to describe external fermions in terms of spinors

$$\left\{ u_s(k), \bar{v}_s(k), \bar{u}_s(k), v_s(k) \right\} . \tag{1.42}$$

   Spin sums are the only way to get rid of spinors in the computation. Equation (1.28) shows that as long as we neglect fermion masses the two spinors u and v for particles and antiparticles are identical.

2. To link external particles to each other and to internal propagators, we need vertices from the interaction Lagrangian or Hamiltonian. If two fermions and a gauge boson interact via a vector current proportional to γ^μ, and adding a conventional factor i, the Feynman rule for a QED vertex reads

$$ieq_f \, \gamma^\mu \qquad\qquad (f - \bar{f} - \gamma). \tag{1.43}$$

   The factor i can be consistently changed for all 3-point and 4-point vertices.

3. Finally, we need to describe the intermediate photon propagation between the γ^μ and the γ^ν vertices. Following Eq.(1.13), and including a conventional factor $-i$, the wave line in the above Feynman diagram corresponds to

$$-i \frac{g^{\mu\nu}}{p^2 + i\epsilon} . \tag{1.44}$$

Blindly combining the Feynman rules for the external spinors, the vertices, and the photon propagator gives us directly $\mathcal{M}$ as written in Eq.(1.34). All we need to do from then on is to square the matrix element, insert the spin sums and compute the Dirac trace.

One QED Feynman rule we are missing and which we need to describe, for instance $e^-\gamma \to e^-\gamma$ scattering, is an internal fermion propagator. It is given by

$$i \frac{\not{p} + m\mathbb{1}}{p^2 - m^2} = i \frac{\not{p} + m\mathbb{1}}{\not{p}^2 - m^2} = i \frac{\not{p} + m\mathbb{1}}{(\not{p} + m\mathbb{1})(\not{p} - m\mathbb{1})} = i \left(\not{p} - m\mathbb{1}\right)^{-1} . \tag{1.45}$$

It lives in the same space as gamma matrices. Because the sign of the 4-momentum matters we assign it in parallel to the fermion arrow. This will work fine until we have to deal with Majorana particles in the neutrino sector or supersymmetry (for those who still remember that).

Finally, we will discuss gauge bosons later and for now just mention the polarization of the external photon $\epsilon_\mu(k, \lambda)$ and the polatization sum

$$\sum_\lambda \epsilon_\mu(k, \lambda) \epsilon^*_\nu(k, \lambda) = -g_{\mu\nu} . \tag{1.46}$$

Whenever we compute such a matrix element starting from a Feynman diagram, nothing tells us that the lines in the Feynman diagrams are not actual physical states propagating from the left to the right. Even loop diagrams will look completely reasonably from a semi–classical point of view. In reality, Feynman rules define an algorithm which hides all field theory input in the calculation of scattering amplitudes and are therefore perfectly suited to compute the differential and total cross sections on the computer.

1.5 Chirality

The vector structure of the QED couplings, for example, mediated by a covariant derivative Eq.(1.26) we did not actually motivate. It happens to work on the Lagrangian level and agrees with data, so it is correct. We can also write a completely general interaction of two fermions with a boson in terms of basis elements

$$g \, \overline{\psi} M \psi = \sum_{\text{basis } j} g_j \, \overline{\psi} M_j \psi . \tag{1.47}$$

For a real (4×4)-matrix M the 16 multivector basis elements can be organized such that they are easy to keep track of using Lorentz transformation properties. This eventually leads to the so-called Fierz transformation. In addition to the scalar unit matrix, we know the 4-dimensional vector γ^μ from the QED interaction. Six more elements we also know, the generators of the spinor representation $[\gamma^\mu, \gamma^\nu]$.

This means that we are missing five basis elements. To define them, we start with another (4×4)-matrix which is invariant under proper Lorentz transformations. We can write it in two equivalent forms

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \frac{i}{4!}\,\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \;, \tag{1.48}$$

using the totally anti-symmetric Levi–Civita tensor $\epsilon_{\mu\nu\rho\sigma}$. This form already shows a major technical complication in dealing with γ_5: in other than four space–time dimensions we do not know how to define the Levi–Civita tensor, which eventually means that for regularization purposes we cannot analytically continue it to $n = 4 - 2\epsilon$ dimensions. The main properties of γ_5 are

$$\gamma_5^2 = \mathbb{1} \qquad \text{and} \qquad \{\gamma_\mu, \gamma_5\} = 0 \;. \tag{1.49}$$

We can combine it with a vector to give us all 16 basis elements for the interaction of two spinors:

	degrees of freedom	basis elements M_j
scalar	1	$\mathbb{1}$
vector	4	γ^μ
pseudoscalar	1	$i\gamma_5$
axialvector	4	$\gamma^\mu\gamma_5$
tensor	6	$\frac{i}{2}[\gamma^\mu, \gamma^\nu]$

In the renormalizable Standard Model as a fundamental theory, tensor interactions do not play any role.

An obvious question is: what does it mean to include a factor γ_5 in the interaction, *i.e.* what distinguishes a scalar from a pseudoscalar and a vector from an axialvector? We can give an easy answer by defining three transformations of our field in space and time. The first is the parity transformation P that mirrors the three spatial coordinates $(t, \vec{x}) \to (t, -\vec{x})$. The second is charge conjugation C, which converts particles into their anti–particles. Both of them leave the Dirac equation intact and can be represented by a unitary transformation. The third transformation is time reversal T which converts $(t, \vec{x}) \to (-t, \vec{x})$, also leaves the Dirac equation intact, but only has an anti–unitary representation. Every one of them is violated in our Standard Model.

Instead of writing out the representation of these transformations in terms of Dirac matrices, we characterize them using the basic interactions from Eq.(1.47):

- Parity symmetry under $(t, \vec{x}) \to (t, -\vec{x})$ does not allow interactions including γ_5, *i.e.* it forbids pseudoscalars and axialvectors.
- Time reversal symmetry under $(t, \vec{x}) \to (-t, \vec{x})$ does not allow complex couplings g_j.
- Because any field theory described by a Lagrangian not including an external field is invariant under CPT, and we have never observed CPT violation, a combined CP-invariance is essentially the same as T invariance.

To look at the parity and CP symmetries more systematically, we rotate the $\{\mathbb{1}, \gamma_5\}$ plane into the two basis elements

$$\mathbb{P}_{R,L} = \frac{1}{2}\,(\mathbb{1} \pm \gamma_5) \;. \tag{1.50}$$

It is easy to show that they are orthogonal projectors,

$$\begin{aligned}\mathbb{P}_L\mathbb{P}_R &= \frac{1}{4}\,(\mathbb{1}-\gamma_5)\,(\mathbb{1}+\gamma_5) = \frac{1}{4}\,\left(\mathbb{1}-\gamma_5^2\right) = 0\\ \mathbb{P}_{R,L}^2 &= \frac{1}{4}\,\left(\mathbb{1}\pm 2\gamma_5+\gamma_5^2\right) = \frac{1}{4}\,(2\mathbb{1}\pm 2\gamma_5) = \frac{1}{2}\,(\mathbb{1}\pm\gamma_5) = \mathbb{P}_{R,L}\;.\end{aligned} \tag{1.51}$$

We first look at what happens when we write a kinetic term with these projectors acting on the fermion fields,

$$\begin{aligned}\overline{\psi}\,\partial\!\!\!/\psi &= \overline{\psi}\,\partial\!\!\!/\,\left(\mathbb{P}_L^2+\mathbb{P}_R^2\right)\psi\\ &= \overline{\psi}\,\left(\mathbb{P}_R\partial\!\!\!/\mathbb{P}_L+\mathbb{P}_L\partial\!\!\!/\mathbb{P}_R\right)\psi\\ &= (\overline{\mathbb{P}_L\psi})\partial\!\!\!/(\mathbb{P}_L\psi)+(\overline{\mathbb{P}_R\psi})\partial\!\!\!/(\mathbb{P}_R\psi)\\ &= \overline{\psi}_L\,\partial\!\!\!/\psi_L+\overline{\psi}_R\,\partial\!\!\!/\psi_R\;.\end{aligned} \tag{1.52}$$

The general kinetic term can be written in terms of independent left-handed and right-handed fields. Their effect on a mass term is different,

$$\begin{aligned}\overline{\psi}\,\mathbb{1}\psi &= \overline{\psi}\,(\mathbb{P}_L+\mathbb{P}_R)\,\psi\\ &= \overline{\psi}\,\left(\mathbb{P}_L^2+\mathbb{P}_R^2\right)\psi\\ &= \psi^\dagger\gamma_0\,\left(\mathbb{P}_L^2+\mathbb{P}_R^2\right)\psi && \text{with} \quad \overline{\psi}=\psi^\dagger\gamma^0\\ &= \psi^\dagger\left(\mathbb{P}_R\gamma^0\mathbb{P}_L+\mathbb{P}_L\gamma^0\mathbb{P}_R\right)\psi && \text{with} \quad \{\gamma_5,\gamma_\mu\}=0\\ &= (\mathbb{P}_R\psi)^\dagger\gamma^0(\mathbb{P}_L\psi)+(\mathbb{P}_L\psi)^\dagger\gamma^0(\mathbb{P}_R\psi) && \text{with} \quad \gamma_5^\dagger=\gamma_5, \mathbb{P}_{L,R}^\dagger=\mathbb{P}_{L,R}\\ &= (\overline{\mathbb{P}_R\psi})\mathbb{1}(\mathbb{P}_L\psi)+(\overline{\mathbb{P}_L\psi})\mathbb{1}(\mathbb{P}_R\psi)\\ &= \overline{\psi}_R\,\mathbb{1}\psi_L+\overline{\psi}_L\,\mathbb{1}\psi_R\;.\end{aligned} \tag{1.53}$$

We see that we can write, for example, QED in terms of independent left-handed and right-handed fields as long as we neglect all fermion masses. This defines the chiral limit where the Lagrangian is symmetric under $\psi_L \leftrightarrow \psi_R$. We call the massless eigenstates right-handed and left-handed fermions $\psi_{L,R}$. Introducing fermion masses breaks this chiral symmetry. In that case, we need to combine a left-handed and a right-handed fermion fields and give them one common Dirac mass.

Moving on to interactions, we define a combined vector–axialvector coupling as

$$\gamma_\mu \pm \gamma_\mu\gamma_5 = 2\gamma_\mu\mathbb{P}_{R,L}\;. \tag{1.54}$$

Sandwiching this coupling between fermion fields gives for example

$$\begin{aligned}\overline{\psi}\,\gamma_\mu\mathbb{P}_L\,\psi &= \overline{\psi}\,\gamma_\mu\mathbb{P}_L^2\,\psi\\ &= \psi^\dagger\mathbb{P}_L\,\gamma_0\,\gamma_\mu\,\mathbb{P}_L\psi && \text{with} \quad \{\gamma_5,\gamma_\mu\}=0\\ &= (\mathbb{P}_L\psi)^\dagger\,\gamma_0\,\gamma_\mu\,\mathbb{P}_L\psi && \text{with} \quad \gamma_5^\dagger=\gamma_5\\ &= \overline{\psi}_L\,\gamma_\mu\,\psi_L && \text{with} \quad \psi_{L,R}\equiv\mathbb{P}_{L,R}\,\psi\;.\end{aligned} \tag{1.55}$$

In terms of chiral eigenstates we define the coupling between left-handed fermions by combining the vector and the axialvector couplings with a relative minus sign. The same is true for right-handed couplings. We can now describe the $\gamma\bar{f}f$ coupling in QED using the Feynman rule

$$-i\gamma^\mu\,(\ell\mathbb{P}_L+r\mathbb{P}_R) \qquad \text{with} \quad \ell=r=qe\;. \tag{1.56}$$

At this stage, it is not obvious at all what chirality means beyond the link between the chiral symmetry forbidding fermion masses. We will see that in the Standard Model the left-handed fermions play a special role: the massive W bosons only couple to them and not to their right-handed counter parts. So chirality is known to a gauge interaction of the Standard Model through its charge.

1.6 Helicity and spin

There exists a property which is identical to the chirality for massless fermions and has an easy physical interpretation, helicity. It is defined as the projection of the particle spin or orbital angular momentum onto its 3-momentum direction

$$h = \vec{s} \cdot \frac{\vec{k}}{|\vec{k}|} = \left(\vec{s} + \vec{L}\right) \cdot \frac{\vec{k}}{|\vec{k}|} \qquad \text{using} \quad \vec{L} = \vec{x} \times \vec{k} \perp \vec{k} \,, \tag{1.57}$$

From quantum mechanics we know that the eigenvalues for the z component of the angular momentum operator are discrete and symmetric around zero. Applied to fermions, this gives us two spin states with the eigenvalues of h being $\pm 1/2$. Unfortunately, there is no really nice way to show this identity.

What we need to know is that the spin operator is given by

$$\vec{s} = \gamma_5 \gamma^0 \, \vec{\gamma} \,. \tag{1.58}$$

We can show this by writing it out in terms of Pauli matrices, but we will skip this here and instead just accept this general form. We then write the solution ψ to the massless Dirac equation after transforming it into momentum space following Eq.(1.27), or in short $\psi(\vec{x}) \to u(\vec{k}) \exp(-ik \cdot x)$

$$\begin{aligned}
\left(\gamma^0 k_0 - \vec{\gamma}\vec{k}\right) \; u(\vec{k}) &= 0 \\
\gamma_5 \gamma^0 \; \gamma^0 k_0 \; u(\vec{k}) &= \gamma_5 \gamma^0 \; \vec{\gamma}\vec{k} \; u(\vec{k}) \\
\gamma_5 k_0 \; u(\vec{k}) &= \vec{s} \cdot \vec{k} \; u(\vec{k}) \qquad\qquad \text{with} \;\; \left(\gamma^0\right)^2 = \mathbb{1} \\
\gamma_5 \; u(\vec{k}) &= \frac{\vec{s} \cdot \vec{k}}{k_0} \; u(\vec{k}) \\
&= \pm \frac{\vec{s} \cdot \vec{k}}{|\vec{k}|} \; u(\vec{k}) = \pm h \; u(\vec{k}) \,.
\end{aligned} \tag{1.59}$$

In other words, the chirality operator γ_5 indeed gives us the helicity of a particle, modulo a sign depending on the sign of the energy. For the helicity, it is easy to argue why for massive particles this property is not Lorentz-invariant and hence not a well defined property: massless particles propagate with the speed of light, which means we can never boost into their rest frame. For massive particles, we can do that and this way switch the sign of $\vec{k}$ and the sign of h. Luckily, for almost all Standard Model fermions we can neglect their masses at the LHC.

2 Extra: helicity amplitudes

In research, transition amplitudes are not computed by squaring matrix elements and computing gamma matrix traces. Instead, all elements of the Feynman rules are evaluated numerically, combined, and then squared. From the previous section we know how to compute the cross section for Z production by writing down all external spinors, external polarization vectors, interaction vertices and propagators and squaring the amplitude analytically. The amplitude itself inherits external indices for example from the polarization vectors, while $|\mathcal{M}|^2$ is a real positive number with a fixed mass dimension depending on the number of external particles.

As an example for a more modern computation of transition amplitudes, we consider again lepton pair production in QED,

$$u\bar{u} \to \gamma^* \to \mu^+\mu^- \; . \tag{2.1}$$

The structure of the amplitude $\mathcal{M}$ with two internal Dirac indices μ and ν involves one vector current on each side $(\bar{u}_f \gamma_\mu u_f)$ where $f = u, \mu$ are to good approximation massless, so we do not have to be careful with the different spinors u and v. The entries in the external spinors are given by the spin of the massless fermions obeying the Dirac equation. For each value of $\mu = 0 \cdots 3$ each current is a complex number, computed from the four component of each spinor and the respective 4×4 gamma matrix γ^μ. The intermediate photon propagator has the form $g_{\mu\nu}/s$, which is a real number for each value of $\mu = \nu$. Summing over μ and ν in both currents forms the matrix element. To square this matrix element we need to sum $\mathcal{M}^* \times \mathcal{M}$ over all possible spin directions of the external fermions.

Instead of squaring this amplitude symbolically we can follow exactly the steps described above and compute an array of numbers for different spin and helicity combinations numerically. Summing over the internal Dirac indices we compute the matrix element; however, to compute the matrix element squared we need to sum over external fermion spin directions or gauge boson polarizations. The helicity basis we have to specify externally. This is why this method is called helicity amplitude approach. To explain the way this method works, we illustrate it for muon pair production based on the implementation in the Madgraph/Helas package.

Madgraph computes matrix elements this way. Other event generators have corresponding codes serving the same purposes. In our case, Madgraph5 automatically produces a Fortran (!!) routine which then calls functions to compute spinors, polarization vectors, currents of all kinds, etc. These functions are available as the Helas library. For our toy process Eq.(2.1) the slightly shortened Madgraph5 output reads

```
      REAL*8 FUNCTION MATRIX1(P,NHEL,IC)
C
C     Generated by Madgraph 5
C
C     Returns amplitude squared summed/avg over colors
C     for the point with external lines W(0:6,NEXTERNAL)
C
C     Process: u u~ > mu+ mu- / z WEIGHTED=4 @1
C
      INTEGER    NGRAPHS, NWAVEFUNCS, NCOLOR
      PARAMETER (NGRAPHS=1, NWAVEFUNCS=5, NCOLOR=1)

      REAL*8 P(0:3,NEXTERNAL)
      INTEGER NHEL(NEXTERNAL), IC(NEXTERNAL)

      INCLUDE 'coupl.inc'

      DATA DENOM(1)/1/
      DATA (CF(I,  1),I=  1,  1) /    3/

      CALL IXXXXX(P(0,1),ZERO,NHEL(1),+1*IC(1),W(1,1))
      CALL OXXXXX(P(0,2),ZERO,NHEL(2),-1*IC(2),W(1,2))
      CALL IXXXXX(P(0,3),ZERO,NHEL(3),-1*IC(3),W(1,3))
      CALL OXXXXX(P(0,4),ZERO,NHEL(4),+1*IC(4),W(1,4))
      CALL FFV1_3(W(1,1),W(1,2),GC_2,ZERO, ZERO, W(1,5))
      CALL FFV1_0(W(1,3),W(1,4),W(1,5),GC_3,AMP(1))
      JAMP(1)=+AMP(1)

      DO I = 1, NCOLOR
        DO J = 1, NCOLOR
          ZTEMP = ZTEMP + CF(J,I)*JAMP(J)
```

```
  ENDDO
  MATRIX1 = MATRIX1 + ZTEMP*DCONJG(JAMP(I))/DENOM(I)
ENDDO

END
```

The input to this function are the external 4-momenta $p(0:3,1:4)$ and the helicities of all fermions $n_{\rm hel}(1:4)$ in the process. Remember that helicity and chirality are identical only for massless fermions because chirality is defined as the eigenvalue of the projectors $(\mathbb{1} \pm \gamma_5)/2$, while helicity is defined as the projection of the spin onto the momentum direction, *i.e.* as the left or right handedness. We give the exact definition of these two properties in Section 1. The entries of $n_{\rm hel}$ will be either $+1$ or -1. For each point in phase space and each helicity combination the Madgraph subroutine `MATRIX1` computes the matrix element using standard Helas routines.

- `IXXXXX`$(p, m, n_{\rm hel}, n_{\rm sf}, F)$ computes the wave function of a fermion with incoming fermion number, so either an incoming fermion or an outgoing anti–fermion. As input it requires the 4-momentum, the mass and the helicity of this fermion. Moreover, $n_{\rm sf} = +1$ marks the incoming fermion u and $n_{\rm sf} = -1$ the outgoing anti–fermion μ^+, because by convention Madgraph defines its particles as u and μ^-.

  The fermion wave function output is a complex array $F(1:6)$. Its first two entries are the left–chiral part of the fermionic spinor, *i.e.* $F(1:2) = (\mathbb{1} - \gamma_5)/2\ u$ or $F(1:2) = (\mathbb{1} - \gamma_5)/2\ v$ for $n_{\rm sf} = \pm 1$. The entries $F(3:4)$ are the right–chiral spinor. These four numbers can directly be computed from the 4-momentum if we know the helicity. The four entries correspond to the size of one γ matrix, so we can compute the trace of the chain of gamma matrices. Because for massless particles helicity and chirality are identical, our quarks and leptons will only have finite entries $F(1:2)$ for $n_{\rm hel} = -1$ and $F(3:4)$ for $n_{\rm hel} = +1$.

  The last two entries of F contain the 4-momentum in the direction of the fermion flow, namely $F(5) = n_{\rm sf}(p(0) + ip(3))$ and $F(6) = n_{\rm sf}(p(1) + ip(2))$.

- `OXXXXX`$(p, m, n_{\rm hel}, n_{\rm sf}, F)$ does the same for a fermion with outgoing fermion flow, *i.e.* our incoming $\bar{u}$ and our outgoing μ^-. The left–chiral and right–chiral components now read $F(1:2) = \bar{u}(\mathbb{1} - \gamma_5)/2$ and $F(3:4) = \bar{u}(\mathbb{1} + \gamma_5)/2$, and similarly for the spinor $\bar{v}$. The last two entries are $F(5) = n_{\rm sf}(p(0) + ip(3))$ and $F(6) = n_{\rm sf}(p(1) + ip(2))$.

- `FFV1_3`$(F_i, F_o, g, m, \Gamma, J_{io})$ computes the (off–shell) current for the vector boson attached to the two external fermions F_i and F_o. The coupling $g(1:2)$ is a complex array with the interaction of the left–chiral and right–chiral fermion in the upper and lower index. For a general Breit–Wigner propagator we need to know the mass m and the width Γ of the intermediate vector boson. The output array J_{io} again has six components which for the photon with momentum q are

$$\begin{aligned} J_{io}(\mu+1) &= -\frac{i}{q^2}\ F_o^T\ \gamma^\mu \left(g(1)\ \frac{\mathbb{1}-\gamma_5}{2}\ + g(2)\ \frac{\mathbb{1}+\gamma_5}{2} \right)\ F_i \qquad\qquad \mu = 0,1,2,3 \\ J_{io}(5) &= -F_i(5) + F_o(5) \sim -p_i(0) + p_o(0) + i\,(-p_i(3) - p_o(3)) \\ J_{io}(6) &= -F_i(6) + F_o(6) \sim -p_i(1) + p_o(1) + i\,(-p_i(2) + p_o(2))\ . \end{aligned} \tag{2.2}$$

  The first four entries in J_{io} correspond to the index μ or the dimensionality of the Dirac matrices in this vector current. The spinor index is contracted between F_o^T and F_i.

  As two more arguments J_{io} includes the 4-momentum flowing through the gauge boson propagator. They allow us to reconstruct q^μ from the last two entries

$$q^\mu = (\mathrm{Re}J_{io}(5), \mathrm{Re}J_{io}(6), \mathrm{Im}J_{io}(6), \mathrm{Im}J_{io}(5))\ . \tag{2.3}$$

- `FFV1_0`(F_i, F_o, J, g, V) computes the amplitude of a fermion–fermion–vector coupling using the two external fermionic spinors F_i and F_o and an incoming vector current J which in our case comes from `FFV1_3`. Again, the coupling $g(1:2)$ is a complex array, so we numerically compute

$$F_o^T\ \not{J} \left(g(1)\ \frac{\mathbb{1}-\gamma_5}{2}\ + g(2)\ \frac{\mathbb{1}+\gamma_5}{2} \right)\ F_i\ . \tag{2.4}$$

All spinor and Dirac indices of the three input arguments are contracted in the final result. Momentum conservation is not enforced by `FFV1_0`, so we have to take care of it by hand.

Given the list above it is easy to follow how Madgraph computes the amplitude for $u\bar{u} \to \gamma^* \to \mu^+\mu^-$. First, it calls wave functions for all external particles and puts them into the array $W(1:6, 1:4)$. The vectors $W(*,1)$ and $W(*,3)$ correspond to $F_i(u)$ and $F_i(\mu^+)$, while $W(*,2)$ and $W(*,4)$ mean $F_o(\bar{u}$ and $F_o(\mu^-)$.

The first vertex we evaluate is the incoming quark–photon vertex. Given the wave functions $F_i = W(*,1)$ and $F_o = W(*,2)$ `FFV1_3` computes the vector current for the massless photon in the s-channel. Not much changes if we instead choose a massive Z boson, except for the arguments m and Γ in the `FFV1_3` call. Its output is the photon current $J_{io} \equiv W(*,5)$.

The last step combines this current with the two outgoing muons coupling to the photon. Since this number gives the final amplitude, it should return a complex number, not an array. Madgraph calls `FFV1_0` with $F_i = W(*,3)$ and $F_o = W(*,4)$, combined with the photon current $J = W(*,5)$. The result `AMP` is copied into `JAMP` without an additional sign which could have come from the relative ordering of external fermions in different Feynman diagrams contributing to the same process.

The only remaining sum left to compute before we square `JAMP` is the color structure, which in our simple case means one color structure with a color factor $N_c = 3$.

As an added bonus Madgraph produces a file with all Feynman diagrams in which the numbering of the external particles corresponds to the second argument of W and the numbering of the Feynman diagrams corresponds to the argument of `AMP`. This helps us identify intermediate results W, each of which is only computed once, even if is appears several times in the different Feynman diagrams.

As mentioned above, to calculate the transition amplitude Madgraph requires all masses and couplings. They are transferred through common blocks in the file coupl.inc and computed elsewhere. In general, Madgraph uses unitary gauge for all vector bosons, because in the helicity amplitude approach it is easy to accommodate complicated tensors, in exchange for a large number of Feynman diagrams.

The function `MATRIX1` described above is not yet the full story. When we square $\mathcal{M}$ symbolically we need to sum over the spins of the outgoing states to transform a spinor product of the kind $u\bar{u}$ into the residue or numerator of a fermion propagator. To obtain the full transition amplitude numerically we correspondingly sum over all helicity combinations of the external fermions, in our case $2^4 = 16$ combinations.

```
      SUBROUTINE SMATRIX1(P,ANS)
C
C     Generated by Madgraph 5
C
C     Returns amplitude squared summed/avg over colors
C     and helicities for the point in phase space P(0:3,NEXTERNAL)
C
C     Process: u u~ > mu+ mu- / z
C
      INTEGER     NCOMB, NGRAPHS, NDIAGS, THEL
      PARAMETER (NCOMB=16, NGRAPHS=1, NDIAGS=1, THEL=2*NCOMB)

      REAL*8 P(0:3,NEXTERNAL)

      INTEGER I,J,IDEN
      INTEGER NHEL(NEXTERNAL,NCOMB),NTRY(2),ISHEL(2),JHEL(2)
      INTEGER JC(NEXTERNAL),NGOOD(2), IGOOD(NCOMB,2)
      REAL*8 T,MATRIX1
      LOGICAL GOODHEL(NCOMB,2)

      DATA NGOOD /0,0/
      DATA ISHEL/0,0/
      DATA GOODHEL/THEL*.FALSE./

      DATA (NHEL(I,   1),I=1,4) /-1,-1,-1,-1/
      DATA (NHEL(I,   2),I=1,4) /-1,-1,-1, 1/
      DATA (NHEL(I,   3),I=1,4) /-1,-1, 1,-1/
      DATA (NHEL(I,   4),I=1,4) /-1,-1, 1, 1/
      DATA (NHEL(I,   5),I=1,4) /-1, 1,-1,-1/
```

```
DATA (NHEL(I,   6),I=1,4) /-1, 1,-1, 1/
DATA (NHEL(I,   7),I=1,4) /-1, 1, 1,-1/
DATA (NHEL(I,   8),I=1,4) /-1, 1, 1, 1/
DATA (NHEL(I,   9),I=1,4) / 1,-1,-1,-1/
DATA (NHEL(I,  10),I=1,4) / 1,-1,-1, 1/
DATA (NHEL(I,  11),I=1,4) / 1,-1, 1,-1/
DATA (NHEL(I,  12),I=1,4) / 1,-1, 1, 1/
DATA (NHEL(I,  13),I=1,4) / 1, 1,-1,-1/
DATA (NHEL(I,  14),I=1,4) / 1, 1,-1, 1/
DATA (NHEL(I,  15),I=1,4) / 1, 1, 1,-1/
DATA (NHEL(I,  16),I=1,4) / 1, 1, 1, 1/
DATA IDEN/36/

DO I=1,NEXTERNAL
  JC(I) = +1
ENDDO

DO I=1,NCOMB
  IF (GOODHEL(I,IMIRROR) .OR. NTRY(IMIRROR).LE.MAXTRIES) THEN
    T = MATRIX1(P ,NHEL(1,I),JC(1))
    ANS = ANS+T
  ENDIF
ENDDO

ANS = ANS/DBLE(IDEN)
END
```

The important part of this subroutine is the list of possible helicity combinations stored in the array $n_{\text{hel}}(1:4, 1:16)$. Adding all different helicity combinations means a loop over the second argument and a call of `MATRIX1` with the respective helicity combination. Because of the naive helicity combinations many are not allowed the array `GOODHEL` keeps track of valid combinations. After an initialization to all 'false' this array is only switched to 'true' if `MATRIX1` returns a finite value, otherwise Madgraph does not waste time to compute the matrix element. At the very end, a complete spin–color averaging factor is included as `IDEN` and in our case given by $2 \times 2 \times N_c^2 = 36$.

Altogether, Madgraph provides us with the subroutine `SMATRIX1` and the function `MATRIX1` which together compute $\overline{|\mathcal{M}|^2}$ for each phase space point given as an external momentum configuration. This helicity method might not seem particularly appealing for a simple $(2 \to 2)$ process, but it makes it possible to compute processes with many particles in the final state and typically up to 10000 Feynman diagrams which we could never square symbolically, no matter how many graduate students' live times we throw in.

3 Electroweak theory

The fundamental structure describing the interactions between the elementary particles of the Standard Model is gauge interactions. We have introduced interactions and Feynman rules for QED, the interaction of leptons and quarks with a photon. Next, we will have a closer look at the gauge symmetry structure and how it is related to the particle interactions. After that, the same concept will lead us the weak interaction.

3.1 QED gauge invariance

Even though we already know how to use Feynman rule to compute QED scattering amplitudes, and we even have an idea how these Feynman rules are related to the underlying quantum field theory, let us have another look at the QED Lagrangian. Specifically, we would like to understand how the symmetry structure of the QED Lagrangian beyond the built-in Lorentz invariance induces the form of the fermion-photon interaction given in Eq.(1.26). The Lagrangian in Eq.(1.6) describes the quantum version of the photons from electrodynamics,

$$\mathscr{L}_{\text{photon}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \text{with} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \; , \tag{3.1}$$

We introduce a so-called gauge transformation of the photon field the way we know it from classical electrodynamics,

$$A_\mu \;\to\; A_\mu - \frac{1}{e}\partial_\mu \alpha \; . \qquad \text{with} \qquad \alpha \equiv \alpha(x) \; . \tag{3.2}$$

The shift is space-time dependent. This defines a local gauge transformation. The field strength $F_{\mu\nu}$ is symmetric under such a local gauge transformation,

$$\begin{aligned} F_{\mu\nu} \;&\to\; \partial_\mu \left(A_\nu - \frac{1}{e}\partial_\nu \alpha \right) - \partial_\nu \left(A_\mu - \frac{1}{e}\partial_\mu \alpha \right) \\ &= F_{\mu\nu} - \frac{1}{e} \left(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu \right) \alpha = F_{\mu\nu} \; . \end{aligned} \tag{3.3}$$

The role of α becomes clearer when we introduce a fermion spinor ψ and a covariant derivative as in Eq.(1.26),

$$\begin{aligned} \mathscr{L}_{\text{fermion-photon}} &= \overline{\psi} \left(i \not{D} - m \mathbb{1} \right) \psi \\ &\equiv \overline{\psi} \left(i (\not{\partial} + ieq \not{A}) - m \mathbb{1} \right) \psi \; , \end{aligned} \tag{3.4}$$

where q is the fermion charge in units of the electron charge and $\overline{\psi} = (\psi^*)^T \gamma^0$. We can rotate the fermion field by the, now, angle $\alpha(x)$,

$$\begin{aligned} \psi \;&\to\; e^{iq\alpha} \psi \\ \overline{\psi} \;&\to\; e^{-iq\alpha} \overline{\psi} \; . \end{aligned} \tag{3.5}$$

Because α is a space-time-dependent angle for for the spinor transformation, we refer to the symmetry as a local $U(1)$ transformation. The fermion mass term is symmetric in itself, which means QED works for massive fermions. The kinetic term with the covariant derivative transforms into

$$\begin{aligned} i\overline{\psi} \not{D} \psi \equiv i\overline{\psi} \gamma^\mu (\partial_\mu + ieqA_\mu) \psi \;&\to\; i\overline{\psi} e^{-iq\alpha} \gamma^\mu \left[\partial_\mu + ieq \left(A_\mu - \frac{1}{e}\partial_\mu \alpha \right) \right] e^{iq\alpha} \psi \\ &= i\overline{\psi} \gamma^\mu e^{-iq\alpha} \left[\partial_\mu e^{iq\alpha} + e^{iq\alpha} ieq \left(A_\mu - \frac{1}{e}\partial_\mu \alpha \right) \right] \psi \\ &= i\overline{\psi} \gamma^\mu e^{-iq\alpha} \left[e^{iq\alpha} \; iq \; \partial_\mu \alpha + e^{iq\alpha} \partial_\mu + e^{iq\alpha} ieq \left(A_\mu - \frac{1}{e}\partial_\mu \alpha \right) \right] \psi \\ &= i\overline{\psi} \gamma^\mu \left(iq \; \partial_\mu \alpha + \partial_\mu + ieqA_\mu - iq \; \partial_\mu \alpha \right) \psi \\ &= i\overline{\psi} \gamma^\mu \left(\partial_\mu + ieqA_\mu \right) \psi \\ &\equiv i\overline{\psi} \not{D} \psi \; . \end{aligned} \tag{3.6}$$

The combined QED Lagrangian is invariant under a local rotation of the fermion field, if we also shift the photon field the way we know it from electrodynamics. In combination, we say that the QED interaction between photons and fermions is defined by the invariance under the Lorentz symmetry and under a local $U(1)$ gauge transformation of both fields.

Combining the kinetic derivative and the interaction into a covariant derivative is useful. Looking at the first and last lines of Eq.(3.6) we see that the gauge transformation of the covariant derivative has to be

$$\not{D} \;\to\; e^{iq\alpha} \not{D} e^{-iq\alpha} \; . \tag{3.7}$$

We can also replace the definition of the field strength in terms of the gauge field by a definition in terms of the covariant derivative, for example acting on a test function $f(x)$

$$\begin{aligned} F_{\mu\nu}\, f &= \frac{1}{ieq}\, [D_\mu, D_\nu]\, f \\ &= \frac{1}{ieq}\, (\partial_\mu + ieqA_\mu)(\partial_\nu + ieqA_\nu)\, f - (\mu \leftrightarrow \nu) \\ &= (\partial_\mu A_\nu) f + A_\nu(\partial_\mu f) + A_\mu(\partial_\nu f) - (\mu \leftrightarrow \nu) \qquad \text{with} \qquad [A_\mu, A_\nu] = 0 \\ &= (\partial_\mu A_\nu) f - (\partial_\nu A_\mu) f \; . \end{aligned} \tag{3.8}$$

We have to assume that the gauge field commutes, so we call an abelian gauge field.

3.2 Massive photon

One of the shortcomings of QED is that it only defines long-range interactions. From the work of Hideki Yukawa in 1935 we know that the mass of the exchange particles changes the form of the interaction potential,

$$\begin{aligned} V(r) &= -\frac{e^2}{r} && \text{massless particle exchange} \\ V(r) &= -g^2\, \frac{e^{-mr}}{r} && \text{massive particle exchange with } m. \end{aligned} \tag{3.9}$$

Yukawa did not actually talk about the weak nuclear force at the quark level. His model was based on fundamental protons and neutrons, and his exchange particles were pions. But his argument applies perfectly to the so-called Fermi interaction between fermions. Here we can observe the short-range Yukawa potential, and one can show that such a potential is generated by a massive particle exchange, with a propagator of the kind $1/(q^2 - m^2)$. This leads us to the challenge of formally including a photon mass in the gauge-invariant QED Lagrangian.

The first step towards defining a massive version of QED is to include a photon mass in the kinetic Lagrangian of Eq.(3.1). We immediately see that just adding a photon mass in analogy to the scalar mass in Eq.(1.10) is not allowed by the gauge symmetry,

$$\frac{1}{2} m^2 A_\mu^2 \;\to\; \frac{1}{2} m^2 \left(A_\mu - \frac{1}{e} \partial_\mu \alpha \right)^2 \neq \frac{1}{2} m^2 A_\mu^2 \; . \tag{3.10}$$

The trick to construct a gauge-invariant photon mass termx is to add an innocent-looking real scalar field without a mass and without a coupling to the photon, but with a scalar–photon mixing term and a well-chosen gauge transformation. The result is called the Boulware–Gilbert model or Stückelberg mass generation,

$$\begin{aligned} \mathscr{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu^2 + \frac{1}{2} \left(\partial_\mu \phi \right)^2 - ef A_\mu \partial^\mu \phi \qquad \text{with} \qquad m = ef \\ &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} e^2 f^2 \left(A_\mu - \frac{1}{ef} \partial_\mu \phi \right)^2 \\ &\equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} e^2 f^2 \tilde{A}_\mu^2 \; . \end{aligned} \tag{3.11}$$

Here, $\tilde{A}_\mu$ is the massive photon field and f is a common mass scale for the photon mass and the mixing. It ensures that all terms in the Lagrangian have mass dimension four — remembering that boson fields like A_μ and ϕ have mass dimension one. The field strength does not change if we define it in terms of the massive photon field,

$$\begin{aligned} F_{\mu\nu}\Big|_{\tilde{A}} &= \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = \partial_\mu \left(A_\nu - \frac{1}{ef} \partial_\nu \phi \right) - \partial_\nu \left(A_\mu - \frac{1}{ef} \partial_\mu \phi \right) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}\Big|_{A} \,, \end{aligned} \tag{3.12}$$

and we can write the massive photon Lagrangian of Eq.(3.11) as

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A^2 \tilde{A}_\mu^2 \,. \tag{3.13}$$

To make this Lagrangian explicitely gauge invariant, a suitable gauge transformation of the scalar field ϕ has to cancel the contribution of the transformed mass term. The simple choice

$$\phi \;\to\; \phi - f\alpha \,. \tag{3.14}$$

indeed gives us together with Eq.(3.2)

$$\begin{aligned} A_\mu - \frac{1}{ef} \partial_\mu \phi \;&\to\; A_\mu - \frac{1}{e} \partial_\mu \alpha - \frac{1}{ef} \partial_\mu \phi + \frac{1}{ef} \partial_\mu (f\alpha) \\ &= A_\mu - \frac{1}{ef} \partial_\mu \phi \end{aligned} \tag{3.15}$$

This Lagrangian describes a massive photon field $\tilde{A}_\mu$, which has absorbed the real scalar ϕ as its additional longitudinal component. This is because a massless gauge boson A_μ has only two on–shell degrees of freedom, the left-handed and right-handed polarizations, while the massive $\tilde{A}_\mu$ has an additional longitudinal polarization. To describe this additional degree of freesom, the massive photon $\tilde{A}$ has 'eaten' the real scalar field ϕ.

What kind of properties does this field ϕ need to have, so that we can use it to provide a photon mass? From the gauge transformation we immediately see that any purely scalar term in the Lagrangian would need to be symmetric under $\phi \to \phi - f\alpha$. This means that we cannot write down polynomial terms ϕ^n, like a mass, a self coupling, or an interaction ϕAA. Only derivative interactions proportional to $\partial\phi$ attached to gauge-invariant currents are allowed, because for those the shift by α turns into an irrelevant total derivative in the Lagrangian.

This example illustrates a few vital properties of Nambu–Goldstone bosons (NGB). Such massless physical states appear in many areas of physics and are described by Goldstone's theorem. It applies to global continuous symmetries of the Lagrangian which are violated by a non–symmetric vacuum state, a mechanism called spontaneous symmetry breaking. Based on Lorentz invariance and states with a positively definite norm we can then prove: *If a global symmetry group is spontaneously broken into a group of lower rank, its broken generators correspond to physical Goldstone modes. They are scalar fields and transform non–linearly under the larger and linearly under the smaller group. They are massless and cannot form a potential, because the non–linear transformation only allows derivative terms in the Lagrangian.*

For our massive QED case we are breaking the $U(1)$ gauge symmetry, which naively introduces a massless scalar degree of freedom ϕ. Following Eq.(3.11) it provides the missing longitudinal polarization for the massive photon. This combines two problems into one solution — we can break the gauge symmetry without creating unobserved massless particles, and our massive photon gets an additional degree of freedom. We will use the same trick for the Higgs later.

3.3 Fermion doublets

The structural element of the Fermi theory and the electroweak Standard Model is the $SU(2)$ doublet structure of paired fermions, like protons and neutrons. We know that they interact with each other, for instance when the neutron

decays into a proton, a positron, and a neutrino, with a life time of 611 s. We want to describe this interaction with a gauge boson exchange.

If one gauge transformation should link the two double components, an obvious choice is to replace the local $U(1)$ gauge invariance by a local $SU(2)$ gauge invariance. We remind ourselves that an $SU(2)$ transformation is very similar to an $O(3)$ rotation, and the generators of the $SU(2)$ transformation are the Pauli matrices. This means that a representation of $SU(2)$ transformations U is

$$U = e^{i\alpha_a \tau_a/2} \qquad \text{with} \qquad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.16}$$

The Pauli matrices satisfy the relation

$$\tau_a \tau_b = \delta_{ab} + i\epsilon_{abc}\tau_c \qquad \Leftrightarrow \qquad [\tau_a, \tau_b] = 2i\epsilon_{abc}\tau_c . \tag{3.17}$$

This means that we can write a commutator of two objects with $SU(2)$ indices as an object with one $SU(2)$ index, for example the field strength $F_a^{\mu\nu}$. For later use we also need a sum rule for the Pauli matrices,

$$\sum_{a,b} \tau_a \tau_b = \sum_{a,b} (\delta_{ab} + i\epsilon_{abc}\tau_c) = \sum \delta_{ab} + i \sum_{a \neq b} \epsilon_{abc}\tau_c = \sum \delta_{ab} + i \sum_{a<b} (\epsilon_{abc} + \epsilon_{bac})\, \tau_c = \sum \delta_{ab} . \tag{3.18}$$

The basis of three Pauli matrices we can write in terms of $\tau_{1,2,3}$ or in terms of

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3.19}$$

If the $SU(2)$ gauge symmety is generated by the Pauli matrices, the corresponding fields will no longer commute, $[A_\mu, A_\nu] \neq 0$. This means we need to think about the definition of the field strength tensor. Motivated by the index structure in Eq.(3.17) we still define

$$\begin{aligned} F_{a,\mu\nu} &\equiv \frac{1}{ieq}[D_\mu, D_\nu]_a = \partial_\mu A_{a,\nu} - \partial_\nu A_{a,\mu} + ieq[A_\mu, A_\nu]_a \\ \Rightarrow \qquad \mathscr{L} &\supset -\frac{1}{4} F_{a,\mu\nu} F_a^{\mu\nu} . \end{aligned} \tag{3.20}$$

The gauge invariance of this field strength and hence this kinetic gauge boson term can be shown from Eq.(3.8).

From the QED Lagrangian we know how to describe fermion masses and interactions with gauge bosons in a dimension-4 Lagrangian. Before we go through this in some detail, let us briefly look at the dimensionality of the Lagrangian, so we can use it to structure the weak Lagrangian. The dimensionality is crucial for the renormalizability or fundamental nature of a Lagrangian. The definition of the transition in terms of the interaction Hamiltonian in Eq.(1.32),

$$\mathcal{S} \sim \exp\left[-i \int dt \mathcal{H}(t)\right] = \exp\left[-i \int d^4x \left(\pi\dot{\phi} - \mathscr{L}\right)\right] , \tag{3.21}$$

suggests that the Hamiltonian should have the unit energy or momentum, and the Lagrangian should have mass dimension four. The elements of the Lagrangian have the mass dimensions.

$$\begin{aligned} &\text{scalar field} && [\phi] = M \\ &\text{explicit mass} && [m] = M \\ &\text{gauge field} && [A_\mu] = M \\ &\text{space derivative} && [\partial_\mu] = M \\ &\text{field strength} && [F_{\mu\nu}] = M^2 \\ &\text{fermion spinor} && [\psi] = M^{3/2} . \end{aligned} \tag{3.22}$$

If we want our theory to be renormalizable and valid to arbitrarily large scales, the Lagrangian cannot include inverse masses, because in that case a momentum or energy in the numerator would eventually lead to an exploding ratio in the ultraviolet. This means all terms in a fundamental Lagragian should have mass dimension two to four, complemented by explicit masses. This is the way we will organize the weak Lagrangian.

3.4 Weak gauge bosons

The main theme of the electroweak theory is that it extends massless QED with its $U(1)$ gauge transformation in two ways: the gauge transformation is $SU(2)$ for fermion doublets and the gauge bosons are massive. This leads to a mixing of the gauge bosons from interaction to mass eigenstates, which we can describe in two different ways:

1. The neutral gauge bosons can mix, so the mass eigenstates are the photon A_μ and the massive Z_μ, but they are related to the interaction eigenstate W^3_μ and a massless B_μ. We can think of the B_μ as the photon of a proto-QED before we combine QED with the weak symmetry to the actual QED. This description leads to the weak mixing angle.

2. The $U(1)$ rotation of the fermion fields and the neutral $SU(2)$ rotation via τ_3 can be combined to individual $U(1)$ rotations of the left-handed and right-handed fermion spinors. This desciption leads to the Gell-Mann–Nishijima formula.

Starting with the mixing from interaction eigenstates to mass eigenstates, we assume that the two ingredients to the neutral electroweak interactions are a massless proto-photon B_μ and the neutral electroweak gauge boson field W^3_μ. Both are neutral particles with the same quantum numbers, so they mix to mass eigenstates. In our case they are A_μ and Z_μ,

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} c_w & s_w \\ -s_w & c_w \end{pmatrix} \begin{pmatrix} B_\mu \\ W^3_\mu \end{pmatrix} \qquad \text{with} \qquad s_w \equiv \sin\theta_w \quad c_w \equiv \cos\theta_w \; . \tag{3.23}$$

The photon describes the $U(1) \equiv U(1)_Q$ charge transformation and couples to electric charge. The field B_μ couples to the so-called hypercharge y via another $U(1) \equiv U(1)_Y$. The angle θ_w is the weak mixing angle or Weinberg angle. Both B_μ and A_μ are massless, because the determinant of the mass matrix keeps its zero eigenvalue after rotations.

As a starting point, the interaction of fermions, our case quarks $Q_{L,R}$, with gauge bosons is again written in terms of covariant derivatives. Just like the kinetic term for the gauge bosons, the kinetic and interaction term for the fermions has mass dimension four,

$$\mathscr{L}_{D4} = \overline{Q}_L i \not{D} Q_L + \overline{Q}_R i \not{D} Q_R - \frac{1}{4} F_{a,\mu\nu} F_a^{\mu\nu} ... \tag{3.24}$$

To write it out we need the covariant derivative. Unlike QED, the weak interaction knows about the chirality of the fermion fields, so we have to distinguish $\not{D}_L$ and $\not{D}_R$. In the interaction basis the covariant derivatives include the massless B_μ and the three massive $W_{a,\mu}$. We know from experiment that the $SU(2) \equiv SU(2)_L$ gauge transformation only acts on the left-handed doublets,

$$\begin{aligned} D_{L\mu} &= \partial_\mu + i g' \frac{y}{2} B_\mu + i g \sum_{a=1,2,3} W_{a,\mu} \frac{\tau_a}{2} \\ D_{R\mu} &= D_{L\mu} \Big|_{\tau_{1,2,3}=0} \; . \end{aligned} \tag{3.25}$$

The right-handed fields can be written as ntuples, but they do not have a doublet structure. This means that the massive charged W-boson only couples doublet like (u_L, d_L). In the mass basis for the neutral states the left-handed covariant derivative has to read

$$D_{L\mu} = \partial_\mu + ieqA_\mu + ig_Z \left(-q s_w^2 \mathbb{1} + \frac{\tau_3}{2} \right) Z_\mu + ig \left(\frac{\tau_1}{2} W^1_\mu + \frac{\tau_2}{2} W^2_\mu \right) \tag{3.26}$$

The comparison with Eq.(3.25) and the rotation matrix in Eq.(3.23) give us the relations between the couplings e and g_Z to g' and g and the weak mixing angle.

To understand the relation of y to q and the τ_3 eigenvalues we remember that according to Eq.(3.23) we should consider the gauge transformation under $U(1)_Q$ and under the neutral component of the $SU(2)_L$ transformation in

combination. Under the full $SU(2)_L$ symmetry the left-handed and right-handed spinors transform differently,

$$\begin{aligned} Q_L &\xrightarrow{U} U Q_L \\ Q_R &\xrightarrow{U} Q_R \qquad \text{with} \qquad U = \exp\left(i\frac{\alpha_a}{2}\tau_a\right) . \end{aligned} \tag{3.27}$$

For the mixing neutral gauge bosons we instead consider the combined transformation

$$\begin{aligned} Q_L &\xrightarrow{V} \exp\left(i\beta q_Q\right) V^\dagger Q_L \\ Q_R &\xrightarrow{V} \exp\left(i\beta q_Q\right) Q_R \qquad \text{with} \qquad V = U\Big|_{\tau_3} = \exp\left(i\frac{\beta}{2}\tau_3\right) . \end{aligned} \tag{3.28}$$

This transformation does not know anything about doublets and can be related to a $U(1)_Y$ transformation as

$$\begin{aligned} \exp(i\beta q)\, V^\dagger &= \exp(i\beta q)\ \exp\left(-\frac{i}{2}\beta\tau_3\right) \\ &= \exp\left(i\frac{\beta}{2} y\mathbb{1}\right) \qquad\qquad \text{with} \quad q\mathbb{1} \equiv \frac{y\mathbb{1}+\tau_3}{2} \qquad y_Q = \frac{1}{3} \quad y_L = -1 . \end{aligned} \tag{3.29}$$

The relation between the charge q, the hypercharge y, and the isospin τ_3 is the same as in Eq.(3.26), called the Gell-Mann–Nishijima formula. The indices Q and L denote quark and lepton doublets. In this notation we do not distinguish $U(1)$ rotations by a real angle and $SU(2)$ rotations proportional to a unit matrix. This is reflected in the Gell-Mann–Nishijima formula, where τ_3 has to be replaced by its eigenvalue ± 1 for up–type and down–type fermions to relate charge and hypercharge.

Next, the Lagrangian for the gauge boson masses is most conveniently written in terms of $\tau_\pm$. To switch bases we only have to make sure we keep the standard normalization of all fields,

$$\begin{aligned} \sqrt{2}\left(\tau_+ W_\mu^+ + \tau_- W_\mu^-\right) &= \sqrt{2}\begin{pmatrix} 0 & W_\mu^+ \\ 0 & 0 \end{pmatrix} + \sqrt{2}\begin{pmatrix} 0 & 0 \\ W_\mu^- & 0 \end{pmatrix} \\ &\stackrel{!}{=} \tau_1 W_\mu^1 + \tau_2 W_\mu^2 = \begin{pmatrix} 0 & W_\mu^1 \\ W_\mu^1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iW_\mu^2 \\ iW_\mu^2 & 0 \end{pmatrix} \\ \Leftrightarrow \qquad & W_\mu^+ = \frac{1}{\sqrt{2}}\left(W_\mu^1 - iW_\mu^2\right) \qquad\qquad W_\mu^- = \frac{1}{\sqrt{2}}\left(W_\mu^1 + iW_\mu^2\right) \\ \Rightarrow \qquad & D_{L\mu} = \partial_\mu + ieqA_\mu + ig_Z\left(-qs_w^2 + \frac{\tau_3}{2}\right) Z_\mu + i\frac{g}{\sqrt{2}}\left(\tau_+ W_\mu^+ + \tau_- W_\mu^-\right) \end{aligned} \tag{3.30}$$

To check this mass basis, we look at the masses of the gauge bosons, which appear as dimension-2 mass terms in the electroweak Lagrangian. Using the above relation, we write them in terms of the charged W-fields,

$$\mathscr{L}_{D2} = \frac{m_W^2}{2}\left(W^{1,\mu}W_\mu^1 + W^{2,\mu}W_\mu^2\right) + \frac{m_Z^2}{2} Z^\mu Z_\mu = m_W^2 W^{+,\mu} W_\mu^- + \frac{m_Z^2}{2} Z^\mu Z_\mu . \tag{3.31}$$

The relative factor two in front of the W mass appears because the Z field is neutral and the W field is charged, again the same as for neutral and charged scalars.

Finally, we look at the Lagrangian terms describing the fermion masses with mass dimension three. From Eq.(1.53) we know that it requires a combination of the left-handed doublet Q_L and the right-handed singlet fields Q_R,

$$\mathscr{L}_{D3} = -\overline{Q}_L m_Q Q_R + ... \tag{3.32}$$

This form will require a doublet structure of the Higgs–Goldstone fields, which we will discuss next term. Moreover, these mass terms can be matrices in generation space, which implies that we have to rotate the fermion fields from an interaction basis into the mass basis, where these mass matrices are diagonal. Flavor physics dealing with such 3×3 mass matrices is its own field of physics. We will also omit this complication for now.

3.5 Sigma model

Next, we need to check the gauge invariance of the different Lagrangian terms. The fermion masses in Eq.(3.32) are clearly not invariant under the $SU(2)_L$ gauge transformation in Eq.(3.27).

$$\overline{Q}_L m_Q Q_R \xrightarrow{U} \overline{Q}_L U^{-1} m_Q Q_R \neq \overline{Q}_L m_Q Q_R \,. \tag{3.33}$$

One way of solving this problem is to introduce an additional field $\Sigma(x)$. This field will play a similar role as the real scalar field we used for the photon mass generation. Its physical properties will become clear step by step. First, we introduce Σ into the fermion mass term, which will tell us what it takes to make this mass term gauge invariant under the weak transformations defined in Eq.(3.27),

$$\begin{aligned} &\overline{Q}_L \Sigma m_Q Q_R \xrightarrow{U} \overline{Q}_L U^{-1} \Sigma^{(U)} m_Q Q_R \stackrel{!}{=} \overline{Q}_L \Sigma m_Q Q_R \\ \Leftrightarrow \qquad &\Sigma \xrightarrow{U} \Sigma^{(U)} = U\Sigma \end{aligned} \tag{3.34}$$

and in Eq. (3.28),

$$\begin{aligned} \overline{Q}_L \Sigma m_Q Q_R &\xrightarrow{V} \overline{Q}_L \exp\left(-i\frac{\beta}{2} y \mathbb{1}\right) \Sigma^{(V)} m_Q \exp\left(i\beta q\right) Q_R \\ &= \overline{Q}_L \Sigma^{(V)} \exp\left(-i\frac{\beta}{2} y \mathbb{1}\right) \exp\left(i\beta q\right) m_Q Q_R \qquad\qquad \exp\left(i\frac{\beta}{2} y \mathbb{1}\right) \text{ always commuting} \\ &= \overline{Q}_L \Sigma^{(V)} V m_Q Q_R \\ &\stackrel{!}{=} \overline{Q}_L \Sigma\, m_Q Q_R \qquad \Leftrightarrow \qquad \Sigma \to \Sigma^{(V)} = \Sigma V^\dagger \,. \end{aligned} \tag{3.35}$$

Together this gives us the transformation property needed for gauge-invariant fermion masses

$$\Sigma \to U \Sigma V^\dagger \,. \tag{3.36}$$

This means Σ is a 2×2 matrix with mass dimension zero and the gauge-invariant fermion mass Lagrangian is

$$\mathscr{L}_{D3} = -\overline{Q}_L \Sigma m_Q Q_R - \overline{L}_L \Sigma m_L L_R + \text{h.c.} + ... \tag{3.37}$$

In a second step, we also use Σ to introduce invariant gauge boson masses. We recall the covariant derivative from Eq.(3.25),

$$D_{L\mu} = \partial_\mu + ig' \frac{y}{2} B_\mu + igW_{a,\mu} \frac{\tau_a}{2} \,. \tag{3.38}$$

We can choose the form of the covariant derivative acting on Σ,

$$\begin{aligned} D_\mu \Sigma &= \partial_\mu \Sigma + ig' \Sigma B_\mu \frac{y}{2}\Big|_{q=0} + igW_{a,\mu} \frac{\tau_a}{2} \Sigma \\ &= \partial_\mu \Sigma - ig' \Sigma B_\mu \frac{\tau_3}{2} + igW_{a,\mu} \frac{\tau_a}{2} \Sigma \,. \end{aligned} \tag{3.39}$$

With the abbreviations

$$V_\mu \equiv \Sigma (D_\mu \Sigma)^\dagger \qquad \text{and} \qquad T \equiv \Sigma \tau_3 \Sigma^\dagger \,, \tag{3.40}$$

we will show in Sec. 3.7 that we can write the gauge boson mass Lagrangian as

$$\mathscr{L}_{D2} = -\frac{v^2}{4} \; \text{Tr}[V_\mu V^\mu] - \Delta\rho \frac{v^2}{8} \; \text{Tr}[T V_\mu] \; \text{Tr}[T V^\mu] \,. \tag{3.41}$$

The trace acts on the 2×2 $SU(2)$ matrices and ensures that this form is gauge invariant. The parameter $\Delta\rho$ is conventional and will be the focus of Section 3.8.

Before we comtinue with the weak boson masses, we discuss which other gauge invariant Lagrangian terms of mass dimension four we can write down using Σ. Our first attempt for a building block

$$\Sigma^\dagger\Sigma \overset{U,V}{\to} (U\Sigma V^\dagger)^\dagger(U\Sigma V^\dagger) = V\Sigma^\dagger U^\dagger U\Sigma V^\dagger = V\Sigma^\dagger\Sigma V^\dagger \neq \Sigma^\dagger\Sigma \tag{3.42}$$

is forbidden. However, the circular trace $\mathrm{Tr}(\Sigma^\dagger\Sigma) \to \mathrm{Tr}(V\Sigma^\dagger\Sigma V^\dagger) = \mathrm{Tr}(\Sigma^\dagger\Sigma)$ allows for the additional potential terms, meaning terms with no derivatives

$$\mathscr{L}_\Sigma = -\frac{\mu^2 v^2}{4}\,\mathrm{Tr}(\Sigma^\dagger\Sigma) - \frac{\lambda v^4}{16}\left(\mathrm{Tr}(\Sigma^\dagger\Sigma)\right)^2 + \cdots\ , \tag{3.43}$$

with properly chosen prefactors μ, v, λ. We will come to it next term. This finalizes our construction of the weak Lagrangian for massive $SU(2)_L$ gauge boson, including the auxiliary Σ field and ordered by mass dimension,

$$\mathscr{L} = \mathscr{L}_{D2} + \mathscr{L}_{D3} + \mathscr{L}_{D4} + \mathscr{L}_\Sigma\ . \tag{3.44}$$

3.6 Weak boson propagators

In Sec. 3.2 we briefly mentioned Goldstone's theorem and how it relates symmetry breaking to new degrees of freedom. We also saw how an additional scalar could be used to make a 'photon' massive. Let us briefly look at how this works for the massive weak gauge bosons, where we break the full electroweak gauge symmetry as

$$SU(2)_L \times U(1)_Y \to U(1)_Q\ . \tag{3.45}$$

We expect three Goldstone bosons, because $SU(2)$ has three generators, the Pauli matrices. They will promote the three massless gauge bosons (with two degrees of freedom each) to massive gauge bosons (with three degrees of freedom each). This is at least the point of view of the unitary gauge, in which we never see Goldstone modes.

In the general renormalizable R_ξ gauge we can see the Goldstone modes in the gauge boson propagators

$$\begin{aligned}
\Delta^{\mu\nu}_{VV}(p) &= \frac{-i}{p^2 - m_V^2 + i\epsilon}\left[g^{\mu\nu} + (\xi-1)\frac{p^\mu p^\nu}{p^2 - \xi m_V^2}\right] \\
&= \begin{cases}
\dfrac{-i}{p^2 - m_V^2 + i\epsilon}\left[g^{\mu\nu} - \dfrac{p^\mu p^\nu}{m_V^2}\right] & \text{unitary gauge } \xi \to \infty \\
\dfrac{-i}{p^2 - m_V^2 + i\epsilon}\, g^{\mu\nu} & \text{Feynman gauge } \xi = 1 \\
\dfrac{-i}{p^2 - m_V^2 + i\epsilon}\left[g^{\mu\nu} - \dfrac{p^\mu p^\nu}{p^2}\right] & \text{Landau gauge } \xi = 0\ .
\end{cases}
\end{aligned} \tag{3.46}$$

If these gauge choices are physically equivalent, something has to compensate for the fact that in Feynman gauge the whole second term vanishes and the polarization sum looks like a massless gauge boson. This is done by the Goldstone propagator for additional scalar degrees of freedom

$$\Delta_{VV}(p^2) = \frac{-i}{p^2 - \xi m_V^2 + i\epsilon}\ , \tag{3.47}$$

The Goldstone mass $\sqrt{\xi} m_V$ depends on the gauge: in unitary gauge the infinitely heavy Goldstones do not propagate ($\Delta_{VV}(p^2) \to 0$), while in Feynman in Landau gauge we include them as particles. In effect, they will cancel the second term of the gauge boson propagators.

These different gauges have different Feynman rules and Green's functions, even a different particle content. For a given problem one or the other might be the most efficient to use in computations or proofs. For example, the proof of renormalizability was first formulated in unitary gauge. Also tree level helicity amplitudes are usually computed in unitary gauge. Loop calculations might be most efficient in Feynman gauge, because of the simplified propagator structure, while many QCD processes benefit from an explicit projection on the physical external gluons, etc.

3.7 Weak boson masses

To check that Eq.(3.41) gives the correct masses in the Standard Model we assume that Σ acquires a vacuum expectation value. The simplest way to achieve this and obtain the correct fermion masses is to just write

$$\Sigma(x) = \mathbb{1} \; . \tag{3.48}$$

This choice is called unitary gauge. It looks like a dirty trick to claim $\Sigma(x) = \mathbb{1}$ helps with the gauge invariance of gauge boson masses, because this constant does not exhibit the correct transformation property under the U and V symmetries. However, we can always work in a specific gauge and check the physical predictions for gauge invariance later.

What we can check is that $\mathscr{L}_{D2}$ from Eq.(3.41) gives the correct gauge boson masses. Using the covariant derivative from Eq.(3.39) acting on a now constant field we can compute V_μ in unitary gauge

$$\begin{aligned} V_\mu = \Sigma(D_\mu\Sigma)^\dagger &= \mathbb{1}(D_\mu\Sigma)^\dagger \\ &= -igW_\mu^+ \frac{\tau_+}{\sqrt{2}} - igW_\mu^- \frac{\tau_-}{\sqrt{2}} - igW_\mu^3 \frac{\tau_3}{2} + ig' B_\mu \frac{\tau_3}{2} \\ &= -i\frac{g}{\sqrt{2}} \left(W_\mu^+ \tau_+ + W_\mu^- \tau_- \right) - ig_Z Z_\mu \frac{\tau_3}{2} \, , \end{aligned} \tag{3.49}$$

with $Z_\mu = c_w W_\mu^3 - s_w B_\mu$ and the two coupling constants

$$g_Z = \frac{g}{c_w} \qquad \text{and} \qquad g' = \frac{g s_w}{c_w} \; . \tag{3.50}$$

This gives us the first of the two terms in $\mathscr{L}_{D2}$, after using $\tau_\pm^2 = 0$ and $\text{Tr}(\tau_3\tau_\pm) = 0$,

$$\begin{aligned} \text{Tr}[V_\mu V^\mu] &= -2\,\frac{g^2}{2} W_\mu^+ W^{-\mu}\,\text{Tr}(\tau_+\tau_-) - \frac{g_Z^2}{4} Z_\mu Z^\mu\,\text{Tr}(\tau_3^2) \\ &= -g^2 W_\mu^+ W^{-\mu} - \frac{g_Z^2}{2} Z_\mu Z^\mu \qquad \text{with} \quad \text{Tr}(\tau_\pm\tau_\mp) = 1, \quad \text{Tr}(\tau_3^2) = \text{Tr}\,\mathbb{1} = 2 \; , \end{aligned} \tag{3.51}$$

The second mass term, proportional to $\Delta\rho$, also simplifies in unitary gauge

$$\begin{aligned} T = \Sigma\tau_3\Sigma^\dagger &= \tau_3 \\ \Rightarrow \qquad \text{Tr}(TV_\mu) &= \text{Tr}\left(-ig_Z Z_\mu \frac{\tau_3^2}{2} \right) = -ig_Z Z_\mu \\ \Rightarrow \quad \text{Tr}(TV_\mu)\;\text{Tr}(TV^\mu) &= -g_Z^2 Z_\mu Z^\mu \; . \end{aligned} \tag{3.52}$$

Combining both terms with the prefactor in Eq.(3.41) yields the complete gauge boson mass term

$$\begin{aligned} \mathscr{L}_{D2} &= -\frac{v^2}{4}\left(-g^2 W_\mu^+ W^{-\mu} - \frac{g_Z^2}{2} Z_\mu Z^\mu \right) - \Delta\rho \frac{v^2}{8}\left(-g_Z^2 Z_\mu Z^\mu \right) \\ &= \frac{v^2 g^2}{4} W_\mu^+ W^{-\mu} + \frac{v^2 g_Z^2}{8}\left(1 + \Delta\rho\right) Z_\mu Z^\mu \; . \end{aligned} \tag{3.53}$$

Identifying the masses with the form given in Eq.(3.31), assuming universality of neutral and charged current interactions ($\Delta\rho = 0$), and introducing a since mass scale v, we find

$$\begin{aligned} m_W &= \frac{gv}{2} \\ m_Z &= \sqrt{1+\Delta\rho}\,\frac{g_Z v}{2} \overset{\Delta\rho=0}{=} \frac{g_Z v}{2} = \frac{gv}{2c_w} \; . \end{aligned} \tag{3.54}$$

The two masses can be expressed in terms of θ_w, assuming that that G_F or g universally govern charged-current and neutral-current interactions. From the known gauge boson masses ($m_W \sim 80$ GeV) and weak coupling ($g \sim 0.7$) we find $v \sim 246$ GeV.

3.8 Custodial symmetry

tp: move custodial symmetry to Higgs model. Put ghosts here instead. At tree level, Eq.(3.54) gives us an experimentally very well tested relation,

$$\frac{m_W^2}{m_Z^2} = \frac{g^2}{g_Z^2} = c_w^2 \qquad \text{corresponding to} \qquad \Delta\rho = 0 \; . \tag{3.55}$$

Ignoring what we already know, we can introduce a free parameter ρ to break this relation

$$\begin{aligned} g_Z^2 &\to g_Z^2 \; \rho \\ m_Z &\to m_Z \; \sqrt{\rho} = m_Z \; \sqrt{1+\Delta\rho} \; , \end{aligned} \tag{3.56}$$

This corresponds to our theoretically derived $\Delta\rho$. Because Eq.(3.54) holds in experimental reality, we need a reason to ensure $\Delta\rho \simeq 0$, and the $SU(2)_L \times U(1)_Y$ gauge symmetry unfortunately does not do the job. In the Standard Model $\rho = 1$ is violated at one-loop level, so we are looking for an approximate symmetry of the Standard Model. We can hope that it is valid in the $SU(2)_L$ gauge sector and slightly broken beyond. To realize this, we replace $SU(2)_L \times U(1)_Y$ with a larger $SU(2)_L \times SU(2)_R$ symmetry, which could even be global,

$$\begin{aligned} \Sigma &\to U \Sigma V^\dagger \qquad\qquad U \in SU(2)_L \qquad\qquad V \in SU(2)_R \\ \text{Tr}(\Sigma^\dagger \Sigma) &\to \text{Tr}\left(V \Sigma^\dagger U^\dagger U \Sigma V^\dagger\right) = \text{Tr}(\Sigma^\dagger \Sigma) \; . \end{aligned} \tag{3.57}$$

In this setup, the three components of W^μ form a triplet under $SU(2)_L$ and a singlet under $SU(2)_R$, so $\rho = 1$.

For the gauge boson masses we set $\Sigma = \mathbb{1}$, such that

$$\langle \Sigma \rangle \to \langle U \Sigma V^\dagger \rangle = \langle U \mathbb{1} V^\dagger \rangle = U V^\dagger \stackrel{!}{=} \mathbb{1} \; . \tag{3.58}$$

This implies $U = V$, so this choice of Σ breaks $SU(2)_L \times SU(2)_R$ to the diagonal or custodial symmetry $SU(2)_{L+R}$.

Even beyond tree level the global $SU(2)_L \times SU(2)_R$ symmetry structure can protect the relation $\rho = 1$. If fermions reside in $SU(2)_L$ and $SU(2)_R$ doublets we cannot generate any difference between up–type and down–type fermions, which implies for instance $m_b = m_t$. The measured masses $m_t \gg m_b$ leads to $\rho \neq 1$, because self energy loops in the W propagator mix a the bottom and top quark, while the Z propagator includes pure bottom and top loops,

$$\begin{aligned} \Delta\rho &\supset \frac{3 G_F}{8\sqrt{2}\pi^2} \left(m_t^2 + m_b^2 - 2 \frac{m_t^2 m_b^2}{m_t^2 - m_b^2} \log \frac{m_t^2}{m_b^2} \right) \\ &= \frac{3 G_F}{8\sqrt{2}\pi^2} \left(2 m_b^2 + m_b^2 \delta - 2 m_b^2 \frac{1+\delta}{\delta} \log(1+\delta) \right) \qquad\qquad \text{defining} \quad m_t^2 = m_b^2 (1+\delta) \\ &= \frac{3 G_F}{8\sqrt{2}\pi^2} \left(2 m_b^2 + m_b^2 \delta - 2 m_b^2 \left(\frac{1}{\delta} + 1 \right) \left(\delta - \frac{\delta^2}{2} + \frac{\delta^3}{3} + \mathcal{O}(\delta^4) \right) \right) \\ &= \frac{3 G_F}{8\sqrt{2}\pi^2} m_b^2 \left(2 + \delta - 2 - 2\delta + \delta + \delta^2 - \frac{2}{3}\delta^2 + \mathcal{O}(\delta^3) \right) \\ &= \frac{3 G_F}{8\sqrt{2}\pi^2} m_b^2 \left(\frac{1}{3}\delta^2 + \mathcal{O}(\delta^3) \right) \\ &= \frac{G_F m_W^2}{8\sqrt{2}\pi^2} \left(\frac{\left(m_t^2 - m_b^2\right)^2}{m_W^2 m_b^2} + \cdots \right) \; . \end{aligned} \tag{3.59}$$

In the Taylor series above the assumption of δ being small is of course not realistic, but the result is nevertheless instructive: the shift vanishes very rapidly towards the symmetric limit $m_t \sim m_b$. For the realistic Standard Model mass ratios the correct result is

$$\Delta\rho \supset \frac{3 G_F}{8\sqrt{2}\pi^2} m_t^2 \left(1 - 2 \frac{m_b^2}{m_t^2} \log \frac{m_t^2}{m_b^2} \right) = \frac{3 G_F m_W^2}{8\sqrt{2}\pi^2} \; \frac{m_t^2}{m_W^2} \; \left(1 + \mathcal{O}\left(\frac{m_b^2}{m_t^2} \right) \right) \; . \tag{3.60}$$

A second contribution to the ρ parameter will arise from Higgs loops, which we will introduce in Sec. 6,

$$\Delta\rho \supset -\frac{11 G_F m_Z^2 s_w^2}{24\sqrt{2}\pi^2} \log\frac{m_H^2}{m_Z^2}. \tag{3.61}$$

There exists another parameterization of the same effect, the T parameter. It is part of an effective theory parameterization of deviations from the tree level relations between gauge boson masses, mixing angles, and neutral and charged current couplings,

$$\{S, T, U\} \tag{3.62}$$

Two of these so-called Peskin–Takeuchi parameters can be understood fairly easily: the S-parameter corresponds to a shift of the Z mass. The T parameter compares contributions to the W and Z masses. The third parameter U is less important for most models. Again, we quote the contributions from the heavy fermion doublet,

$$\begin{aligned}
\Delta S &= \frac{N_c}{6\pi} \left(1 - 2Y \log\frac{m_t^2}{m_b^2}\right) \\
\Delta T &= \frac{N_c}{4\pi s_w^2 c_w^2 m_Z^2} \left(m_t^2 + m_b^2 - \frac{2 m_t^2 m_b^2}{m_t^2 - m_b^2} \log\frac{m_t^2}{m_b^2}\right) ,
\end{aligned} \tag{3.63}$$

with $Y = 1/6$ for quarks and $Y = -1/2$ for leptons. While the parameter S has nothing to do with our custodial symmetry, ρ and $T \sim \Delta\rho/\alpha$ are closely linked. Their main difference is the reference point, where $\rho = 1$ refers to its tree level value and $T = 0$ is often chosen for some kind of light Higgs mass and including the Standard Model top-bottom corrections.

Typical experimental constraints form an ellipse in the S vs T plane along the diagonal. They are usually quoted as ΔT with respect to a reference Higgs mass. Compared to a 125 GeV Standard Model Higgs boson the measured values range around $T \sim 0.1$ and $S \sim 0.05$. Additional contributions $\Delta T \sim 0.1$ tend to be within the experimental errors, much larger contributions are in contradiction with experiment.

There are two reasons to discuss these loop contributions breaking the custodial symmetry in the Standard Model. First, $\Delta\rho$ is experimentally very strongly constrained by electroweak precision measurements, which means that alternative models for electroweak symmetry breaking usually include the same kind of approximate custodial symmetry by construction. Second, in the Standard Model we can measure the symmetry violations from the heavy quarks and from the Higgs sector shown in Eqs.(3.59) and (3.61) in electroweak precision measurements. Even though the Higgs contributions depend on the Higgs mass only logarithmically, we can then derive an upper bound on the Higgs mass of the order of $\mathcal{O}(200\ \text{GeV})$. Since the Higgs discovery , studying electroweak precision data given the measured Higgs mass is one of the most sensitive consistency tests of the Standard Model.

4 Quantum Chromodynamics

After introducing QED as a $U(1)$-gauge interaction and the weak force as an $SU(2)$-gauge interaction, the step to the strong interaction or QCD with based on $SU(3)$ gauge invariance is at least conceptually simple. In particular, we assume, in agreement with all existing measurements, that the gluons as the QCD gauge bosons as massless and that their interaction is not affected by the weak quantum numbers or the fermion doublet structure. This means we can build QCD as an independent non-abelian version of QED. We will see that QCD, in spite of the structural similarity to QED, has especially interesting properties and equally interesting experimental consequences.

We postulate the number of QCD colors to be three,

$$SU(N_c) \equiv SU(3) \; , \tag{4.1}$$

and follow the same argument we made for $SU(2)$ from Eq.(3.16) on. Quarks live in the fundamental representation of $SU(3)$ and therefore have three colors. The $SU(N_c)$ transformations are given by

$$U = e^{i\alpha_a T_a} \qquad \text{with} \qquad T_{1,...,8} = \begin{pmatrix} \tau_1/2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \tau_2/2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \tau_3/2 & 0 \\ 0 & 0 \end{pmatrix} \cdots \tag{4.2}$$

with $a = 1 \; ... \; N_c^2 - 1 = 1 \; ... \; 8$. Note the conventional factor $1/2$ relative to the Pauli matrices. The $(N_c \times N_c)$ matrices T_a are the traceless and hermitian Gell-Mann matrices. Gluons live in this adjoint representation of $SU(3)$, so they have 8 color states. We give the first three of them in terms of the Pauli matrices, but there is no point in writing them down because we only ever need two algebraic properties to compute the color factors of scattering amplitudes,

$$[T_a, T_b] = i f_{abc} T_c \qquad \text{and} \qquad \text{Tr}(T_a T_b) = T_R \delta_{ab} \equiv \frac{1}{2} \delta_{ab} \; . \tag{4.3}$$

Here, f_{abc} are the antisymmetric structure constants of $SU(N_c)$ with

$$f_{acd} f_{bcd} = N_c \delta_{ab} \; . \tag{4.4}$$

The one formula we need to compute color factors for quark processes will be

$$(T_a)_{ij} (T_a)_{k\ell} = \frac{1}{2} \left(\delta_{i\ell} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{k\ell} \right) \; . \tag{4.5}$$

4.1 QCD Lagrangian

The QCD Lagrangian can be constructed as a non-abelian massless version of the QED Lagrangian. For the dimension-4 Lagrangian we follow Eq.(3.24), limit ourselves to the quarks as fermions charged under $SU(3)$, and write

$$\mathscr{L}_{\text{QCD}} = \bar{q}_i i \gamma^\mu (D_\mu)_{ij} q_j - \frac{1}{4} F_{a,\mu\nu} F_a^{\mu\nu} \qquad \text{with} \quad i,j = 1,...,3 \qquad a = 1,...,8 \; . \tag{4.6}$$

The good news is that we can ignore the doublet structure of the fermions if the color and flavor indices do not mix. The $SU(3)$-covariant derivative follows from its $SU(2)$ counterpart in Eq.(3.4) with $e \to -g_s$

$$(D_\mu)_{ij} = \partial_\mu \mathbb{1}_{ij} - i g_s A_{\mu,a} (T_a)_{ij} \tag{4.7}$$

and the non-abelian gluon field strength tensor the same way from Eq.(3.20),

$$F_{a,\mu\nu} = \partial_\mu A_{a,\nu} - \partial_\nu A_{a,\mu} - i g_s [A_\mu, A_\nu]_a \; , \tag{4.8}$$

With the non-abelian gauge group the field strength gives rise to the gluon propagator and to triple and quartic gluon self-interactions. We also see that the quark propagators are the same as for QED. The quark-quark-gluon interaction is very similar to the quark-quark-photon vertex in Eq.(1.43),

$$-i g_s \, \gamma^\mu \, (T_a)_{ij} \qquad\qquad (q_j - \bar{q}_i - g_{a,\mu}). \tag{4.9}$$

The only additional element is the $SU(3)$-matrix, and we already see that all color matrices in a scattering process will form traces along the quark line(s). The resulting color factors factorize from the Dirac algebra and can be computed independently.

Let us start with the electroweak processes

$$e^+e^- \to q\bar{q} \qquad \text{and} \qquad q\bar{q} \to e^+e^- \,, \tag{4.10}$$

described by the Feynman diagrams



No gluons appear at leading order, which means we do not have any matrices T^a in our calculations. Nevertheless, a color factor appears because we sum over the color states in the final state or average over the color states in the initial state. We can write this color sum for the squared matrix element formally as

$$\delta_{ij}\delta_{ji} = \delta_{ii} = N_c \tag{4.11}$$

add color factor for $q\bar{q} \to b\bar{b}$ Let us now radiate a gluon from either process



First, we have to deal with another gluon in the Dirac traces in complete analogy of radiating a photon. In addition, there is the color trace from the Feynman rule in Eq.(4.9), where we omit the trivial external delta distributions from Eq.(4.11)

$$(T_a)_{ij}(T_a)_{ji} = \mathrm{Tr}(T_aT_a) = \frac{1}{2}\delta_{aa} = \frac{N_c^2-1}{2} \,. \tag{4.12}$$

The color trace can be expressed in terms of the fundamental Casimir as

$$\mathrm{Tr}(T_aT_a) = N_cC_F \qquad \text{with} \qquad C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3} \,, \tag{4.13}$$

Another example is the successive radiation of two gluons from a hard quark line,

$$q \to qg_ag_b \,. \tag{4.14}$$

If we square the diagram without crossing gluon lines we find the planar color factor,

$$\begin{aligned}
\mathrm{Tr}(T^aT^aT^bT^b) &= (T^aT^a)_{il}(T^bT^b)_{li} = (T^a)_{ij}(T^a)_{jl}\,(T^b)_{lk}(T^b)_{ki} \\
&= \frac{1}{4}\left(\delta_{il}\delta_{jj} - \frac{\delta_{ij}\delta_{jl}}{N_c}\right)\left(\delta_{li}\delta_{kk} - \frac{\delta_{lk}\delta_{ki}}{N_c}\right) \\
&= \frac{1}{4}\delta_{il}\delta_{li}\left(N_c - \frac{1}{N_c}\right)^2 \\
&= N_c\left(\frac{N_c^2-1}{2N_c}\right)^2 = N_cC_F^2 = \frac{16}{3} = \mathcal{O}(N_c^3) \,.
\end{aligned} \tag{4.15}$$

If we cross the gluon lines we eventually find

$$\mathrm{Tr}(T_a T_b T_a T_b) = -\frac{C_F}{2} = -\frac{2}{3} = \mathcal{O}(N_c) \; . \tag{4.16}$$

That contribution is suppressed by a factor eight, which means that two gluons have a significant preference for ordered emission.

We can also compute the color factor for the purely gluonic theory, *i.e.* radiating gluons off two hard gluons in the final state. For instance, planar double gluon emission with the exchanged gluon indices b and f gives us the color factor

$$f^{abd} f^{abe} f^{dfg} f^{efg} = N_c \delta^{de} \; N_c \delta^{de} = \mathcal{O}(N_c^3) \; , \tag{4.17}$$

4.2 Ghosts

The main complication of QCD compared to QED comes from the kinetic term of the gluons. From Sec. 3.2 we remember that photons have only two transverse degree of freedom and we still write them as A_μ, effectively consisting of four degrees of freedom. How does QED ensure that in the actual calculation the longitudinal and the scalar degrees of freedom do not contribute?

To learn how to remove the unwanted degrees of freedom we start with the different gauge choices for the photon propagator in Eq.(3.46), but in the massless limit,

$$\Delta^{\mu\nu}(p) = \frac{-i}{p^2 + i\epsilon} \left[g^{\mu\nu} + (\xi - 1) \frac{p^\mu p^\nu}{p^2} \right] = \begin{cases} \dfrac{-i}{p^2 + i\epsilon} \, g^{\mu\nu} & \text{Feynman gauge } \xi = 1 \\ \dfrac{-i}{p^2 + i\epsilon} \left[g^{\mu\nu} - \dfrac{p^\mu p^\nu}{p^2} \right] & \text{Landau/Lorenz gauge } \xi = 0 \; . \end{cases} \tag{4.18}$$

The unitary gauge makes no sense for massless particles. We can also write the photon Lagrangian from Eq.(3.1) such that it gives us the ξ-dependent photon propagator. We only quote the result as

$$\mathscr{L}_{\text{photon, gf}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \; . \tag{4.19}$$

In analogy to Eq.(1.8) this Lagrangian gives us the equation of motion

$$\partial^\mu \partial_\mu A_\nu - \left(1 + \frac{1}{\xi} \right) \partial^\mu \partial_\nu A_\mu = 0 \; . \tag{4.20}$$

It is equivalent to the version without the so-called gauge fixing term, in that it still requires the d'Alembert equation for the photon and the Lorenz gauge condition. The parameter ξ defines the gauge choice for our theory description, where predicted observables cannot dependent on ξ.

Let us now see if the Lagrangian in Eq.(4.19) is or can be made invariant under local $U(1)$ transformations,

$$\begin{aligned} A^\mu &\to A^\mu - \frac{1}{e} \partial^\mu \alpha \\ -\frac{1}{2\xi} (\partial_\mu A^\mu)^2 &\to -\frac{1}{2\xi} \left(\partial_\mu A^\mu - \frac{1}{e} \partial^2 \alpha \right)^2 \approx -\frac{1}{2\xi} (\partial_\mu A^\mu)^2 + \frac{1}{e\xi} (\partial_\mu A^\mu)(\partial^2 \alpha) \; . \end{aligned} \tag{4.21}$$

In the last step we ignore higher powers of α and work with infinitesimal gauge transformations. To turn this Lagrangian $U(1)$-invariant we add an auxiliary field $\bar{c}$ in a specific way,

$$\mathscr{L}_{\text{photon-ghost}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 - \bar{c} \, \partial^2 \alpha \; . \tag{4.22}$$

The combination of the gauge fixing term and this new term is $U(1)$-invariant if

$$\bar{c} \to \bar{c} + \frac{1}{e\xi}(\partial_\mu A^\mu) \; . \tag{4.23}$$

To give a meaning to this auxiliary term we move one ∂_μ via integration by parts. Now the additional term in the Lagrangian makes sense if we promote α to another field, $\alpha \to c$ and combine $\bar{c}$ and c to a complex scalar,

$$\mathscr{L}_{\text{photon-ghost}} \to -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + (\partial_\mu \bar{c})\,(\partial^\mu c) \; . \tag{4.24}$$

This complex scalar is called a ghost, and it has to be computed in, both, Lorenz and Feynman gauge. However, what saves us in QED is that they do not appear elsewhere in the Lagrangian, so they are propagating but non-interacting fields and can be ignored for experimental predictions.

For QCD we can follow the same steps, but it turns out that we have to replace the standard derivative with a covariant derivative in the gauge transformation and also in the kinetic term for the ghosts. The gauge-fixed QCD Lagrangian with the compensating ghost fields then reads

$$\mathscr{L}_{\text{gluon-ghost}} = -\frac{1}{4}F_{a,\mu\nu}F_a^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A_a^\mu)^2 - (\partial^\mu \bar{c}_a)(\partial_\mu c_a) + g_s f_{abc} A_a^\mu (\partial_\mu \bar{c}_b) c_c \; . \tag{4.25}$$

The gluon propagator in the R_ξ gauge is the same as the photon propagator in Eq.(4.18), just with a factor δ_{ab}. The ghost propagator follows from Eq.(4.22)

$$-\frac{\delta_{ab}}{p^2 + i\epsilon} \tag{4.26}$$

and the ghost-ghost-gluon interaction

$$-igf_{abc}p_\mu \qquad\qquad (c - \bar{c} - g) \; , \tag{4.27}$$

where p_μ is the 4-momentum of the incoming ghost field.

Because we derived the ghosts through gauge invariance, we can at least for external gluons follow a slightly different direction to ensure the correct degrees of freedom contribute to the matrix element. For a matrix element with two external gluons $\mathcal{M}^{\mu\nu}$ the explicit gauge invariance condition is

$$p_\mu \mathcal{M}^{\mu\nu} = 0 = p_\nu \mathcal{M}^{\mu\nu} \; . \tag{4.28}$$

We can enforce it by projecting $\mathcal{M}^{\mu\nu}$ onto the allowed tensor structures. Following the same line of thought for propagators leads us back to Eq.(4.18), where we find for the so-called transverse tensor

$$T^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \qquad \Rightarrow \qquad p_\mu T^{\mu\nu} = 0 = p_\nu T^{\mu\nu} \; . \tag{4.29}$$

It is a projector on the transverse degrees of freedom because

$$\begin{aligned} T^{\mu\nu}T^\rho_\nu &= \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\left(g^\rho_\nu - \frac{p_\nu p^\rho}{p^2}\right) \\ &= g^{\mu\rho} - \frac{p^\mu p^\rho}{p^2} - \frac{p^\rho p^\mu}{p^2} + \frac{p^2 p^\mu p^\rho}{p^4} = T^{\mu\rho} \; . \end{aligned} \tag{4.30}$$

This makes is look like the propagator in Lorenz gauge is transverse in the covariant sense. However, for calculations even the Lorenz gauge does not ensure the correct gluon polarization, we always have to include the ghosts explicitly to account for the correct degrees of freedom.

4.3 Ultraviolet divergences

Renormalization as the proper treatment of ultraviolet divergences is a crucial concept in quantum field theory. Ultraviolet divergences appear in loop calculations, and we first regularize them, *i.e.* describe in a well-defined manner, and then renormalize them away through counter terms. This leads to the appearance of the renormalization scale.

Scales automatically arise from divergences. We can see this by writing down a simple scalar loop integral, with virtual scalar propagators with mass m and an external momentum p flowing through a diagram,

$$B(p^2;m,m) \equiv \int \frac{d^4q}{16\pi^2} \, \frac{1}{q^2-m^2} \frac{1}{(q+p)^2-m^2} \, . \tag{4.31}$$

Such two-point functions appear for example in the gluon self energy with virtual gluons, ghosts, and quarks. The integration measure $1/(16\pi^2)$ is dictated by the Feynman rule for the integration over loop momenta. Counting powers of q, we see that the integral behaves like

$$B(p^2;m,m)\Big|_{\text{UV}} \sim \frac{1}{16\pi^2} \int \frac{d^4q}{q^4} \tag{4.32}$$

in the ultraviolet, so it is logarithmically divergent. One regularization scheme is a momentum cutoff Λ, for example through the so-called Pauli—Villars regularization. Because the ultraviolet behavior of the integrand or integral cannot depend on any parameter living at low energy scales, like m or p^2. The scalar two-point function has mass dimension zero, so its divergence has to be proportional to a log-ratio with a new scale,

$$B(p^2;m,m) \sim \frac{1}{16\pi^2} \int^\Lambda \frac{d^4q}{q^4} \propto \log \frac{\Lambda}{\mu_R} \, . \tag{4.33}$$

This scale is an artifact of the regularization, so it has to eventually vanish from our theory prediction.

A more elegant regularization scheme is dimensional regularization. It is designed not to break gauge invariance and naively seems to not introduce a mass scale μ_R. When we shift the momentum integration from 4 to $4-2\epsilon$ dimensions and use analytic continuation in the number of space–time dimensions to renormalize the theory, a renormalization scale μ_R appears when we ensure the two-point function and with it observables like cross sections keep their correct mass dimension

$$\begin{aligned} \int \frac{d^4q}{16\pi^2} \, \frac{1}{q^2-m^2} \frac{1}{(q+p)^2-m^2} &\to \mu_R^{2\epsilon} \int \frac{d^{4-2\epsilon}q}{16\pi^2} \, \frac{1}{q^2-m^2} \frac{1}{(q+p)^2-m^2} \\ &= \frac{i\mu_R^{2\epsilon}}{(4\pi)^2} \left[\frac{C_{-1}}{\epsilon} + C_0 + C_1 \, \epsilon + \mathcal{O}(\epsilon^2) \right] \, . \end{aligned} \tag{4.34}$$

The constants C_i in the series in $1/\epsilon$ depend on the loop integral. Defining scalar integrals with the integration measure $1/(i\pi^2)$ will make for example C_{-1} come out as of the order $\mathcal{O}(1)$. This is the reason we usually find factors $1/(4\pi)^2 = \pi^2/(2\pi)^4$ in front of loop integrals.

We can keep track of the renormalization scale best by expanding the prefactor of the regularized but not yet renormalized integral in Eq.(4.34) in a Taylor series in ϵ, no question asked about convergence radii

$$\begin{aligned} \mu_R^{2\epsilon} \left[\frac{C_{-1}}{\epsilon} + C_0 + \mathcal{O}(\epsilon) \right] &= e^{2\epsilon \log \mu_R} \left[\frac{C_{-1}}{\epsilon} + C_0 + \mathcal{O}(\epsilon) \right] \\ &= \left[1 + 2\epsilon \log \mu_R + \mathcal{O}(\epsilon^2) \right] \left[\frac{C_{-1}}{\epsilon} + C_0 + \mathcal{O}(\epsilon) \right] \\ &= \frac{C_{-1}}{\epsilon} + C_0 + C_{-1} \, \log \mu_R^2 + \mathcal{O}(\epsilon) \\ &\to \frac{C_{-1}}{\epsilon} + C_0 + C_{-1} \, \log \frac{\mu_R^2}{M^2} + \mathcal{O}(\epsilon) \, . \end{aligned} \tag{4.35}$$

In the last step we correct by hand for the fact that $\log \mu_R^2$ with a mass dimension inside the logarithm cannot appear in our calculations. From somewhere else in our calculation the logarithm will be matched with a $\log M^2$, such that M^2 is the typical mass or energy scale in our process. Carefully counting, there are three scales we need to consider:

1. the physical scale in the process, like the top or Higgs mass;
2. the renormalization scale μ_R, a reference scale as part of the definition of the counter term;
3. The scale M separating the counter term from the process dependent result. The role of M will become clear when look at the running strong coupling α_s.

We would prefer to choose all three scales the same, but in a complex physical process this will not be possible.

4.4 Counter terms

The ultraviolet poles $1/\epsilon$ will cancel with universal counter terms once we renormalize the theory. We include counter terms by shifting parameters in the Lagrangian and the leading order matrix element,

$$\begin{aligned} |\mathcal{M}_{\text{LO}}(g) + \mathcal{M}_{\text{virt}}|^2 &= |\mathcal{M}_{\text{LO}}(g)|^2 + 2\,\text{Re}\,\mathcal{M}_{\text{LO}}(g)\mathcal{M}_{\text{virt}} + \cdots \\ &\to |\mathcal{M}_{\text{LO}}(g+\delta g)|^2 + 2\,\text{Re}\,\mathcal{M}_{\text{LO}}(g)\mathcal{M}_{\text{virt}} + \cdots \\ \text{with} \qquad g \to g^{\text{bare}} &= g + \delta g \qquad \text{and} \quad \delta g \propto \frac{\alpha_s}{\epsilon}\,. \end{aligned} \tag{4.36}$$

The counter terms do not come with a factor $\mu_R^{2\epsilon}$, so this factor will not be matched between the actual ultraviolet divergence and the counter term.

Counter terms are not uniquely defined. They remove a divergence to return finite observables, but we are free to add any finite contribution we want. We can, define a counter term based on physical processes, to only cancel the pole but also finite contributions at a given order in perturbation theory. An example is the on–shell scheme for masses, where we define a counter term such that external on–shell particles do not receive any corrections to their masses. For the top mass this means

$$\begin{aligned} m_t^{\text{bare}} &= m_t + \delta m_t \\ &= m_t + m_t \frac{\alpha_s C_F}{4\pi}\left[3\left(-\frac{1}{\epsilon} + \gamma_E - \log(4\pi) - \log\frac{\mu_R^2}{M^2}\right) - 4 + 3\log\frac{m_t^2}{M^2}\right] \\ &\equiv m_t + m_t \frac{\alpha_s C_F}{4\pi}\left(-\frac{3}{\tilde{\epsilon}\left(\frac{\mu_R^2}{M^2}\right)} - 4 + 3\log\frac{m_t^2}{M^2}\right) \qquad \text{with} \qquad \frac{1}{\tilde{\epsilon}\left(\frac{\mu_R^2}{M^2}\right)} \equiv \frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu_R^2}{M^2}\,. \end{aligned} \tag{4.37}$$

The Euler constant $\gamma_E \approx 0.577$ comes from the Gamma function $\Gamma(\epsilon) = 1/\epsilon + \gamma_E + \mathcal{O}(\epsilon)$. The convenient scale dependent pole $1/\tilde{\epsilon}$ includes the universal additional terms like the Euler gamma function and the scaling logarithm. This logarithm is a result of our universality argument, forcing us to introduce an energy scale M to separate the universal logarithm of the renormalization scale and the parameter-dependent logarithm of the physical process.

One common feature of all mass counter terms is $\delta m \propto m$, which means that our renormalization is multiplicative,

$$m^{\text{bare}} = Z_m\, m = (1 + \delta Z_m)\, m = \left(1 + \frac{\delta m}{m}\right) m = m + \delta m \qquad \text{with} \qquad \delta Z_m = \frac{\delta m}{m}\,, \tag{4.38}$$

This form implies that particles with zero mass will not obtain a finite mass through renormalization. If we remember that chiral symmetry protects a Lagrangian from acquiring fermion masses this means that on–shell renormalization does not break this symmetry. A massless theory cannot become massive by mass renormalization. Regularization and renormalization schemes which do not break symmetries of the Lagrangian are ideal.

Another way of introducing counter terms is by defining a renormalization point. This can be the energy scale at which the counter terms cancels all higher order contributions, divergent as well as finite. The best example is the electric charge which we renormalize in the Thomson limit of zero momentum transfer through the photon propagator

$$e \to e^{\text{bare}} = e + \delta e \; . \tag{4.39}$$

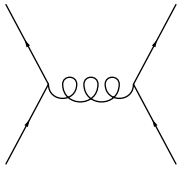
Finally, we can define a general counter term: if dimensional regularization does not break any of the symmetries of our Lagrangian, we can just subtract the ultraviolet pole. The only question is: do we subtract $1/\epsilon$ in the MS scheme or $1/\tilde{\epsilon}$ in the $\overline{\text{MS}}$ scheme. In that case the counter term becomes scale dependent.

4.5 Running coupling

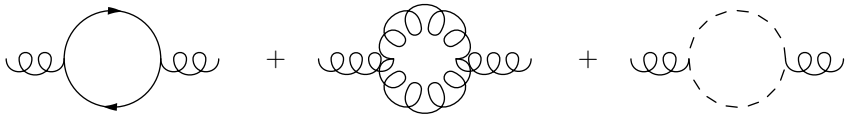
To get an idea what these different scales mean we compute the running strong coupling $\alpha_s(\mu_R^2)$. A simple process where we can study it is bottom pair production, where at some energy range we will be dominated by valence quarks

$$q\bar{q} \to b\bar{b} \; . \tag{4.40}$$

The only Feynman diagram is an s-channel off–shell gluon with momentum $p^2 \equiv s$,



At next–to–leading order this gluon propagator will be corrected by self energy loops. aThe Feynman diagrams include a quark loop, a gluon loop, and the ghost loop which removes the unphysical degrees of freedom of the gluon:



All fermion lines close in the Feynman diagram and the Dirac trace is computed inside the loop. In color space the self energy will (hopefully) be diagonal, just like the gluon propagator itself, so we can ignore the color indices for now. In Lorenz gauge the gluon propagator is proportional to the transverse tensor defined in Eq.(4.29). The same should be true for the gluon self energy, which we therefore write as $\Pi^{\mu\nu} \equiv \Pi \, T^{\mu\nu}$. The result for the regularized gluon self energy with momentum flow p^2 through the propagator reads

$$\begin{aligned}
-\frac{1}{p^2}\,\Pi\left(\frac{\mu_R^2}{p^2}\right) &= \frac{\alpha_s}{4\pi}\,\left(-\frac{1}{\tilde{\epsilon}\left(\frac{\mu_R^2}{M^2}\right)} + \log\frac{p^2}{M^2}\right)\,\left(\frac{13}{6}N_c - \frac{2}{3}n_f\right) \\
&\equiv \alpha_s\,\left(-\frac{1}{\tilde{\epsilon}\left(\frac{\mu_R^2}{M^2}\right)} + \log\frac{p^2}{M^2}\right)\,b_0 \\
&\text{with} \qquad b_0 = \frac{1}{4\pi}\left(\frac{13}{6}N_c - \frac{2}{3}n_f\right) \\
&\text{but really} \qquad b_0 = \frac{1}{4\pi}\left(\frac{11}{3}N_c - \frac{2}{3}n_f\right) \overset{\text{SM}}{>} 0 \; .
\end{aligned} \tag{4.41}$$

The number of fermions coupling to the gluons is n_f. The factor b_0 reflects the one-loop corrections. Strictly speaking, it gives the first term in a perturbative series in the strong coupling $\alpha_s = g_s^2/(4\pi)$.

In this form we the $\log M^2$ from the definition of $\tilde{\epsilon}$ in Eq.(4.37) remains. While we know at least one physical scale p^2, we have not identified it with M^2, because the self-energy diagrams are always embedded in Feynman diagrams with additional scales, like for example particle masses.

In the last step of Eq.(4.41) we have snuck in additional contributions by replacing the factor 13/6 by a factor 11/3. This is related to the fact that there are actually three types of divergent virtual gluon diagrams in the physical process $q\bar{q} \to b\bar{b}$: the external quark self energies with renormalization factors $Z_f^{1/2}$, the internal gluon self energy Z_A, and the vertex corrections Z_{Aff}. The physical parameters we can renormalize in this process are the strong coupling and the quark masses. Wave function renormalization constants are not physical. The entire divergence which eventually needs to be absorbed in Z_g is given by the combination

$$Z_{Aff} = Z_g Z_A^{1/2} Z_f \qquad \Leftrightarrow \qquad \frac{Z_{Aff}}{Z_A^{1/2} Z_f} \equiv Z_g \; . \tag{4.42}$$

This changes the factor from 13/6 to 11/3 in the running of the strong coupling.

We can check this definition of Z_g by comparing all vertices in which the strong coupling g_s appears, namely the gluon coupling to quarks and ghosts, as well as the triple and quartic gluon vertex. All of them need to have the same divergence structure

$$\frac{Z_{Aff}}{Z_A^{1/2} Z_f} \stackrel{!}{=} \frac{Z_{Acc}}{Z_A^{1/2} Z_c} \stackrel{!}{=} \frac{Z_{3A}}{Z_A^{3/2}} \stackrel{!}{=} \sqrt{\frac{Z_{4A}}{Z_A^2}} \; . \tag{4.43}$$

If we had done the same calculation in QED and looked for a running electric charge, we would have found that the vacuum polarization diagrams for the photon do account for the entire counter term of the electric charge. The other two renormalization constants Z_{Aff} and Z_f cancel because of gauge invariance.

Unlike QED, the strong coupling diverges in the low-energy Thomson limit because QCD is confined towards large distances and weakly coupled at small distances. Lacking a good reference point we renormalize $\alpha_s = g_s^2/(4\pi)$ in the $\overline{\text{MS}}$ scheme,

$$\alpha_s^{\text{bare}} = \left(1 + \frac{\delta\alpha_s}{\alpha_s}\right) \alpha_s \stackrel{\overline{\text{MS}}}{=} \left(1 - \frac{\Pi}{p^2}\bigg|_{\text{pole}}\right) \alpha_s(M^2) \; . \tag{4.44}$$

We have already written Eq.(4.41) in terms of $1/\tilde{\epsilon}$, so we can read off the $\overline{\text{MS}}$ counter term at the physical scale M^2,

$$\alpha_s^{\text{bare}} = \left(1 - \frac{\alpha_s b_0}{\tilde{\epsilon}\left(\frac{\mu_R^2}{M^2}\right)}\right) \alpha_s(M^2) \; . \tag{4.45}$$

As mentioned above, we could have assumed that there is only one physical scale and evaluated the strong coupling in Eq.(4.41) at the scale p^2,

$$\alpha_s^{\text{bare}} \equiv \alpha_s(p^2) \left(1 - \frac{\alpha_s b_0}{\tilde{\epsilon}\left(\frac{\mu_R^2}{p^2}\right)} + \alpha_s b_0 \log\frac{p^2}{M^2}\right) \; . \tag{4.46}$$

For the appearances of α_s in the parenthesis we skip the argument, because its effect is of higher order in pertrubation theory. Identifying Eqs.(4.45) and (4.46) and dropping the $1/\tilde{\epsilon}$ term which always just cancels the explicit pole we find for the finite terms

$$\alpha_s(M^2) = \alpha_s(p^2) \left(1 + \alpha_s b_0 \log\frac{p^2}{M^2}\right) \; . \tag{4.47}$$

This defines a running of the strong coupling between one physical scale $\alpha_s(p^2)$ and another general scale $\alpha_s(M^2)$.

4.6 Resummation

For the loop contributions from the gluon self enegry we can do better than fixed order in perturbation theory: instead of simply including the gluon self energy bubble at a given order in perturbation theory we can include chains of one-loop diagrams with Π appearing many times in the gluon propagator. It means we replace the off–shell gluon propagator by

$$\begin{aligned}\frac{T^{\mu\nu}}{p^2} \to & \frac{T^{\mu\nu}}{p^2} + \left(\frac{T}{p^2}\cdot(-T\,\Pi)\cdot\frac{T}{p^2}\right)^{\mu\nu} \\ & + \left(\frac{T}{p^2}\cdot(-T\,\Pi)\cdot\frac{T}{p^2}\cdot(-T\,\Pi)\cdot\frac{T}{p^2}\right)^{\mu\nu} + \cdots \\ = & \frac{T^{\mu\nu}}{p^2}\sum_{j=0}^{\infty}\left(-\frac{\Pi}{p^2}\right)^j = \frac{T^{\mu\nu}}{p^2}\,\frac{1}{1+\dfrac{\Pi}{p^2}}\,, \end{aligned} \tag{4.48}$$

schematically written without the factors i. To avoid indices we abbreviate $T^{\mu\nu}T^{\rho}_{\nu} = T\cdot T$ and

$$(T\cdot T\cdot T)^{\mu\nu} = T^{\mu\rho}T^{\sigma}_{\rho}T^{\nu}_{\sigma} = T^{\mu\sigma}T^{\nu}_{\sigma} = T^{\mu\nu}\,. \tag{4.49}$$

This (re-)summation moves the finite shift in α_s from Eqs.(4.41) and (4.46) into the denominator. The identification of the finite terms in Eq.(4.47) now gives us

$$\begin{aligned}& \alpha_s(M^2) \equiv \alpha_s(p^2)\frac{1}{1-\alpha_s b_0\,\log\dfrac{p^2}{M^2}} \\ \Leftrightarrow \qquad & \frac{1}{\alpha_s(M^2)} = \frac{1}{\alpha_s(p^2)}\left(1-\alpha_s\,b_0\,\log\frac{p^2}{M^2}\right) = \frac{1}{\alpha_s(p^2)} - b_0\,\log\frac{p^2}{M^2}\,. \end{aligned} \tag{4.50}$$

If we taylor the denominator we see that the leading terms in α_s have not changed. The additional higher-order terms define a resummed evolution of physical parameters from one scale to another.

If we are interested in the running of the strong coupling as a function of the relevant scale, p^2, we can write

$$\frac{1}{\alpha_s(p^2)} = \frac{1}{\alpha_s(M^2)}\left(1+\alpha_s(M^2)b_0\log\frac{p^2}{M^2}\right)\,. \tag{4.51}$$

Differentiating it with respect to p^2 we find with $d/dx(1/\alpha_s) = -1/\alpha_s^2\; d\alpha_s/dx$

$$\begin{aligned} p^2\frac{d\alpha_s(p^2)}{dp^2} = \frac{d\alpha_s(p^2)}{d\log p^2} &= -\alpha_s^2\frac{d}{d\log p^2}\,\frac{1}{\alpha_s(p^2)} \\ &= -\alpha_s^2\frac{d}{d\log p^2}\,\left[\frac{1}{\alpha_s(M^2)} + b_0\log\frac{p^2}{M^2}\right] \\ &= -\alpha_s^2\,b_0 + \mathcal{O}(\alpha_s^2) \\ &\to -\alpha_s^2\sum_{n=0} b_n\alpha_s^n \equiv \beta\,. \end{aligned} \tag{4.52}$$

This is the renormalization group equation (RGE) describing the running of the strong coupling constant including all higher order terms b_n. From Eq.(4.41) we know that $b_0 > 0$, which means that towards larger scales the strong coupling has a negative slope. The ultraviolet limit of the strong coupling is zero and QCD is asymptotically free, while in the infrared it hits a Landau pole.

Going back to Eq.(4.51) the term in parenthesis can become zero at $p^2 \ll M^2$ for $b_0 > 0$. This defines a so-called Landau pole defining a small energy scale at which the strong coupling diverges and our theory switches degrees of

freedom from quarks and gluon to mesons and baryons. We turn this problem into a feature by using the Landau pole as a reference scale Λ_{QCD} at a given loop order,

$$\begin{aligned} 1 + \alpha_s(M^2)\, b_0\, \log \frac{\Lambda^2_{\text{QCD}}}{M^2} \stackrel{!}{=} 0 \quad &\Leftrightarrow \quad \log \frac{\Lambda^2_{\text{QCD}}}{M^2} = -\frac{1}{\alpha_s(M^2) b_0} \\ &\Leftrightarrow \quad \log \frac{p^2}{M^2} = \log \frac{p^2}{\Lambda^2_{\text{QCD}}} - \frac{1}{\alpha_s(M^2) b_0} \; . \end{aligned} \tag{4.53}$$

We can use it in the running as

$$\begin{aligned} \frac{1}{\alpha_s(p^2)} &\stackrel{\text{Eq.(4.51)}}{=} \frac{1}{\alpha_s(M^2)} + b_0\, \log \frac{p^2}{M^2} \\ &\stackrel{\text{Eq.(4.53)}}{=} \frac{1}{\alpha_s(M^2)} + b_0 \log \frac{p^2}{\Lambda^2_{\text{QCD}}} - \frac{1}{\alpha_s(M^2)} \qquad \Leftrightarrow \qquad \alpha_s(p^2) = \frac{1}{b_0 \log \dfrac{p^2}{\Lambda^2_{\text{QCD}}}} \; . \end{aligned} \tag{4.54}$$

An interesting aspect of Λ_{QCD} is that we introduce a scale into our theory without ever setting it. All we do is renormalize a coupling which becomes strong at large energies and search for the mass scale of this strong interaction. This trick is called dimensional transmutation.

4.7 Resumming scaling logarithms

One question we still need to answer is — how does the resummation of the one-loop diagrams for the s-channel gluon improve our prediction of the LHC observables? As an illustration we need a simple observable which depends on just one energy scale p^2. The first observable coming to mind is again the Drell–Yan cross section $\sigma(q\bar{q} \to \mu^+\mu^-)$, but since we are not really sure what to do with the parton densities we resort to simpler e^+e^- collisions. A simple observable which includes α_s at least in the one-loop corrections is

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_{\text{quarks}} Q_q^2 = \frac{11 N_c}{9} \; . \tag{4.55}$$

Including higher order corrections we can express the result in a power series in α_s,

$$R\left(\frac{p^2}{M^2}, \alpha_s\right) = \sum_{n=0} r_n \left(\frac{p^2}{M^2}\right)\, \alpha_s^n(M^2) \qquad \text{with} \qquad r_0 = \frac{11 N_c}{9} \; . \tag{4.56}$$

The $\overline{\text{MS}}$ scheme introduces an unphysical dependence on M in the individual r_n. We assume the r_n to be dimensionless, such that they depend on the ratio of the external p^2 and the artificial M^2.

As an observable, R cannot depend on an artificial scale M once we include all orders in perturbation theory. Writing this dependence as a total derivative and setting it to zero we find an equation which would be called a Callan–Symanzik equation if instead of the running coupling we had included a running mass

$$\begin{aligned} 0 &\stackrel{!}{=} M^2 \frac{d}{dM^2} R\left(\frac{p^2}{M^2}, \alpha_s(M^2)\right) \\ &= \left[M^2 \frac{\partial}{\partial M^2} + \beta\, \frac{\partial}{\partial \alpha_s}\right] \sum_{n=0} r_n \left(\frac{p^2}{M^2}\right)\, \alpha_s^n && \text{with} \quad \beta = M^2 \frac{\partial \alpha_s}{\partial M^2} \\ &= \sum_{n=1} M^2 \frac{\partial r_n}{\partial M^2}\, \alpha_s^n + \sum_{n=1} \beta\, r_n\, n \alpha_s^{n-1} && \text{with} \quad r_0 = \frac{11 N_c}{9} = \text{const} \\ &= M^2 \sum_{n=1} \frac{\partial r_n}{\partial M^2}\, \alpha_s^n - \sum_{n=1} \sum_{m=0} n r_n\, \alpha_s^{n+m+1}\, b_m && \text{with} \quad \beta = -\alpha_s^2 \sum_{m=0} b_m \alpha_s^m \\ &= M^2 \frac{\partial r_1}{\partial M^2}\, \alpha_s + \left(M^2 \frac{\partial r_2}{\partial M^2} - r_1 b_0\right)\, \alpha_s^2 + \left(M^2 \frac{\partial r_3}{\partial M^2} - 2 r_2 b_0 - r_1 b_1\right)\, \alpha_s^3 \quad + \mathcal{O}(\alpha_s^4) \; . \end{aligned} \tag{4.57}$$

This perturbative series in α_s has to vanish at any order of perturbation theory,

$$\begin{aligned}
\frac{\partial r_1}{\partial \log M^2} &= 0 \\
\frac{\partial r_2}{\partial \log M^2} &= r_1 b_0 \\
\frac{\partial r_3}{\partial \log M^2} &= r_1 b_1 + 2 r_2(M^2) b_0 \\
&\vdots
\end{aligned} \tag{4.58}$$

The mix of r_n derivatives and the perturbative terms in the β function can be seen at order α_s^3: first, we have the appropriate NNNLO corrections r_3; next, we have one loop in the gluon propagator b_0 and two loops for example in the vertex r_2; and finally, we need the two-loop diagram for the gluon propagator b_1 and a one-loop vertex correction r_1.

The M^2-dependence vanishes for r_0 and r_1. Keeping in mind the integration constants c_n we find the solutions

$$\begin{aligned}
r_0 &= c_0 = \frac{11 N_c}{9} \\
r_1 &= c_1 \\
r_2 &= c_2 + r_1 b_0 \log \frac{M^2}{p^2} = c_2 + c_1 b_0 \log \frac{M^2}{p^2} \\
r_3 &= \int d \log \frac{M'^2}{p^2} \left(c_1 b_1 + 2 \left(c_2 + c_1 b_0 \log \frac{M'^2}{p^2} \right) b_0 \right) = c_3 + (c_1 b_1 + 2 c_2 b_0) \log \frac{M^2}{p^2} + c_1 b_0^2 \log^2 \frac{M^2}{p^2} \\
&\vdots
\end{aligned} \tag{4.59}$$

This form for the r_n values suggests to interpret the fixed-order perturbative series in Eq.(4.56) as implicitly including terms $\log^{n-1} M^2/p^2$ in each r_n. They become problematic if a large logarithm spoils the convergence in terms of $\alpha_s \sim 0.1$, *i.e.* measuring R at scales p^2 far away from the scale choice for the strong coupling constant, M^2.

However, we can also use Eq.(4.59) to express R in terms of the c_n,

$$\begin{aligned}
R = \sum_n r_n \left(\frac{p^2}{M^2} \right) \alpha_s^n(M^2) = c_0 &+ c_1 \left(1 + \alpha_s(M^2) b_0 \log \frac{M^2}{p^2} + \alpha_s^2(M^2) b_0^2 \log^2 \frac{M^2}{p^2} + \cdots \right) \alpha_s(M^2) \\
&+ c_2 \left(1 + 2 \alpha_s(M^2) b_0 \log \frac{M^2}{p^2} + \cdots \right) \alpha_s^2(M^2) + \cdots
\end{aligned} \tag{4.60}$$

and resum this geometric series to

$$R = c_0 + c_1 \frac{\alpha_s(M^2)}{1 - \alpha_s(M^2) b_0 \log \frac{M^2}{p^2}} + c_2 \left(\frac{\alpha_s(M^2)}{1 - \alpha_s(M^2) b_0 \log \frac{M^2}{p^2}} \right)^2 + \cdots \equiv \sum c_n \, \alpha_s^n(p^2) \, . \tag{4.61}$$

In the last step we use Eq.(4.50) with flipped arguments p^2 and M^2, derived from the resummation of the vacuum polarization bubbles. In contrast to the r_n, the c_n are just numbers and therefore more suitable as a perturbative series. The re-organization of the perturbation series for R can be interpreted as resumming all logarithms of the kind $\log M^2/p^2$ in a new organization of the perturbative series.

Some of the c_n are known, for example inserting $N_c = 3$ and five quark flavors just as we assume in Eq.(4.55)

$$R = \frac{11}{3} \left(1 + \frac{\alpha_s(p^2)}{\pi} + 1.4 \left(\frac{\alpha_s(p^2)}{\pi} \right)^2 - 12 \left(\frac{\alpha_s(p^2)}{\pi} \right)^3 + \mathcal{O}\left(\frac{\alpha_s(p^2)}{\pi} \right)^4 \right) \, . \tag{4.62}$$

This alternating series with increasing perturbative prefactors indicates the asymptotic instead of convergent behavior of perturbative QCD. At the bottom mass scale the relevant coupling factor is only $\alpha_s(m_b^2)/\pi \sim 1/14$, so a further increase of the c_n would become dangerous. However, a detailed look into the calculation shows that the dominant contributions to c_n arise from the analytic continuation of logarithms, which are large finite terms for example from $\text{Re}(\log^2(-E^2)) = \log^2 E^2 + \pi^2$. In the literature such π^2 terms arising from the analytic continuation of loop integrals are often phrased in terms of $\zeta_2 = \pi^2/6$.

Before moving on we collect the logic of the argument given in this section: when we regularize an ultraviolet divergence we automatically introduce a reference scale μ_R. Naively, this could be an ultraviolet cutoff scale, but even the seemingly scale invariant dimensional regularization in the conformal limit of our field theory cannot avoid the introduction of a scale. There are several ways of dealing with such a scale: first, we can renormalize our parameter at a reference point. Secondly, we can define a running parameter and this way absorb the scale logarithm into the $\overline{\text{MS}}$ counter term. For the asymptotically free strong coupling Λ_{QCD} leaves us with a compact form of the running coupling $\alpha_s(M^2)$.

Strictly speaking, at each order in perturbation theory the scale dependence should vanish together with the ultraviolet poles, as long as there is only one scale affecting a given observable. However, defining the running strong coupling we sum one-loop vacuum polarization graphs. Even when we compute an observable at a given loop order, we implicitly include higher order contributions. They lead to a dependence of our perturbative result on the artificial scale M^2, which phenomenologists refer to as renormalization scale dependence.

Using the R ratio we see what our definition of the running coupling means in terms of resumming logarithms: reorganizing our perturbative series to get rid of the ultraviolet divergence $\alpha_s(p^2)$ resums the scale logarithms $\log p^2/M^2$ to all orders in perturbation theory.

5 Partons and DGLAP equation

During our introduction to QCD and the running coupling we focused on the ultraviolet or high-energy behavior of the theory and noticed how using a running coupling resums potentially large logarithms. In this section we will follow a similar approach to the infrared behavior, where we will find that collinear divergences lead to the DGLAP equations for parton densities, which resum collinear logarithms.

5.1 Incoming partons

To describe scattering processes at hadron colliders, we need to introduce parton distribution functions to describe the probability of finding a collinear parton with momentum fraction x in a proton. Such a pdf is not an observable, but a distribution in the mathematical sense: it has to produce reasonable results when we integrate it with a differential partonic cross section as the test function. Using parton densities we compute a hadronic cross section as

$$\boxed{\sigma_{\text{tot}} = \int_0^1 dx_1 \int_0^1 dx_2 \sum_{ij} f_i(x_1)\, f_j(x_2)\, \hat{\sigma}_{ij}(x_1 x_2 S)} \,, \tag{5.1}$$

where i, j are the incoming partons with the momentum factions $x_{i,j}$. The partonic energy of the scattering process is $s = x_1 x_2 S$ with the LHC proton energy of $\sqrt{S} = 13.6$ TeV. The partonic cross section $\hat{\sigma}$ includes all θ and δ functions needed for energy–momentum conservation. When we express an n–particle cross section $\hat{\sigma}$ including its phase space integration, the x_i integrations and the phase space integrations can be exchanged, but Jacobians will make life hard.

Different parton densities of the proton have very different behavior — for the valence quarks (uud) they peak below $x < 1/3$, while the gluon pdf is small at $x \sim 1$ and grows rapidly towards small x. For typical LHC kinematics, the gluon density scales roughly like

$$f_g(x) \propto \frac{1}{x^2} \qquad \text{for} \qquad x = 10^{-3} \cdots 10^{-1} \,. \tag{5.2}$$

We cannot compute parton distribution functions $f_i(x)$ as a function of the longitudinal momentum fraction x, but there are a few predictions we can make based on symmetries and properties of the hadrons, leading to sum rules:

1. The antiproton and proton parton densities are related through the CP-symmetry, which is exact for QCD,

$$f_q^{\bar{p}}(x) = f_{\bar{q}}(x) \qquad\qquad f_{\bar{q}}^{\bar{p}}(x) = f_q(x) \qquad\qquad f_g^{\bar{p}}(x) = f_g(x) \,. \tag{5.3}$$

2. The proton consists of three valence quarks uud, plus quantum fluctuations leading to gluons or quark–antiquark pairs. This sea-quark contribution is symmetric between quarks and antiquarks, so the expectation values for the up and down quark densities in a proton have to fulfill

$$\langle N_u \rangle = \int_0^1 dx \;\; (f_u(x) - f_{\bar{u}}(x)) = 2 \qquad\qquad \langle N_d \rangle = \int_0^1 dx \;\; (f_d(x) - f_{\bar{d}}(x)) = 1 \,. \tag{5.4}$$

3. The expectation value for the proton momentum is split between all parton,

$$\left\langle \sum x_i \right\rangle = \int_0^1 dx \; x \left(\sum_q f_q(x) + \sum_{\bar{q}} f_{\bar{q}}(x) + f_g(x) \right) = 1 \,. \tag{5.5}$$

   If we only take into account the measured quark and antiquark parton densities, we find

$$\int_0^1 dx \; x \left(\sum_q f_q(x) + \sum_{\bar{q}} f_{\bar{q}}(x) \right) \approx \frac{1}{2} \,, \tag{5.6}$$

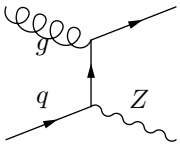
   so half of the proton momentum is then carried by quantum-fluctuation gluons.

5.2 Infrared divergences

As the simplest LHC process, let us look at the Drell–Yan process to next-to-leading order or $\sigma_{\text{tot}} \propto G_F \alpha_s$,

$$q\bar{q} \to Zg \; . \tag{5.7}$$

Beyond leading order, processes involve renormalization of ultraviolet divergences as well as loop diagrams, which we have to include before we can say anything reasonable, *i.e.* ultraviolet and infrared finite. To make life easy we look at the crossed NLO contribution



It behaves like a generic other $(2 \to 2)$ process. Because it has a different incoming state than the leading order process, it does not involve virtual corrections or renormalization. Modulo prefactors its amplitude is

$$|\mathcal{M}|^2 \sim -\frac{t}{s} - \frac{s^2 - 2m_Z^2(s+t-m_Z^2)}{st} \; . \tag{5.8}$$

The Mandelstam variable t for one massless final–state particle can be expressed in terms of the scattering angle

$$t = -s(1-\tau)y \qquad \text{with} \qquad y = \frac{1-\cos\theta}{2} \in [0,1] \qquad \text{and} \qquad \tau = \frac{m_Z^2}{s} < 1 \; . \tag{5.9}$$

Similarly, we obtain $u = -s(1-\tau)(1-y)$, so we can confirm that $t + u = -s(1-\tau) = -s + m_Z^2$. The collinear limit, when the gluon splits into a quark-antiquark pair in the beam direction, is given by

$$\begin{aligned} & y \to 0 \qquad \Leftrightarrow \qquad t \to 0 \qquad \Leftrightarrow \qquad u = -s + m_Z^2 < 0 \\ \Rightarrow \qquad & |\mathcal{M}|^2 \to \frac{s^2 - 2m_Z^2 s + 2m_Z^4}{s(s-m_Z^2)y} + \mathcal{O}(y^0) \; . \end{aligned} \tag{5.10}$$

The cross section diverges for small angles y. We can translate this $1/y$ divergence into the transverse momentum of the gluon or Z using

$$p_T^2 = \frac{tu}{s} = s(1-\tau)^2\, y(1-y) = \frac{(s-m_Z^2)^2}{s} y + \mathcal{O}(y^2) \tag{5.11}$$

In terms of p_T, the collinear limit of Eq.(5.10) reads

$$|\mathcal{M}|^2 \sim \frac{s^2 - 2sm_Z^2 + 2m_Z^4}{s^2} \; \frac{s-m_Z^2}{p_T^2} + \mathcal{O}(p_T^0) \; . \tag{5.12}$$

It has a leading divergence proportional to $1/p_T^2$. To compute the total cross section we need to integrate the matrix element over the two-particle phase space. Approximating the matrix element as C'/y or C/p_T^2 this gives us

$$\begin{aligned} \sigma_{\text{tot}} &\sim \int_{y^{\text{min}}}^{y^{\text{max}}} dy \frac{C'}{y} \\ &= \int_{p_T^{\text{min}}}^{p_T^{\text{max}}} dp_T^2 \frac{C}{p_T^2} = 2\int_{p_T^{\text{min}}}^{p_T^{\text{max}}} dp_T \; p_T \; \frac{C}{p_T^2} \simeq \; 2C \int_{p_T^{\text{min}}}^{p_T^{\text{max}}} dp_T \frac{1}{p_T} = 2C \; \log \frac{p_T^{\text{max}}}{p_T^{\text{min}}} \end{aligned} \tag{5.13}$$

The approximate form C/p_T^2 is valid in the collinear limit; over the non–collinear phase space C is not constant. For this not integrable divergence we follow the same strategy as for the ultraviolet divergence: (i) we regularize it for example using dimensional regularization, and (ii) we find a well–defined way to get rid of it.

Dimensional regularization means writing the two-particle phase space in $n = 4 - 2\epsilon$ dimensions. Just for reference, the complete formula for the y-distribution reads

$$s\,\frac{d\sigma}{dy} = \frac{\pi(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)}\,\left(\frac{\mu_F^2}{m_Z^2}\right)^\epsilon\,\frac{\tau^\epsilon(1-\tau)^{1-2\epsilon}}{y^\epsilon(1-y)^\epsilon}\,|\mathcal{M}|^2 \sim \left(\frac{\mu_F^2}{m_Z^2}\right)^\epsilon\,\frac{|\mathcal{M}|^2}{y^\epsilon(1-y)^\epsilon}\;. \tag{5.14}$$

The factor $1/y^\epsilon$ regularizes the integral at $y \to 0$, as long as $\epsilon < 0$ slightly increases the suppression of the integrand in the infrared. Integrating the leading collinear divergence $1/y^{1+\epsilon}$ gives us a pole $1/(-\epsilon)$. This means that for infrared divergences we choose $n = 4 + 2|\epsilon|$ space–time dimensions.

As for the ultraviolet case, a scale $\mu_F^{2\epsilon}$ appears with the n-dimensional integral. When it arises from an infrared regularization we call it factorization scale. The actual removal of the infrared pole — corresponding to the renormalization in the ultraviolet case — is called mass factorization and works exactly the same way as renormalizing a parameter: in a well–defined scheme we subtract the pole from the fixed-order result.

5.3 Parton splitting

Let us understand the patterns behind infrared divergences for massless quarks and gluons in the initial or final state. At the LHC, the most important case is the splitting of one gluon into two,

$$g(p_a) \to g(p_b) + g(p_c) \qquad \text{with} \qquad p_a^2 \equiv (-p_b - p_c)^2 \gg p_b^2, p_c^2\;. \tag{5.15}$$

The two daughter gluons are close to mass shell while the mother has to have a finite positive invariant mass.

We begin with the phase space, common to all parton splittings. In our case we assume an outgoing gluon as part of an n-particle process, leading to an $(n+1)$-particle process with the two outgoing gluons. We describe the kinematics of this approximately collinear process in terms of the energy fractions z and $1-z$ defined as

$$\begin{aligned} z = \frac{|E_b|}{|E_a|} = 1 - \frac{|E_c|}{|E_a|} \qquad & p_a^2 = (-p_b - p_c)^2 = 2(p_b p_c) = 2z(1-z)(1-\cos\theta)E_a^2 \\ \Leftrightarrow \qquad & \theta \equiv \theta_b + \theta_c\;. \end{aligned} \tag{5.16}$$

in the collinear limit and in terms of the opening angle θ between $\vec{p}_b$ and $\vec{p}_c$. Using this parameterization we split the $(n+1)$-particle phase space into an n-particle phase space and a splitting process of quarks and gluons,

$$\begin{aligned} d\Phi_{n+1} &= \cdots \frac{d^3\vec{p}_b}{2(2\pi)^3|E_b|}\,\frac{d^3\vec{p}_c}{2(2\pi)^3|E_c|} = \cdots \frac{d^3\vec{p}_a}{2(2\pi)^3|E_a|}\,\frac{d^3\vec{p}_c}{2(2\pi)^3|E_c|}\,\frac{|E_a|}{|E_b|} \\ &\equiv d\Phi_n\,\frac{dp_{c,3}dp_T p_T d\phi}{2(2\pi)^3|E_c|}\,\frac{1}{z} \\ &= d\Phi_n\,\frac{dp_{c,3}dp_T^2 d\phi}{4(2\pi)^3|E_c|}\,\frac{1}{z} \end{aligned} \tag{5.17}$$

This separation does not include any (collinear) approximation. Next, we translate $p_{c,3}$ and p_T^2 into z and $p_a^2 \neq 0$ in the collinear limit,

$$\begin{aligned} p_a^2 &= z(1-z)E_a^2\theta^2 + \mathcal{O}(\theta^4) \\ \theta &\simeq \frac{1}{|E_a|}\,\sqrt{\frac{p_a^2}{z(1-z)}} \\ \Rightarrow \qquad \frac{dp_{c,3}}{|E_c|} &= \frac{dz}{1-z}\,(1+\mathcal{O}(\theta)) \qquad \text{and} \qquad dp_T^2 = z(1-z)dp_a^2\;. \end{aligned} \tag{5.18}$$

Assuming azimuthal symmetry in an addition step this gives us the the separated collinear phase space

$$d\Phi_{n+1} = d\Phi_n\,\frac{dz\,dp_a^2\,d\phi}{4(2\pi)^3}\,(1+\mathcal{O}(\theta)) = d\Phi_n\,\frac{dz\,dp_a^2}{4(2\pi)^2}\,(1+\mathcal{O}(\theta))\;. \tag{5.19}$$

Adding the transition matrix elements to the factorized phase space gives us a full factorization in the collinear approximation

$$\begin{aligned} d\sigma_{n+1} &= \overline{|\mathcal{M}_{n+1}|^2}\; d\Phi_{n+1} \\ &= \overline{|\mathcal{M}_{n+1}|^2}\; d\Phi_n \frac{dp_a^2\, dz}{4(2\pi)^2}\;\; (1+\mathcal{O}(\theta)) \\ &\simeq \frac{2g_s^2}{p_a^2}\; \hat{P}(z)\; \overline{|\mathcal{M}_n|^2}\; d\Phi_n \frac{dp_a^2\, dz}{16\pi^2} \qquad \text{assuming} \quad \boxed{\overline{|\mathcal{M}_{n+1}|^2} \simeq \frac{2g_s^2}{p_a^2}\; \hat{P}(z)\; \overline{|\mathcal{M}_n|^2}} \\ &= d\sigma_n\; \frac{dp_a^2}{p_a^2}\; dz\; \frac{\alpha_s}{2\pi} \hat{P}(z)\; . \end{aligned} \tag{5.20}$$

For the case of incoming parton splitting we replace $p_a^2 \to t$, the usual Mandelstam variable introduced in Sec. 5.8. We can show this assumed factorization by constructing the appropriate splitting kernels $\hat{P}(z)$ for all quark and gluon configurations:

- For gluon splitting into two gluons we can compute the transition amplitude in terms of the gluon momenta and polarizations. We only quote the result

$$\begin{aligned} \overline{|\mathcal{M}_{n+1}|^2} &= \frac{2g_s^2}{p_a^2}\; \frac{N_c}{2}\; 2\left[\frac{z}{1-z} + z(1-z) + \frac{1-z}{z}\right]\; \overline{|\mathcal{M}_n|^2} \\ &\equiv \frac{2g_s^2}{p_a^2}\; \hat{P}_{g\leftarrow g}(z)\; \overline{|\mathcal{M}_n|^2} \\ \Leftrightarrow \qquad \hat{P}_{g\leftarrow g}(z) &= C_A\left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right]\; , \end{aligned} \tag{5.21}$$

  using $C_A = N_c$. The splitting kernel is symmetric when we exchange the two gluons z and $(1-z)$. It diverges if either gluon becomes soft. The notation $\hat{P}_{i\leftarrow j} \sim \hat{P}_{ij}$ is inspired by a matrix notation which we can use to multiply the splitting matrix from the right with the incoming parton vector to get the final parton vector.

- A second kernel describes the splitting of a gluon into two quarks. Again, we omit the calculation and quote

$$\hat{P}_{q\leftarrow g}(z) = T_R\left[z^2 + (1-z)^2\right]\; . \tag{5.22}$$

  It is symmetric under $z \leftrightarrow (1-z)$ because QCD does not distinguish between the outgoing quark and antiquark.

- The third splitting is gluon radiation off a quark line,

$$\hat{P}_{q\leftarrow q}(z) = C_F \frac{1+z^2}{1-z}\; . \tag{5.23}$$

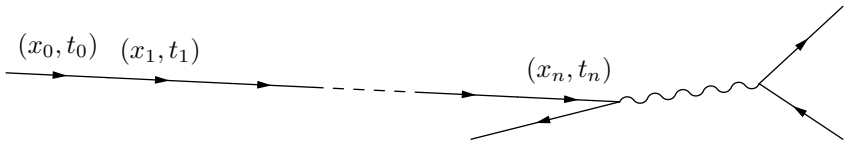
- Just switching $z \leftrightarrow (1-z)$ we can read off the kernel for a quark splitting into the final–state gluon

$$\hat{P}_{g\leftarrow q}(z) = C_F \frac{1+(1-z)^2}{z}\; . \tag{5.24}$$

A key property of QCD is that, similar to ultraviolet divergences, these collinear splitting kernels are universal. They factorize in the sense that they do not depend on the hard n-particle matrix element as part of the full $(n+1)$-particle process.

5.4 DGLAP equation

Our idea to get rid of collinear divergences is to absorb them into the parton densities and define the partons entering the hard process at the point when we can resolve the splittings. To describe such successive splittings we simplify our argument by considering only successive gluon radiation off an incoming quark in an abelian version of QCD,



As the incoming quark with energy fraction x_0 is confined inside the proton, it can only have small transverse momentum, so the 4-momentum squared is negative and its absolute value is smaller than typical scales,

$$-t_0 = |t_0| \ll m_Z^2 \; . \tag{5.25}$$

Collinear gluon radiation will decrease the quark energy and increase the virtuality through recoil,

$$x_{j+1} < x_j \qquad \text{and} \qquad |t_{j+1}| = -t_{j+1} > -t_j = |t_j| \; . \tag{5.26}$$

We know the successive splitting probabilities and can describe how the probabilistic parton density $f(x, -t)$ evolves in the $(x, |t|)$ plane in Figure 1. Starting at (x_0, t_0) we interpret each branching as a discrete step down in $x_j \to x_{j+1}$ and up in $|t_j| \to |t_{j+1}|$. The evolved parton density at (x_n, t_n) enters the hard scattering, including its energy–momentum conservation. Each splitting probability is given by Eq.(5.20),

$$\frac{\alpha_s}{2\pi} \, \hat{P}(z) \, \frac{dt}{t} \, dz \equiv \frac{\alpha_s}{2\pi} \, \hat{P}_{q \leftarrow q}(z) \, \frac{dt}{t} \, dz \; . \tag{5.27}$$

In our $(x, |t|)$ plane we can compute the probabilistic flows into and out of the square

$$[x_j, x_j + \delta x] \qquad \text{and} \qquad [|t_j|, |t_j| + \delta t] \; . \tag{5.28}$$

Their combination defines the net change in f

$$\delta f_{\text{in}} - \delta f_{\text{out}} = \delta f(x, -t) \; . \tag{5.29}$$

The way we define the path between two splittings in Figure 1 it can enter and leave the square either vertically or horizontally. Because we want to arrive at a differential equation in t we choose the vertical drop, such that the area the incoming and outgoing flows see is given by δt. t_{j+1}, An incoming path entering the square can come from any

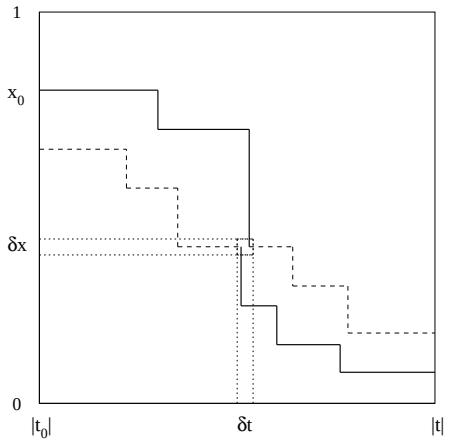


Figure 1: Path of an incoming quark in the $(x - |t|)$ plane. Because $t < 0$, the axis is labelled $|t|$.

x-value above the square. This gives us for the vertical flow into the square

$$\begin{aligned}
\delta f_{\text{in}}(-t) &= \delta t \left(\frac{\alpha_s \hat{P}}{2\pi t} \otimes f\right)(x,-t) \\
&= \frac{\delta t}{t} \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}(z) f\left(\frac{x}{z},-t\right) \\
&\equiv \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}(z) f\left(\frac{x}{z},-t\right) \qquad \text{assuming } f(x',-t)=0 \text{ for } x'>1 \,, \qquad (5.30)
\end{aligned}$$

where we use the definition of a convolution

$$(f\otimes g)(x) = \int_0^1 dx_1 dx_2 f(x_1) g(x_2)\, \delta(x-x_1 x_2) = \int_0^1 \frac{dx_1}{x_1} f(x_1) g\left(\frac{x}{x_1}\right) = \int_0^1 \frac{dx_2}{x_2} f\left(\frac{x}{x_2}\right) g(x_2) \,. \qquad (5.31)$$

The outgoing flow leaves the infinitesimal square vertically,

$$\delta f_{\text{out}}(-t) = \delta t \int_0^1 dy \frac{\alpha_s \hat{P}(y)}{2\pi t} f(x,-t) = \frac{\delta t}{t} f(x,-t) \int_0^1 dy \frac{\alpha_s}{2\pi} \hat{P}(y) \,. \qquad (5.32)$$

Here, the y-integration is not a convolution, because we know the starting condition and integrate over all final configurations. Combining Eq.(5.30) and Eq.(5.32) the net change in the quark probability is

$$\begin{aligned}
\delta f(x,-t) = \delta f_{\text{in}} - \delta f_{\text{out}} &= \frac{\delta t}{t} \left[\int_0^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}(z)\, f\left(\frac{x}{z},-t\right) - \int_0^1 dy \frac{\alpha_s}{2\pi} \hat{P}(y)\, f(x,-t)\right] \\
&= \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \left[\hat{P}(z) - \delta(1-z) \int_0^1 dy \hat{P}(y)\right] f\left(\frac{x}{z},-t\right) \\
&\equiv \frac{\delta t}{t} \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}(z)_+ f\left(\frac{x}{z},-t\right) \\
\Leftrightarrow \qquad \frac{\delta f(x,-t)}{\delta(-t)} &= \frac{1}{(-t)} \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}(z)_+ f\left(\frac{x}{z},-t\right) \qquad (5.33)
\end{aligned}$$

Strictly speaking, we require α_s to only depend on t and introduce the plus subtraction

$$F(z)_+ \equiv F(z) - \delta(1-z) \int_0^1 dy\, F(y) \qquad \text{or} \qquad \int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \left(\frac{f(z)}{1-z} - \frac{f(1)}{1-z}\right) \,. \qquad (5.34)$$

For the second definition we choose $F(z) = 1/(1-z)$, multiply it with an arbitrary test function $f(z)$ and integrate over z. The plus–subtracted integral is regularized in the soft limit $z \to 1$, where some splitting kernels diverge. It can be related to dimensional regularization

$$\int_0^1 dz \frac{1}{(1-z)^{1-\epsilon}} = \int_0^1 dz \frac{1}{z^{1-\epsilon}} = \frac{z^\epsilon}{\epsilon}\bigg|_0^1 = \frac{1}{\epsilon} \qquad \text{with } \epsilon > 0 \,, \qquad (5.35)$$

corresponding to $4+2\epsilon$ dimensions. We can compare the two regularization methods through

$$\begin{aligned}
\int_0^1 dz \frac{f(z)}{(1-z)^{1-\epsilon}} &= \int_0^1 dz \frac{f(z)-f(1)}{(1-z)^{1-\epsilon}} + f(1) \int_0^1 dz \frac{1}{(1-z)^{1-\epsilon}} \\
&= \int_0^1 dz \frac{f(z)-f(1)}{1-z} (1+\mathcal{O}(\epsilon)) + \frac{f(1)}{\epsilon} \\
&= \int_0^1 dz \frac{f(z)}{(1-z)_+} (1+\mathcal{O}(\epsilon)) + \frac{f(1)}{\epsilon} \qquad \text{by definition} \\
\Leftrightarrow \qquad \int_0^1 dz \frac{f(z)}{(1-z)^{1-\epsilon}} - \frac{f(1)}{\epsilon} &= \int_0^1 dz \frac{f(z)}{(1-z)_+} (1+\mathcal{O}(\epsilon)) \,. \qquad (5.36)
\end{aligned}$$

The finite part of the dimensionally regularized integral, is the same as the plus–subtracted integral modulo terms of the order ϵ. This means the plus-subtraction plays a similar role as $\overline{\text{MS}}$ renormalization for UV-divergences.

To regularize our $\hat{P}_{q\leftarrow q}$ in Eq.(5.23) we can choose between two subtraction schemes with the finite difference

$$\begin{aligned}\left(\frac{1+z^2}{1-z}\right)_+ - (1+z^2)\,\left(\frac{1}{1-z}\right)_+ &= \frac{1+z^2}{1-z} - \delta(1-z)\int_0^1 dy\,\frac{1+y^2}{1-y} - \frac{1+z^2}{1-z} + \delta(1-z)\int_0^1 dy\,\frac{1+z^2}{1-y} \\ &= -\delta(1-z)\int_0^1 dy\,\left(\frac{1+y^2}{1-y} - \frac{2}{1-y}\right) \\ &= \delta(1-z)\int_0^1 dy\,\frac{y^2-1}{y-1} = \delta(1-z)\int_0^1 dy\,(y+1) = \frac{3}{2}\delta(1-z)\,. \end{aligned} \tag{5.37}$$

This means we can write the regularized quark splitting kernel in two equivalent ways, where $\delta(1-z)$ implies that the difference is a contribution to a soft–radiation phase space integral. We write

$$\begin{aligned}P_{q\leftarrow q}(z) &\equiv \hat{P}_{q\leftarrow q}(z)_+ \\ &= C_F\left(\frac{1+z^2}{1-z}\right)_+ = C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]\end{aligned} \tag{5.38}$$

and find the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi or DGLAP equation for quarks radiating gluons,

$$\frac{df_q(x,-t)}{d\log(-t)} = -t\,\frac{df_q(x,-t)}{d(-t)} = \int_x^1 \frac{dz}{z}\,\frac{\alpha_s}{2\pi}\,P_{q\leftarrow q}(z)\,f_q\left(\frac{x}{z},-t\right)\,. \tag{5.39}$$

Because in QCD quarks can also be created from a splitting gluon, we have to generalize it to

$$\boxed{\frac{df_q(x,-t)}{d\log(-t)} = -t\,\frac{df_q(x,-t)}{d(-t)} = \sum_{j=q,g}\int_x^1 \frac{dz}{z}\,\frac{\alpha_s}{2\pi}\,P_{q\leftarrow j}(z)\,f_j\left(\frac{x}{z},-t\right)}\,. \tag{5.40}$$

The splitting from gluon to quark only affects the incoming flow in our $(x,|t|)$ plane, proportional to the gluon pdf. Because quarks can be produced in gluon splitting but cannot vanish into it, $P_{q\leftarrow g}$ does not include a plus–regulator

$$P_{q\leftarrow g}(z) \equiv \hat{P}_{q\leftarrow g}(z) = T_R\left[z^2+(1-z)^2\right]\,. \tag{5.41}$$

This kernel is indeed missing a soft-radiation divergence for $z\to 1$.

Next we look at the gluon density. The incoming contribution to the infinitesimal square is given by the sum of four splitting scenarios

$$\begin{aligned}\delta f_{\text{in}}(-t) &= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\,\frac{\alpha_s}{2\pi}\,\left[\hat{P}_{g\leftarrow g}(z)\left(f_g\left(\frac{x}{z},-t\right) + f_g\left(\frac{x}{1-z},-t\right)\right) + \hat{P}_{g\leftarrow q}(z)\left(f_q\left(\frac{x}{z},-t\right) + f_{\bar{q}}\left(\frac{x}{z},-t\right)\right)\right] \\ &= \frac{\delta t}{t}\int_0^1 \frac{dz}{z}\,\frac{\alpha_s}{2\pi}\,\left[2\hat{P}_{g\leftarrow g}(z) f_g\left(\frac{x}{z},-t\right) + \hat{P}_{g\leftarrow q}(z)\left(f_q\left(\frac{x}{z},-t\right) + f_{\bar{q}}\left(\frac{x}{z},-t\right)\right)\right]\,,\end{aligned} \tag{5.42}$$

using $P_{g\leftarrow\bar{q}} = P_{g\leftarrow q}$ in the first line and $P_{g\leftarrow g}(1-z) = P_{g\leftarrow g}(z)$ in the second. To leave the volume element in $(x,|t|)$-space a gluon can either split into two gluons or radiate one of n_f light-quark flavors, giving the net change

$$\begin{aligned}\delta f_g(x,-t) = &\frac{\delta t}{t}\int_0^1 \frac{dz}{z}\,\frac{\alpha_s}{2\pi}\,\left[2\hat{P}_{g\leftarrow g}(z) f_g\left(\frac{x}{z},-t\right) + \hat{P}_{g\leftarrow q}(z)\left(f_q\left(\frac{x}{z},-t\right) + f_{\bar{q}}\left(\frac{x}{z},-t\right)\right)\right] \\ &- \frac{\delta t}{t}\int_0^1 dy\,\frac{\alpha_s}{2\pi}\,\left[\hat{P}_{g\leftarrow g}(y) + n_f\hat{P}_{q\leftarrow g}(y)\right] f_g(x,-t)\end{aligned} \tag{5.43}$$

Unlike in the quark case these terms do not immediately correspond to regularizing the diagonal splitting kernel using the plus prescription.

First, the contribution to δf_{in} proportional to f_q or $f_{\bar{q}}$ is not matched by the outgoing flow. In analogy to the quark case, the corresponding splitting kernel does not need regularization,

$$P_{g\leftarrow q}(z) \equiv \hat{P}_{g\leftarrow q}(z) = C_F \frac{1+(1-z)^2}{z} \; . \tag{5.44}$$

The structure of the DGLAP equation implies that the two off-diagonal splitting kernels do not include any plus prescription $\hat{P}_{i\leftarrow j} = P_{i\leftarrow j}$, because the kernels are finite in the soft limit, $z \to 1$.

Next, we can compute the outgoing y-integral describing the gluon splitting into a quark pair explicitly,

$$\begin{aligned}
-\int_0^1 dy \; \frac{\alpha_s}{2\pi} \; n_f \; \hat{P}_{q\leftarrow g}(y) &= -\frac{\alpha_s}{2\pi} \; n_f \; T_R \; \int_0^1 dy \; \left[1-2y+2y^2\right] \qquad \text{using Eq.(5.41)} \\
&= -\frac{\alpha_s}{2\pi} \; n_f \; T_R \; \left[y - y^2 + \frac{2y^3}{3}\right]_0^1 \\
&= -\frac{2}{3} \; \frac{\alpha_s}{2\pi} \; n_f \; T_R \; .
\end{aligned} \tag{5.45}$$

Finally, the terms proportional to $P_{g\leftarrow g}$ in Eq.(5.43) require some work. The outgoing y-integral has to give us a finite term and a term defining the plus prescription for $\hat{P}_{g\leftarrow g}$,

$$\begin{aligned}
-\int_0^1 dy \; \frac{\alpha_s}{2\pi} \; \hat{P}_{g\leftarrow g}(y) &= -\frac{\alpha_s}{2\pi} \; C_A \; \int_0^1 dy \; \left[\frac{y}{1-y} + \frac{1-y}{y} + y(1-y)\right] \qquad \text{using Eq.(5.21)} \\
&= -\frac{\alpha_s}{2\pi} \; C_A \; \int_0^1 dy \; \left[\frac{2y}{1-y} + y(1-y)\right] \\
&= -\frac{\alpha_s}{2\pi} \; C_A \; \int_0^1 dy \; \left[\frac{2(y-1)}{1-y} + y(1-y)\right] - \frac{\alpha_s}{2\pi} \; C_A \; \int_0^1 dy \; \frac{2}{1-y} \\
&= -\frac{\alpha_s}{2\pi} \; C_A \; \int_0^1 dy \; \left[-2+y-y^2\right] - \frac{\alpha_s}{2\pi} \; 2C_A \; \int_0^1 dz \; \frac{1}{1-z} \\
&= -\frac{\alpha_s}{2\pi} \; C_A \; \left[-2+\frac{1}{2}-\frac{1}{3}\right] - \frac{\alpha_s}{2\pi} \; 2C_A \; \int_0^1 dz \; \frac{1}{1-z} \\
&= \frac{\alpha_s}{2\pi} \; \frac{11}{6} \; C_A \; - \frac{\alpha_s}{2\pi} \; 2C_A \; \int_0^1 dz \; \frac{1}{1-z} \; .
\end{aligned} \tag{5.46}$$

The second term is what we need to regularize the the first term in the splitting kernel of Eq.(5.21) to $1/(1-z)_+$. We can see this using $f(z) = z$ and correspondingly $f(1) = 1$ in Eq.(5.34). The two finite terms in Eq.(5.45) and Eq.(5.46) have to be included in the definition of $\hat{P}_{g\leftarrow g}$ ad hoc. Because the regularized splitting kernel appears in a convolution, they require an explicit factor $\delta(1-z)$. This gives us

$$P_{g\leftarrow g}(z) = 2C_A \; \left(\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right) + \frac{11}{6} \; C_A \; \delta(1-z) - \frac{2}{3} \; n_f \; T_R \; \delta(1-z) \; . \tag{5.47}$$

In its standard form and in terms of the factorization scale $\mu_F^2 \equiv -t$ the complete DGLAP equation reads

$$\boxed{\frac{df_i(x,\mu_F^2)}{d\log\mu_F^2} = \sum_j \int_x^1 \frac{dz}{z} \; \frac{\alpha_s}{2\pi} \; P_{i\leftarrow j}(z) \; f_j\left(\frac{x}{z},\mu_F\right) = \frac{\alpha_s}{2\pi} \; \sum_j \; \left(P_{i\leftarrow j} \otimes f_j\right)(x,\mu_F)} \; . \tag{5.48}$$

This DGLAP equation is formulated in terms of four splitting kernels of gluons and quarks which are linked to the splitting probabilities, but have to be regularized. The finite off-diagonal quark–gluon and gluon–quark splittings are unchanged, while the diagonal kernels become finite in the soft limit $z \to 1$. Altogether, the plus prescription and the parton densities obeying the DGLAP equation act as a soft and collinear kind of renormalization, moving universal infrared divergences into the definition of the parton densities.

5.5 Solving the DGLAP equation

To understand the structure of the solutions of the DGLAP equation let us write a simplified version

$$\frac{df(x,\mu_F^2)}{d\log\mu_F^2} = \int_x^1 \frac{dz}{z}\,\frac{\alpha_s}{2\pi}\,P(z)\,f\left(\frac{x}{z},\mu_F^2\right)\,. \tag{5.49}$$

To solve it, we need a transformation that simplifies a convolution, leading us to the Mellin transform. Starting from a function $f(x)$ of a real variable x we define the Mellin transform into moment space m as

$$\mathcal{M}[f](m) \equiv \int_0^1 dx\, x^{m-1} f(x) \qquad\qquad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dm\,\frac{\mathcal{M}[f](m)}{x^m}\,, \tag{5.50}$$

where for the back transformation we choose an arbitrary appropriate constant $c > 0$, such that the integration contour for the inverse transformation lies to the right of all singularities of the analytic continuation of $\mathcal{M}[f](m)$. The key property is that the Mellin transform of a convolution is the product of the two Mellin transforms, which gives us

$$\begin{aligned}\frac{d\mathcal{M}[f](m,\mu_F^2)}{d\log\mu_F^2} &= \frac{\alpha_s}{2\pi}\,\mathcal{M}\left[\int_x^1 \frac{dz}{z}P(z)f\left(\frac{x}{z},\mu_F^2\right)\right](m) \\ &= \frac{\alpha_s}{2\pi}\,\mathcal{M}[P](m)\,\mathcal{M}[f](m,\mu_F^2)\,. \end{aligned}\tag{5.51}$$

In Mellin space it has the simple solution

$$\begin{aligned}\mathcal{M}[f](m,\mu_F^2) &= \mathcal{M}[f](m,\mu_{F,0}^2)\,\exp\left(\frac{\alpha_s}{2\pi}\,\mathcal{M}[P](m)\log\frac{\mu_F^2}{\mu_{F,0}^2}\right) \\ &\equiv \mathcal{M}[f](m,\mu_{F,0}^2)\,\left(\frac{\mu_F^2}{\mu_{F,0}^2}\right)^{\frac{\alpha_s}{2\pi}\gamma(m)} \qquad \text{with} \qquad \gamma(m) = \mathcal{M}[P](m)\,. \end{aligned}\tag{5.52}$$

To combine μ_F and α_s into one parameter, we identify the renormalization and factorization scales

$$\mu_F \equiv \mu_R \equiv \mu\,, \tag{5.53}$$

and write the derivative as

$$\frac{d}{d\log\mu^2} = \frac{d\log\alpha_s}{d\log\mu^2}\,\frac{d}{d\log\alpha_s} = \frac{1}{\alpha_s}\,\frac{d\alpha_s}{d\log\mu^2}\,\frac{d}{d\log\alpha_s} = -\alpha_s b_0\,\frac{d}{d\log\alpha_s}\,. \tag{5.54}$$

The pre-factor α_s cancels the factor α_s on the right hand side of the DGLAP equation Eq.(5.51), giving us

$$\mathcal{M}[f](m,\mu^2) = \mathcal{M}[f](m,\mu_0^2)\,\left(\frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)}\right)^{\frac{\gamma(m)}{2\pi b_0}}\,. \tag{5.55}$$

Among other things, this kind of parton density is not physical, we have neglected that some splitting functions have singularities and therefore the Mellin transform is not obviously well defined, our convolution is not really a convolution with a cutoff at Q_0^2 etc; but the final structure in Eq.(5.55) really holds.

5.6 Resumming collinear logarithms

In our discussion of the DGLAP equation and its solution we encounter the splitting probability in the exponent, which reminds of the resummed series of bubble diagrams to a running α_s. Powers of $(\mathcal{M}[\hat{P}])^n$ summed over n corresponds to a sum over splittings, hence a variable number of partons in the final state. This suggests that parton

densities cannot be formulated in terms of a fixed final state because they include effects from any number of collinear partons summed over the number of such partons, like

$$\boxed{pp \to \mu^+\mu^- + X} \qquad \text{where } X \text{ includes any number of collinear jets.} \tag{5.56}$$

We start with a model based on collinear splittings, but without using the DGLAP equation, solve it, and compare its solution to the DGLAP solutions in Eq.(5.55).

To treat initial state splittings, we need a definition of the virtuality t. We introduce a positive transverse momentum variable

$$-t = \frac{\vec{p}_T^{\,2}}{1-z} > 0 \qquad \Rightarrow \qquad \frac{dt}{t} = \frac{d\vec{p}_T^{\,2}}{\vec{p}_T^{\,2}} \; , \tag{5.57}$$

where $\vec{p}_T^{\,2}$ is the transverse 3-momentum of of the parton pair after splitting. Equation(5.20) with a running strong coupling and a generic splitting kernel $P(z)$ then becomes

$$\sigma_{n+1}(x,\mu_F) \approx \int_{x_0}^1 \frac{dx_n}{x_n} \; P\left(\frac{x}{x_n}\right) \sigma_n(x_n,\mu_0) \int_{\mu_0}^{\mu_F} \frac{d\vec{p}_{T,n}^{\,2}}{\vec{p}_{T,n}^{\,2}} \frac{\alpha_s(\mu_R^2)}{2\pi} \; . \tag{5.58}$$

Because the splitting kernel is infrared divergent we cut off the convolution integral at x_0. Similarly, the transverse momentum integral is bounded by an infrared cutoff μ_0 and the physical external scale μ_F. They give the range in which an additional collinear radiation is included in σ_{n+1}. For splitting the two integrals in Eq.(5.58) it is crucial that μ_0 is the only scale on which the matrix element σ_n depends. The other integration variable, the transverse momentum, does not appear in σ_n because collinear factorization is defined in the limit $\vec{p}_T^{\,2} \to 0$.

If μ_F is the global upper limit of the transverse momentum integration for collinear splitting, we can apply the recursion formula in Eq.(5.58) iteratively

$$\begin{aligned}\sigma_{n+1}(x,\mu_F) \approx & \int_{x_0}^1 \frac{dx_n}{x_n} \; P\left(\frac{x}{x_n}\right) \; \cdots \; \int_{x_0}^1 \frac{dx_1}{x_1} \; P\left(\frac{x_2}{x_1}\right) \; \sigma_1(x_1,\mu_0) \\ & \times \int_{\mu_0}^{\mu_F} \frac{d\vec{p}_{T,n}^{\,2}}{\vec{p}_{T,n}^{\,2}} \frac{\alpha_s(\mu_R^2)}{2\pi} \; \cdots \; \int_{\mu_0}^{\mu_F} \frac{d\vec{p}_{T,1}^{\,2}}{\vec{p}_{T,1}^{\,2}} \frac{\alpha_s(\mu_R^2)}{2\pi} \; .\end{aligned} \tag{5.59}$$

Next, we will look for assumptions that allow us to solve Eq.(5.58). To develop this physics picture of the DGLAP equation we make two assumptions:

1. We identify the scale of the strong coupling α_s with the transverse momentum scale of the splitting,

$$\mu_R^2 = \vec{p}_T^{\,2} \; . \tag{5.60}$$

2. We assume strongly ordered splittings in the transverse momentum. If the ordering of the splitting is fixed externally by the chain of momentum fractions x_j, this means

$$\mu_0^2 < \vec{p}_{T,1}^{\,2} < \vec{p}_{T,2}^{\,2} < \cdots < \mu_F^2 \tag{5.61}$$

Under these two assumptions the transverse momentum integrals in Eq.(5.59) become

$$\begin{aligned}& \int_{\mu_0}^{\mu_F} \frac{d\vec{p}_{T,n}^{\,2}}{\vec{p}_{T,n}^{\,2}} \frac{\alpha_s(\vec{p}_{T,n}^{\,2})}{2\pi} \; \cdots \int_{\mu_0}^{p_{T,3}} \frac{d\vec{p}_{T,2}^{\,2}}{\vec{p}_{T,2}^{\,2}} \frac{\alpha_s(\vec{p}_{T,2}^{\,2})}{2\pi} \int_{\mu_0}^{p_{T,2}} \frac{d\vec{p}_{T,1}^{\,2}}{\vec{p}_{T,1}^{\,2}} \frac{\alpha_s(\vec{p}_{T,1}^{\,2})}{2\pi} \\ & = \int_{\mu_0}^{\mu_F} \frac{d\vec{p}_{T,n}^{\,2}}{\vec{p}_{T,n}^{\,2}} \frac{1}{2\pi b_0 \log \dfrac{\vec{p}_{T,n}^{\,2}}{\Lambda_{\rm QCD}^2}} \; \cdots \int_{\mu_0}^{p_{T,3}} \frac{d\vec{p}_{T,2}^{\,2}}{\vec{p}_{T,2}^{\,2}} \frac{1}{2\pi b_0 \log \dfrac{\vec{p}_{T,2}^{\,2}}{\Lambda_{\rm QCD}^2}} \int_{\mu_0}^{p_{T,2}} \frac{d\vec{p}_{T,1}^{\,2}}{\vec{p}_{T,1}^{\,2}} \frac{1}{2\pi b_0 \log \dfrac{\vec{p}_{T,1}^{\,2}}{\Lambda_{\rm QCD}^2}} \\ & = \frac{1}{(2\pi b_0)^n} \int_{\mu_0}^{\mu_F} \frac{d\vec{p}_{T,n}^{\,2}}{\vec{p}_{T,n}^{\,2}} \frac{1}{\log \dfrac{\vec{p}_{T,n}^{\,2}}{\Lambda_{\rm QCD}^2}} \; \cdots \int_{\mu_0}^{p_{T,3}} \frac{d\vec{p}_{T,2}^{\,2}}{\vec{p}_{T,2}^{\,2}} \frac{1}{\log \dfrac{\vec{p}_{T,2}^{\,2}}{\Lambda_{\rm QCD}^2}} \int_{\mu_0}^{p_{T,2}} \frac{d\vec{p}_{T,1}^{\,2}}{\vec{p}_{T,1}^{\,2}} \frac{1}{\log \dfrac{\vec{p}_{T,1}^{\,2}}{\Lambda_{\rm QCD}^2}} \; .\end{aligned} \tag{5.62}$$

We can solve the individual integrals by switching variables, for example in the last integral

$$\int_{\mu_0}^{p_{T,2}} \frac{d\vec{p}_{T,1}^2}{\vec{p}_{T,1}^2} \frac{1}{\log \dfrac{\vec{p}_{T,1}^2}{\Lambda_{\rm QCD}^2}} = \int_{\log\log \mu_0^2/\Lambda^2}^{\log\log p_{T,2}^2/\Lambda^2} d\log\log \frac{\vec{p}_{T,1}^2}{\Lambda_{\rm QCD}^2} \qquad \text{with} \qquad \frac{d(ax)}{(ax)\log x} = d\log\log x$$

$$= \log \frac{\log \vec{p}_{T,2}^2/\Lambda_{\rm QCD}^2}{\log \mu_0^2/\Lambda_{\rm QCD}^2} \; . \tag{5.63}$$

This gives us for the chain of transverse momentum integrals, shifted to get rid of the lower boundaries,

$$\begin{aligned}
&\int^{p_{T,n}\equiv\mu_F} d\log \frac{\log \vec{p}_{T,n}^2/\Lambda_{\rm QCD}^2}{\log \mu_0^2/\Lambda_{\rm QCD}^2} \; \cdots \int^{p_{T,2}\equiv p_{T,3}} d\log \frac{\log \vec{p}_{T,2}^2/\Lambda_{\rm QCD}^2}{\log \mu_0^2/\Lambda_{\rm QCD}^2} \int^{p_{T,1}\equiv p_{T,2}} d\log \frac{\log \vec{p}_{T,1}^2/\Lambda_{\rm QCD}^2}{\log \mu_0^2/\Lambda_{\rm QCD}^2} \\
&= \int^{p_{T,n}\equiv\mu_F} d\log \frac{\log \vec{p}_{T,n}^2/\Lambda_{\rm QCD}^2}{\log \mu_0^2/\Lambda_{\rm QCD}^2} \; \cdots \int^{p_{T,2}\equiv p_{T,3}} d\log \frac{\log \vec{p}_{T,2}^2/\Lambda_{\rm QCD}^2}{\log \mu_0^2/\Lambda_{\rm QCD}^2} \left(\log \frac{\log \vec{p}_{T,2}^2/\Lambda_{\rm QCD}^2}{\log \mu_0^2/\Lambda_{\rm QCD}^2} \right) \\
&= \int^{p_{T,n}\equiv\mu_F} d\log \frac{\log \vec{p}_{T,n}^2/\Lambda_{\rm QCD}^2}{\log \mu_0^2/\Lambda_{\rm QCD}^2} \; \cdots \frac{1}{2} \left(\log \frac{\log \vec{p}_{T,3}^2/\Lambda_{\rm QCD}^2}{\log \mu_0^2/\Lambda_{\rm QCD}^2} \right)^2 \\
&= \int^{p_{T,n}\equiv\mu_F} d\log \frac{\log \vec{p}_{T,n}^2/\Lambda_{\rm QCD}^2}{\log \mu_0^2/\Lambda_{\rm QCD}^2} \left(\frac{1}{2} \cdots \frac{1}{n-1} \right) \left(\log \frac{\log \vec{p}_{T,n}^2/\Lambda_{\rm QCD}^2}{\log \mu_0^2/\Lambda_{\rm QCD}^2} \right)^{n-1} \\
&= \frac{1}{n!} \left(\log \frac{\log \mu_F^2/\Lambda_{\rm QCD}^2}{\log \mu_0^2/\Lambda_{\rm QCD}^2} \right)^n = \frac{1}{n!} \left(\log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu_F^2)} \right)^n \; .
\end{aligned} \tag{5.64}$$

This is the final result for the chain of transverse momentum integrals in Eq.(5.59). After integrating over the transverse momenta, the strong coupling is evaluated at $\mu_R \equiv \mu_F$.

Next, we look at the convolution integrals in Eq.(5.58),

$$\sigma_{n+1}(x,\mu) \approx \frac{1}{n!} \left(\frac{1}{2\pi b_0} \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right)^n \int_{x_0}^1 \frac{dx_n}{x_n} \, P\left(\frac{x}{x_n}\right) \cdots \int_{x_0}^1 \frac{dx_1}{x_1} \, P\left(\frac{x_2}{x_1}\right) \, \sigma_1(x_1,\mu_0) \; . \tag{5.65}$$

To solve it we again use the Mellin transformation with the key property the Mellin transform of a convolution is the product of the two Mellin transforms, us

$$\begin{aligned}
\mathcal{M}[\sigma_{n+1}](m,\mu) &\sim \frac{1}{n!} \left(\frac{1}{2\pi b_0} \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right)^n \; \mathcal{M}\left[\int_{x_0}^1 \frac{dx_n}{x_n} \, P\left(\frac{x}{x_n}\right) \cdots \int_{x_0}^1 \frac{dx_1}{x_1} \, P\left(\frac{x_2}{x_1}\right) \, \sigma_1(x_1,\mu_0) \right](m) \\
&= \frac{1}{n!} \left(\frac{1}{2\pi b_0} \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \, \gamma(m) \right)^n \; \mathcal{M}[\sigma_1](m,\mu_0) \qquad\qquad \text{using } \gamma(m) \equiv \mathcal{M}[P](m)
\end{aligned} \tag{5.66}$$

Next, we sum the production cross sections for up to n collinear jets,

$$\begin{aligned}
\sum_{n=0}^\infty \mathcal{M}[\sigma_{n+1}](m,\mu) &= \mathcal{M}[\sigma_1](m,\mu_0) \; \sum_n \frac{1}{n!} \left(\frac{1}{2\pi b_0} \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \, \gamma(m) \right)^n \\
&= \mathcal{M}[\sigma_1](m,\mu_0) \; \exp\left(\frac{\gamma(m)}{2\pi b_0} \; \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right) \\
&= \mathcal{M}[\sigma_1](m,\mu_0) \; \left(\frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right)^{\frac{\gamma(m)}{2\pi b_0}} \; .
\end{aligned} \tag{5.67}$$

This is exactly the same as Eq.(5.55), so we can understand the physics of the DGLAP equation using our model calculation of a successive splitting, including the variable number of collinear jets in the form of $pp \to \mu^+\mu^- + X$,

	renormalization scale μ_R	factorization scale μ_F
source	ultraviolet divergence	collinear (infrared) divergence
poles cancelled	counter terms (renormalization)	parton densities (mass factorization)
summation	resum self energy bubbles	resum parton splittings
parameter	running coupling $\alpha_s(\mu_R^2)$	running parton density $f_j(x, \mu_F)$
evolution	RGE for α_s	DGLAP equation
large scales	decrease of σ_{tot}	increase of σ_{tot} for gluons/sea quarks
theory background	renormalizability proven for gauge theories	factorization proven all orders for DIS proven order-by-order DY...

Table 2: Comparison of renormalization and factorization scales appearing in LHC cross sections.

as shown in Eq.(5.56). In Eq.(5.67) we sum over any number of additional collinear partons; on the right hand side we see fixed order Drell–Yan production without any additional partons, but with an exponentiated correction factor. Comparing this to the running parton densities we can draw the analogy that any process computed with a scale dependent parton density where the scale dependence is governed by the DGLAP equation includes any number of collinear partons.

We can also identify the logarithms which are resummed by scale dependent parton densities. Going back to Eq.(5.13) reminds us that we start from the divergent collinear logarithms $\log p_T^{\text{max}}/p_T^{\text{min}}$ arising from the collinear phase space integration. In our model for successive splitting we replace the upper boundary by μ_F. The collinear logarithm of successive initial–state parton splitting diverges for $\mu_0 \to 0$, but it gets absorbed into the parton densities and determines the structure of the DGLAP equation and its solutions. The upper boundary μ_F tells us to what extent we assume incoming quarks and gluons to be a coupled system of splitting partons and what the maximum momentum scale of these splittings is. Transverse momenta $p_T > \mu_F$ generated by hard parton splitting are not covered by the DGLAP equation and hence not a feature of the incoming partons anymore. They belong to the hard process and have to be consistently simulated. While this scale can be chosen freely we have to make sure that it does not become too large, because at some point the collinear approximation $C \simeq$ constant in Eq.(5.13) ceases to hold.

6 Higgs physics

In the first part of the lecture we introduced massive electroweak gauge bosons and fermions. We showed a way to add a third, longitudinal degree of freedom to a massless photon in Sec. 3.2. That lead us to the Goldstone theorem and Nambu-Goldstone modes. In Sec. 3.5 we introduced the Σ-field with the properties:

- the 2×2 matrix field $\Sigma(x)$ provides the longitudinal $W^\pm$ and Z modes, as we never showed explicitly;
- the transformation $\Sigma \to U\Sigma V^\dagger$ gives us a gauge-invariant Lagrangian with fermion masses, Eq.(3.37);
- the same transformation gives us a gauge-invariant Lagrangian for massive weak bosons, Eq.(3.41);
- for the mass terms it is sufficient to replace $\Sigma(x)$ by a constant matrix $\mathbb{1}$, Eq.(3.54).

The prize we paid for finite fermion and gauge boson masses is that our theory includes the additional field Σ which we cannot make sense of in terms of quantum field theory. It also ruins renormalizability of our theory, because it does not follow our usual power counting.

6.1 Higgs boson

If we consider our Σ model a fundamental theory, we need to turn Σ into a quantum field, *i.e.* we need to allow for quantum fluctuations. Unlike for all other fields, they now have to be around a finite (vacuum) expectation value. Omitting the Goldstone modes for now we write this quantum field as a real scalar field H with mass dimension one and an appropriate normalization factor,

$$\boxed{\Sigma(x) \;\to\; \left(1 + \frac{H(x)}{v}\right) \Sigma(x)} \; . \tag{6.1}$$

This physical Higgs field H does not have a vacuum expectation value and behaves like any other fundamental scalar field,

$$\frac{1}{2}\langle \mathrm{Tr}\, \Sigma\rangle = \frac{1}{2}\, \langle \mathrm{Tr}(\Sigma^\dagger \Sigma)\rangle = \left\langle \left(1 + \frac{H}{v}\right)^2 \right\rangle = 1 \qquad \Leftrightarrow \qquad \langle H\rangle = 0 \; . \tag{6.2}$$

The normalization of H/v is not fixed until we define the kinetic terms for the Higgs field.

We now re-write the mass terms in the electroweak Lagrangian using this completed Σ-field. We start with the fermion masses in Eq.(3.37) and replace the relevant contribution as

$$\Sigma \sim \mathbb{1} \quad \to \quad \left(1 + \frac{H}{v}\right) \Sigma \sim \left(1 + \frac{H}{v}\right) \mathbb{1} \; . \tag{6.3}$$

We define the Yukawa coupling as $y_f = \sqrt{2} m_f / v$ and find the combination of mass and Higgs coupling

$$\mathscr{L}_{D3} = -m_f \Sigma \overline{\psi}_f \psi_f \to -y_f \frac{(v+H)}{\sqrt{2}}\, \overline{\psi}_f \psi_f \; . \tag{6.4}$$

Without showing it in detail, we claim that the same replacement works for Eq.(3.53) with the massive gauge bosons, assuming custodial symmetry,

$$\begin{aligned} \mathscr{L}_{D2} &= \frac{g^2}{4}(v+H)^2\, W_\mu^+ W^{-\mu} + \frac{g_Z^2}{8}(v+H)^2\, Z_\mu Z^\mu \\ &\supset g m_W\, H\, W_\mu^+ W^{-\mu} + \frac{g_Z m_Z}{2}\, H\, Z_\mu Z^\mu \\ &\text{with} \quad m_W = \frac{gv}{2} \quad \text{and} \quad m_Z = \frac{g_Z v}{2} \; . \end{aligned} \tag{6.5}$$

From Eq.(3.54) we already know the relation between the electroweak scale v and the weak boson masses. The couplings of the Higgs boson are completely determined by the masses of the particles it is coupling to. Finally, we re-write the last remaining Lagrangian term given in Eq.(3.43) and find a pure Higgs Lagrangian

$$\mathscr{L}_\Sigma = -\frac{\mu^2 v^2}{2}\left(1+\frac{H}{v}\right)^2 - \frac{\lambda v^4}{4}\left(1+\frac{H}{v}\right)^4 + ... \tag{6.6}$$

The dots stand for possible higher–dimensional terms. In the renormalizable Standard Model with a proper Higgs field, all terms in these Lagrangians have mass dimension four.

6.2 Higgs potential to dimension six

Introducing $SU(2)_L$ doublets for the Higgs field and for now just keeping track of the Higgs VEV and Higgs field

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+H \end{pmatrix} \tag{6.7}$$

we can formally write the Lagrangian of Eq.(6.6) as

$$\mathscr{L}_\Sigma = -V_{\text{SM}} = -\mu^2|\phi|^2 - \lambda|\phi|^4 + \text{const} \, . \tag{6.8}$$

Here, μ^2 and λ are prefactors which we will relate to the Higgs mass and the Higgs self-couplings.

To be general and allow for an effective theory extension of the renormalizable Standard Model (SMEFT), we allow for two additional operators in the Higgs potential. There are exactly two gauge–invariant operators of dimension six in terms of the Higgs doublet $|\phi|^2$, *i.e.* before electroweak symmetry breaking

$$\mathcal{O}_1 = \frac{1}{2}\partial_\mu(\phi^\dagger\phi)\,\partial^\mu(\phi^\dagger\phi) \qquad \text{and} \qquad \mathcal{O}_2 = -\frac{1}{3}(\phi^\dagger\phi)^3 \, . \tag{6.9}$$

One more possible operator $(D_\mu\phi)^\dagger\phi\;\phi^\dagger(D^\mu\phi)$ violates custodial symmetry, so we ignore it. These two dimension-6 operators are part of a Lagrangian, for which we multiply them with so-called Wilson coefficients f_i and an unknown mass scale Λ

$$\mathscr{L}_{D6} = \sum_{i=1}^{2499}\frac{f_i}{\Lambda^2}\mathcal{O}_i \, . \tag{6.10}$$

As long as the typical energy scales E or v in the numerator in a given matrix element is small compared to Λ, the corrections from the additional operators are small as well.

First, we look at the effect of the dimension-6 operator $\mathcal{O}_1$ on the kinetic term of the Higgs field in the Lagrangian, for now denoted as $\hat{H} \neq H$,

$$\begin{aligned}\mathcal{O}_1 &= \frac{1}{2}\partial_\mu\left(\frac{(\hat{H}+v)^2}{2}\right)\partial^\mu\left(\frac{(\hat{H}+v)^2}{2}\right) \\ &= \frac{1}{2}\,(\hat{H}+v)^2\,\partial_\mu\hat{H}\,\partial^\mu\hat{H} \, .\end{aligned} \tag{6.11}$$

With the additional contribution from $\mathcal{O}_1$ the kinetic term reads

$$\mathscr{L}_{\text{kin}} = \frac{1}{2}\partial_\mu\hat{H}\partial^\mu\hat{H}\left(1+\frac{f_1 v^2}{\Lambda^2}\right) \stackrel{!}{=} \frac{1}{2}\partial_\mu H\,\partial^\mu H \qquad \Leftrightarrow \qquad \boxed{H = \sqrt{1+\frac{f_1 v^2}{\Lambda^2}}\hat{H}} \, . \tag{6.12}$$

This is a simple rescaling to define the canonical kinetic term in the Lagrangian, corresponding to a finite wave function renormalization which ensures that the residuum of the Higgs propagator is one. This means we have to

replace $\hat{H}$ with H in the entire SM-Lagrangian. Often a wave function renormalization does not lead to observable effects, because it can be absorbed for example in a coupling renormalization. However, the electroweak sector does not give us enough freedom to absorb this rescaling, so it will appear in observable couplings.

Next, we write the Higgs potential including the operator $\mathcal{O}_2$,

$$V_{\text{SMEFT}} = \mu^2|\phi|^2 + \lambda|\phi|^4 + \frac{f_2}{3\Lambda^2}|\phi|^6 \,. \tag{6.13}$$

The positive sign in the last term ensures that for $f_2 > 0$ the potential is bounded from below for large field values ϕ. The non–trivial minimum at $\phi \neq 0$ is given by

$$\begin{aligned} \frac{\partial V_{\text{SMEFT}}}{\partial\,|\phi|^2} &= \mu^2 + 2\lambda|\phi|^2 + \frac{3f_2}{3\Lambda^2}|\phi|^4 \stackrel{!}{=} 0 \\ \Leftrightarrow \qquad & |\phi|^4 + \frac{2\lambda\Lambda^2}{f_2}|\phi|^2 + \frac{\mu^2\Lambda^2}{f_2} \stackrel{!}{=} 0 \,. \end{aligned} \tag{6.14}$$

The minimum of the potential defines v via

$$|\phi|^2 \stackrel{!}{=} \frac{v^2}{2} \,. \tag{6.15}$$

The two solutions of the quadratic equation in Eq.(6.14) then become

$$\begin{aligned} \frac{v^2}{2} &= -\frac{\lambda\Lambda^2}{f_2} \pm \left[\left(\frac{\lambda\Lambda^2}{f_2}\right)^2 - \frac{\mu^2\Lambda^2}{f_2}\right]^{\frac{1}{2}} \\ &= \frac{\lambda\Lambda^2}{f_2}\left[-1 \pm \sqrt{1 - \frac{\mu^2 f_2}{\Lambda^2\lambda^2}}\right] \\ &= \frac{\lambda\Lambda^2}{f_2}\left[-1 \pm \left(1 - \frac{f_2\mu^2}{2\lambda^2\Lambda^2} - \frac{f_2^2\mu^4}{8\lambda^4\Lambda^4} + \mathcal{O}\left(\frac{1}{\Lambda^6}\right)\right)\right] \\ &= \begin{cases} -\dfrac{\mu^2}{2\lambda} - \dfrac{f_2\mu^4}{8\lambda^3\Lambda^2} + \mathcal{O}\left(\dfrac{1}{\Lambda^4}\right) = -\dfrac{\mu^2}{2\lambda}\left(1 + \dfrac{f_2\mu^2}{4\lambda^2\Lambda^2} + \mathcal{O}\left(\dfrac{1}{\Lambda^4}\right)\right) \equiv \dfrac{v_{\text{SM}}^2}{2}\left(1 + \dfrac{f_2 v_{\text{SM}}^2}{4\lambda\Lambda^2} + \mathcal{O}\left(\dfrac{1}{\Lambda^4}\right)\right) \\ -\dfrac{2\lambda\Lambda^2}{f_2^2} + \mathcal{O}(\Lambda^0) \end{cases} \end{aligned} \tag{6.16}$$

The first solution we have expanded around the Standard Model minimum,

$$\boxed{v_{\text{SM}}^2 = -\frac{\mu^2}{\lambda} = (246\text{ GeV})^2} \,. \tag{6.17}$$

We know its value from the gauge boson masses, and we see that it only appears if $\mu^2 < 0$. We will talk about this in more detail in Sec. 6.4. The second, high–scale solution is of the order Λ and not consistent with our effective theory.

To compute the Higgs mass we need to collect all quadratic terms contributing to the Lagrangian from Eq.(6.13)

$$\begin{aligned} V_{\text{SMEFT}} &= \mu^2\frac{(\hat{H}+v)^2}{2} + \lambda\frac{(\hat{H}+v)^4}{4} + \frac{f_2}{3\Lambda^2}\frac{(\hat{H}+v)^6}{8} \\ &= \frac{\mu^2}{2}\left(\hat{H}^2\cdots\right) + \frac{\lambda}{4}\left(\cdots 6\hat{H}^2v^2\cdots\right) + \frac{f_2}{24\Lambda^2}\left(\cdots 15\hat{H}^2v^4\cdots\right) \,. \end{aligned} \tag{6.18}$$

This gives us the quadratic mass term to dimension six

$$\begin{aligned}
\mathscr{L} \supset & -\frac{1}{2}\left(\mu^2 + 3\lambda v^2 + \frac{5}{4}\frac{f_2 v^4}{\Lambda^2}\right)\hat{H}^2 \\
&= -\frac{1}{2}\left(-\lambda v^2\left(1+\frac{f_2 v^2}{4\lambda\Lambda^2}\right) + 3\lambda v^2 + \frac{5}{4}\frac{f_2 v^4}{\Lambda^2}\right)\hat{H}^2 && \text{replacing } \mu^2 \text{ using Eq.(6.16)} \\
&= -\frac{1}{2}\left(2\lambda v^2 - \frac{f_2 v^4}{4\Lambda^2} + \frac{5}{4}\frac{f_2 v^4}{\Lambda^2}\right)\left(1+\frac{f_1 v^2}{\Lambda^2}\right)^{-1} H^2 && \text{replacing } \hat{H} \text{ using Eq.(6.12)} \\
&= -\lambda v^2\left(1+\frac{f_2 v^2}{2\lambda\Lambda^2}\right)\left(1-\frac{f_1 v^2}{\Lambda^2} + \mathcal{O}\left(\frac{1}{\Lambda^4}\right)\right)H^2 \\
&= -\lambda v^2\left(1-\frac{f_1 v^2}{\Lambda^2} + \frac{f_2 v^2}{2\Lambda^2\lambda} + \mathcal{O}\left(\frac{1}{\Lambda^4}\right)\right)H^2 \stackrel{!}{=} -\frac{m_H^2}{2}H^2 \\
\Leftrightarrow \qquad & m_H^2 = 2\lambda v^2\left(1-\frac{f_1 v^2}{\Lambda^2} + \frac{f_2 v^2}{2\Lambda^2\lambda}\right) . && (6.19)
\end{aligned}$$

We see that the Higgs mass can be computed from the Higgs potential and depend on the operators we take into account. Including dimension-6 operators the relation between the vacuum expectation value, the Higgs mass and the factor in front of the $|\phi|^4$ term in the potential changes. In the Standard Model we only consider dimension-4 operators, which give us the Higgs mass

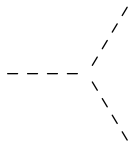
$$\boxed{m_H^2 = 2\lambda v_{\text{SM}}^2 = -2\mu^2 = (125\text{ GeV})^2} \qquad \Rightarrow \qquad \lambda_{\text{SM}} \simeq \frac{1}{8} . \tag{6.20}$$

6.3 Higgs self-couplings

After the Higgs mass we extract the predicted trilinear and quadrilinear Higgs self couplings through the right powers of H in the Higgs potential. We find

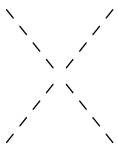
$$\begin{aligned}
\mathscr{L} \supset & -\frac{m_H^2}{2v}\left[\left(1-\frac{f_1 v^2}{2\Lambda^2} + \frac{2 f_2 v^4}{3\Lambda^2 m_H^2}\right)H^3 - \frac{2 f_1 v^2}{\Lambda^2 m_H^2} H\left(\partial_\mu H\right)\left(\partial^\mu H\right)\right] \\
& -\frac{m_H^2}{8v^2}\left[\left(1-\frac{f_1 v^2}{\Lambda^2} + \frac{4 f_2 v^4}{\Lambda^2 m_H^2}\right)H^4 - \frac{4 f_1 v^2}{\Lambda^2 m_H^2} H^2\left(\partial_\mu H\right)\left(\partial^\mu H\right)\right] . \qquad (6.21)
\end{aligned}$$

This gives the Feynman rules



$$-i\frac{3m_H^2}{v}\left[1-\frac{f_1 v^2}{2\Lambda^2} + \frac{2 f_2 v^4}{3\Lambda^2 m_H^2} + \frac{2 f_1 v^2}{3\Lambda^2 m_H^2}\sum_{j<k}^{3}(p_j p_k)\right] \tag{6.22}$$

and



$$-i\frac{3m_H^2}{v^2}\left[1-\frac{f_1 v^2}{\Lambda^2} + \frac{4 f_2 v^4}{\Lambda^2 m_H^2} + \frac{2 f_1 v^2}{3\Lambda^2 m_H^2}\sum_{j<k}^{4}(p_j p_k)\right] \tag{6.23}$$

Just like the Higgs mass, the Higgs self couplings can be computed from the Higgs potential and depend on the operators we take into account.

These Higgs self couplings are structurally different from their dimension-4 counterparts in that their higher dimensional modifications are momentum dependent, driven by $\mathcal{O}_1$. For large momenta such terms will cause problems, because the momentum in the numerator can exceed the suppression $1/\Lambda$. An interesting question is if we

can define a wave function renormalization which removes the momentum dependent terms for the Higgs self couplings. Like in Eq.(6.12) we relate the native Higgs field $\hat{H}$ and the physical H using a general parameterization

$$H = \left(1 + \frac{a_0 v^2}{\Lambda^2}\right) \hat{H} + \frac{a_1 v}{\Lambda^2} \hat{H}^2 + \frac{a_2}{\Lambda^2} \hat{H}^3 \; . \tag{6.24}$$

Unlike before, we now include powers of the Higgs field in the wave function renormalization. Higher powers would be allowed, but it will turn out that we do not need them. The canonically normalized kinetic term for the real scalar field H is

$$\begin{aligned}
\mathscr{L}_{\text{kin}} &= \frac{1}{2} \partial_\mu H \; \partial^\mu H \\
&= \frac{1}{2} \; \partial_\mu \left[\left(1 + \frac{a_0 v^2}{\Lambda^2}\right) \hat{H} + \frac{a_1 v}{\Lambda^2} \hat{H}^2 + \frac{a_2}{\Lambda^2} \hat{H}^3\right] \partial^\mu \left[\left(1 + \frac{a_0 v^2}{\Lambda^2}\right) \hat{H} + \frac{a_1 v}{\Lambda^2} \hat{H}^2 + \frac{a_2}{\Lambda^2} \hat{H}^3\right] \\
&= \frac{1}{2} \left[1 + \frac{a_0 v^2}{\Lambda^2} + \frac{2 a_1 v}{\Lambda^2} \hat{H} + \frac{3 a_2}{\Lambda^2} \hat{H}^2\right]^2 (\partial_\mu \hat{H}) \; (\partial^\mu \hat{H}) \\
&= \left[1 + \frac{2 a_0 v^2}{\Lambda^2} + \frac{a_0^2 v^4}{\Lambda^4} + \left(1 + \frac{a_0 v^2}{\Lambda^2}\right) \frac{4 a_1 v}{\Lambda^2} \hat{H} + \left(\frac{6 a_2}{\Lambda^2} + \frac{2 a_0 v^2}{\Lambda^2} \frac{3 a_2}{\Lambda^2} + \frac{4 a_1^2 v^2}{\Lambda^4}\right) \hat{H}^2 + \mathcal{O}(\hat{H}^3)\right] \frac{\partial_\mu \hat{H} \partial^\mu \hat{H}}{2} \\
&= \left[1 + \frac{2 a_0 v^2}{\Lambda^2} + \frac{4 a_1 v}{\Lambda^2} \hat{H} + \frac{6 a_2}{\Lambda^2} \hat{H}^2 + \mathcal{O}(\hat{H}^3) + \mathcal{O}\left(\frac{1}{\Lambda^4}\right)\right] \frac{\partial_\mu \hat{H} \partial^\mu \hat{H}}{2} \; .
\end{aligned} \tag{6.25}$$

We want this general form to remove all contributions from $\mathcal{O}_1$ to the kinetic term of $\hat{H}$ in Eq.(6.11),

$$\begin{aligned}
\mathscr{L}_{\text{kin}} &= \frac{\partial_\mu \hat{H} \partial^\mu \hat{H}}{2} + \frac{f_1}{\Lambda^2} \; \mathcal{O}_1 \\
&= \left[1 + \frac{f_1}{\Lambda^2} \; \left(v^2 + 2 v \hat{H} + \hat{H}^2\right)\right] \; \frac{\partial_\mu \hat{H} \partial^\mu \hat{H}}{2} \; .
\end{aligned} \tag{6.26}$$

Comparing Eq.(6.25) and Eq.(6.26) we can identify the general pre-factors a_j with the specific f_1 from our dimension-6 ansatz, $a_0 = f_1/2$, $a_1 = f_1/2$, and $a_2 = f_1/6$. The wave function renormalization Eq.(6.24) then reads

$$H = \left(1 + \frac{f_1 v^2}{2\Lambda^2}\right) \hat{H} + \frac{f_1 v}{2\Lambda^2} \hat{H}^2 + \frac{f_1}{6\Lambda^2} \hat{H}^3 + \mathcal{O}(\hat{H}^4) + \mathcal{O}\left(\frac{1}{\Lambda^4}\right) \; . \tag{6.27}$$

In this alternative, generalized canonical normalization of the Higgs field we avoid any momentum dependent contributions to the Higgs self couplings. The prize we pay is that we have to apply the shift defined in Eq.(6.27) throughout the entire Standard Model Lagrangian. This means that the higher dimensional operator $\mathcal{O}_1$ leads to multiple Higgs couplings to any pair of massive gauge bosons or massive fermions.

6.4 Mexican hat

Until now we have only looked at the real scalar Higgs field and written the Higgs potential in Eq.(6.7) in a silly manner. Altogether, Σ includes four degrees of freedom, just like a complex doublet. Three are eaten Goldstones, and one is the fundamental, real Higgs scalar. We introduce a Goldstone triplet $\vec{w}$ in analogy to the weak gauge boson triplet and write

$$\Sigma \; \to \; \left(1 + \frac{H}{v}\right) \mathbb{1} \; \to \; \frac{1}{v} \begin{pmatrix} v + H - i w_3 & -w_2 - i w_1 \\ w_2 - i w_1 & v + H + i w_3 \end{pmatrix} = \frac{\sqrt{2}}{v} \; (\tilde{\phi} \phi) \; . \tag{6.28}$$

The last step is just another way to write the (2×2)-matrix as a bi-doublet in terms of a $SU(2)_L$ doublet containing the physical Higgs field and the Goldstone modes for the massive vector bosons W and Z,

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} -w_2 - i w_1 \\ v + H + i w_3 \end{pmatrix} \qquad \tilde{\phi} = -i \tau_2 \; \phi^* = \frac{1}{\sqrt{2}} \begin{pmatrix} v + H - i w_3 \\ w_2 - i w_1 \end{pmatrix} \; . \tag{6.29}$$

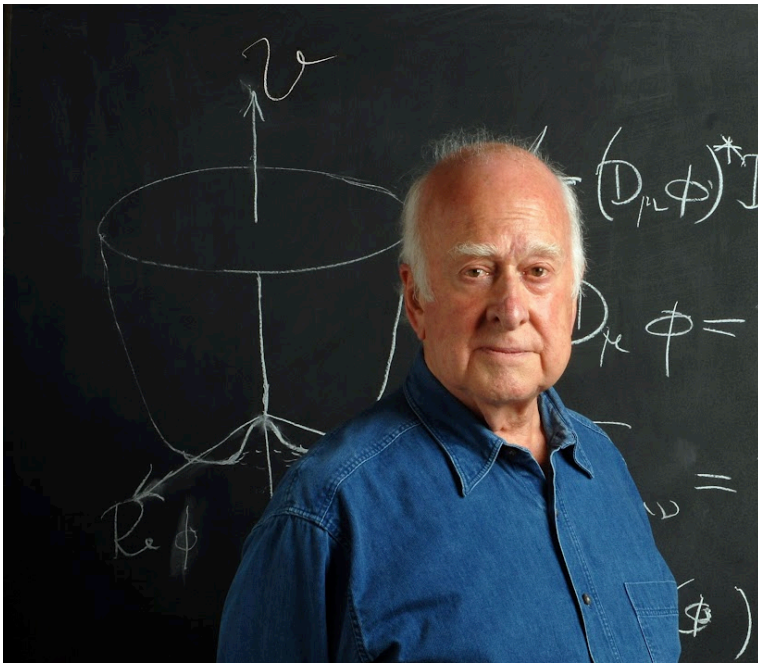


Figure 2: Mexican hat form of the Higgs potential.

For our illustration we consider all neutral fields

$$\phi \;\to\; \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H + i w_3 \end{pmatrix} . \tag{6.30}$$

In this approximation we compute the potential from Eq.(6.13) in the Standard Model,

$$V_{\text{SM}} = \frac{\mu^2}{2} \left((v+H)^2 + w_3^2 \right) + \frac{\lambda}{4} \left((v+H)^2 + w_3^2 \right)^2 . \tag{6.31}$$

In the unbroken phase, *i.e.* assuming $\mu^2 > 0$ and hence $v = 0$, the potential only depends on the radius in the two-dimensional field plane, $|\phi| = \sqrt{H^2 + w_3^2}/\sqrt{2}$. The minimum is at $|\phi| = 0$ or $H = w_3 = 0$, towards larger field values the potential increases first like $|\phi|^2$ and then like $|\phi|^4$.

Fom the SM-solution we know that a real Higgs mass implies a broken phase with $\mu^2 < 0$, leading to a finite VEV v. The minimum in the complex plane is a circle with $|\phi| = v/\sqrt{2}$. It is rotationally symmetric, on the real H axis it becomes $H = \pm v$. This is shown in Figure 2.

To determine the masses of the particles corresponding to the Higgs and Goldstone fields we compute the mass matrix of a field or particle in a potential, as we know it from mechanics,

$$\boxed{\left(\mathcal{M}_H^2\right)_{jk} = \left. \frac{\partial^2 V_{\text{SM}}}{\partial^2 \{H w_3\}} \right|_{\text{minimum}}} . \tag{6.32}$$

We choose one point on the degenerate vacuum circle, *e.g.* $\phi = v/\sqrt{2}$. For real fields, where the mass term is proportional to $m^2 H^2/2$ there is no factor $1/2$ in this relation. Inserting the neutral potential from in Eq.(6.31) we

first find

$$\begin{aligned}
\frac{\partial V_{\text{SM}}}{\partial H} &= \mu^2(v+H) + \frac{\lambda}{2}\left((v+H)^2 + w_3^2\right) 2(v+H) \\
&= \mu^2(v+H) + \lambda(v+H)\left((v+H)^2 + w_3^2\right) \\
\Rightarrow \qquad \left.\frac{\partial^2 V_{\text{SM}}}{\partial H^2}\right|_{\text{minimum}} &= \mu^2 + \lambda\left((v+H)^2 + w_3^2\right) + 2\lambda(v+H)^2\Big|_{\text{minimum}} \\
&= \mu^2 + 3\lambda v^2 = 2\lambda v^2 \ .
\end{aligned} \tag{6.33}$$

In the last step we use the relation $\mu^2 = -\lambda v^2$ at the minimum. The second diagonal entry is

$$\begin{aligned}
\frac{\partial V_{\text{SM}}}{\partial w_3} &= \mu^2 w_3 + \frac{\lambda}{2}\left((v+H)^2 + w_3^2\right) 2w_3 \\
&= \mu^2 w_3 + \lambda w_3\left((v+H)^2 + w_3^2\right) \\
\left.\frac{\partial^2 V_{\text{SM}}}{\partial w_3^2}\right|_{\text{minimum}} &= \mu^2 + \lambda\left((v+H)^2 + w_3^2\right) + 2\lambda w_3^2\Big|_{\text{minimum}} \\
&= \mu^2 + \lambda v^2 = 0 \ .
\end{aligned} \tag{6.34}$$

The off-diagonal entry of the symmetric mass matrix is

$$\left.\frac{\partial^2 V_{\text{SM}}}{\partial H \partial w_3}\right|_{\text{minimum}} = \frac{\partial}{\partial w_3}\left[\mu^2(v+H) + \lambda(v+H)\left((v+H)^2 + w_3^2\right)\right]\Big|_{\text{minimum}} = 2\lambda(v+H)w_3\Big|_{\text{minimum}} = 0 \tag{6.35}$$

Altogether, the Higgs and Goldstone basis corresponds to mass eigenstates of the symmetric mass matrix

$$\mathcal{M}_H^2 = \begin{pmatrix} 2\lambda v^2 & 0 \\ 0 & 0 \end{pmatrix} \ , \tag{6.36}$$

with the known Higgs mass and vanishing Goldstone mass. If we fix our vacuum to the positive real axis, the Higgs mode living on the real axis is massive while the Goldstone mode living orthogonally in the direction of the flat potential valley is massless. The fact that we had to fix the vacuum to the real axis is a direct consequence of our linear representation in Eq.(6.28).

6.5 Unitarity

Goldstone modes are useful when we want to compute transition amplitudes at very high energies. In the rest frame of a $V = W^\pm, Z$ we write the three polarization vectors of the massive gauge boson as

$$\epsilon_{T,1}^\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \epsilon_{T,2}^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \epsilon_L^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \ . \tag{6.37}$$

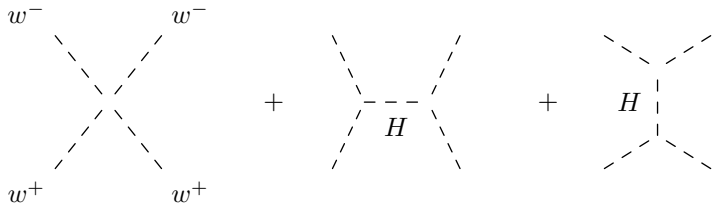
If we boost V into z-direction, giving it a 4-momentum $p^\mu = (E, 0, 0, |\vec{p}|)$, the polarization vectors become

$$\epsilon_{T,1}^\mu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \epsilon_{T,2}^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \epsilon_L^\mu = \frac{1}{m_V}\begin{pmatrix} |\vec{p}| \\ 0 \\ 0 \\ E \end{pmatrix} \overset{E \gg m_V}{\longrightarrow} \frac{1}{m_V}\begin{pmatrix} |\vec{p}| \\ 0 \\ 0 \\ |\vec{p}| \end{pmatrix} \equiv \frac{1}{m_V}\, p^\mu \ . \tag{6.38}$$

Very relativistic gauge bosons are dominated by their longitudinal polarization $|\vec{\epsilon}_L| \sim E/m_V \gg 1$. This longitudinal degree of freedom is the Goldstone mode, so at high energies we can approximate the vector bosons $Z, W^\pm$ as scalar

Goldstones $w_0, w_\pm$. The question which gauge–dependent mass value ξm_V to assign to it does not occur, because in the high energy limit we always assume $m_V \to 0$. This relation between Goldstones and gauge bosons at very high energies is called equivalence theorem.

Based on the equivalence theorem we can compute the amplitude for $W^+W^- \to W^+W^-$ scattering for $E \gg m_W$ in terms of scalar Goldstones. Three diagrams contribute to this processes: a 4-point vertex, the s-channel Higgs exchange and the t-channel Higgs exchange:



To confirm these Feynman diagrams and to compute the corresponding amplitude we need the Feynman rules for Goldstone couplings to the Higgs and 4-Goldstone interactions. We start with the complete Higgs–Goldsone doublet from Eq.(6.29)

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} -w_2 - i w_1 \\ v + H + i w_3 \end{pmatrix} \qquad \Rightarrow \qquad \phi^\dagger \phi = \frac{1}{2}\left(w_1^2 + w_2^2 + w_3^2 + (v+H)^2\right) \tag{6.39}$$

$$\begin{aligned} \left(\phi^\dagger \phi\right)^2 &= \frac{1}{4}\left(\sum_i w_i^2\right)^2 + \frac{1}{2}(v+H)^2 \sum_i w_i^2 + \frac{1}{4}(v+H)^4 \\ &= \frac{1}{4}\left(\sum_i w_i^2\right)^2 + \left(v\,H + \frac{v^2}{2} + \frac{H^2}{2}\right) \sum_i w_i^2 + \mathcal{O}(w^0)\,. \end{aligned}$$

There are no 3-Goldstone vertices. Only keeping the relevant terms contributing to the 4-Goldstone and Higgs–Goldstone–Goldstone couplings at dimension four the potential becomes

$$\begin{aligned} V_{\text{SM}} \supset \lambda |\phi|^4 &= \frac{m_H^2}{2v^2}\, |\phi|^4 \\ &= \frac{m_H^2}{2v^2}\left[\frac{1}{4}\left(\sum_i w_i^2\right)^2 + vH \sum_i w_i^2 + \mathcal{O}(w^0)\right] \\ &= \frac{m_H^2}{8v^2}\left(\sum_i w_i^2\right)^2 + \frac{m_H^2}{2v}\, H \sum_i w_i^2 + \mathcal{O}(w^0)\,. \end{aligned} \tag{6.40}$$

Focussing on the scattering of charged Goldstones $w_\pm w_\pm \to w_\pm w_\pm$ we use the corresponding fields $w_\pm = (w_1 \pm i w_2)/\sqrt{2}$. In the potential they appear as $w_1^2 + w_2^2 = 2 w_+ w_-$, giving

$$V_{\text{SM}} \supset \frac{m_H^2}{2v^2} w_+ w_- w_+ w_- + \frac{m_H^2}{v} H w_+ w_-\,. \tag{6.41}$$

Linking the Lagrangian to the Feynman rule for the quartic coupling involves one complication: for each positively charged Goldstone in the vertex there are two ways we can identify them with the Lagrangian fields. There are also two choices to identify the two negatively charged Goldstones, which implies an additional combinatorial factor four in the Feynman rule. Including a common factor $(-i)$ the two Feynman rules then become $-2i m_H^2/v^2$ and $-i m_H^2/v$.

The amplitude for the Goldstone scattering is given in terms of s and t and made real by an additional factor i,

$$\begin{aligned} A &= i\left[\frac{-2i m_H^2}{v^2} + \left(\frac{-i m_H^2}{v}\right)^2 \frac{i}{s - m_H^2} + \left(\frac{-i m_H^2}{v}\right)^2 \frac{i}{t - m_H^2}\right] \\ &= \frac{m_H^2}{v^2}\left[2 + \frac{m_H^2}{s - m_H^2} + \frac{m_H^2}{t - m_H^2}\right]\,. \end{aligned} \tag{6.42}$$

Now, we test the unitarity of the S-matrix for this process. For this purpose write it terms of a transition amplitude $S = \mathbb{1} + iA$. It should be unitary to conserve probability

$$\mathbb{1} \stackrel{!}{=} S^\dagger S = (\mathbb{1} - iA^\dagger)(\mathbb{1} + iA) = \mathbb{1} + i(A - A^\dagger) + A^\dagger A \qquad \Leftrightarrow \qquad A^\dagger A = -i(A - A^\dagger) \; . \tag{6.43}$$

If we sandwich this equation between identical asymptotically free fields, unitarity implies the so-called optical theorem for the cross section for scattering at high energies

$$\begin{aligned} \sigma \equiv \frac{1}{2s} \langle j | A^\dagger A | j \rangle &= -\frac{i}{2s} \langle j | A - A^{*T} | j \rangle \\ &= -\frac{i}{2s} \langle j | A - A^* | j \rangle = \frac{1}{s} \, \text{Im} \langle j | A(\theta = 0) | j \rangle \; . \end{aligned} \tag{6.44}$$

For the cross section the square $A^\dagger A$ implies a sum over intermediate states, whereas the imaginary part is evaluated between identical states, or at zero scattering angle.

To evaluate this condition for our Goldstone scattering we follow the standard approach from electrodynamics, namely decomposing the transition amplitude into partial waves ordered by the orbital angular momentum ℓ,

$$A = 16\pi \sum_{\ell=0}^{\infty} (2\ell + 1) \, a_\ell \, P_\ell(c) \qquad \text{with} \qquad \int_{-1}^{1} dc \, P_\ell(c) P_{\ell'}(c) = \frac{2}{2\ell + 1} \delta_{\ell\ell'} \; , \tag{6.45}$$

The $P_\ell(c)$ with $c \equiv \cos\theta$ are Legendre polynomials, which obey an orthogonality condition. The scattering cross section including all prefactors and the phase space integration is then given by

$$\begin{aligned} \sigma &= \int d\Omega \, \frac{|A|^2}{64\pi^2 s} \\ &= \frac{(16\pi)^2}{64\pi^2 s} \, 2\pi \int_{-1}^{1} dc \sum_\ell \sum_{\ell'} (2\ell + 1) a_\ell P_\ell(c) \; (2\ell' + 1) a_{\ell'}^* P_{\ell'}(c) \\ &= \frac{16\pi}{s} \sum_\ell (2\ell + 1) \; |a_\ell|^2 \; . \end{aligned} \tag{6.46}$$

Applied to each term in the partial wave expansion the optical theorem from Eq.(6.44) requires

$$\begin{aligned} \frac{16\pi}{s} (2\ell + 1) \; |a_\ell|^2 &= \frac{1}{s} \; \text{Im} A(\theta = 0) \Big|_\ell \\ &= \frac{1}{s} \; 16\pi (2\ell + 1) \; \text{Im} \, a_\ell \qquad \Leftrightarrow \qquad |a_\ell|^2 \stackrel{!}{=} \text{Im} \, a_\ell \; , \end{aligned} \tag{6.47}$$

using $P_\ell(c = 1) = 1$. This condition on Im a_ℓ and Re a_ℓ describes a circle around $a_\ell = (0, 1/2)$ with radius $1/2$, so we can write the optical theorem for partial waves as

$$(\text{Re} \; a_\ell)^2 + \left(\text{Im} \; a_\ell - \frac{1}{2} \right)^2 = \frac{1}{4} \qquad \Rightarrow \qquad \boxed{|\text{Re} \; a_\ell| \; \leq \; \frac{1}{2}} \; . \tag{6.48}$$

For Goldstone scattering we assume the first term to be leading and compute in the high energy limit $s \gg m_H^2$

$$\begin{aligned} a_0 = \frac{1}{16\pi s} \int_{-s}^{0} dt \, |A| \; &= \; \frac{1}{16\pi s} \int_{-s}^{0} dt \, \frac{m_H^2}{v^2} \left[2 + \frac{m_H^2}{s - m_H^2} + \frac{m_H^2}{t - m_H^2} \right] \\ &= \; \frac{m_H^2}{16\pi v^2} \left[2 + \frac{m_H^2}{s - m_H^2} - \frac{m_H^2}{s} \log \left(1 + \frac{s}{m_H^2} \right) \right] \\ &= \; \frac{m_H^2}{16\pi v^2} \left[2 + \mathcal{O} \left(\frac{m_H^2}{s} \right) \right] \leq \; \frac{1}{2} \; . \end{aligned} \tag{6.49}$$

The leading term is the 4-Goldstone coupling, the two Higgs diagrams are sub-leading in m_H^2/s. The perturbative unitarity constraint translates into an upper limit on the Higgs mass which in Eq.(6.42) enters as the Goldstone coupling

$$\boxed{m_H^2 < 4\pi v^2 = (870\ \text{GeV})^2}\ . \tag{6.50}$$

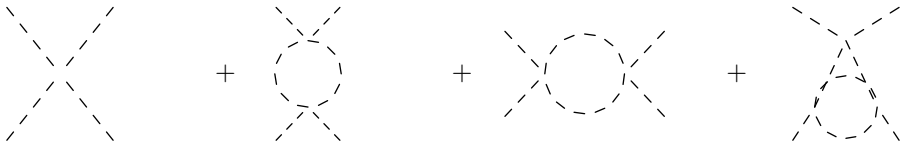
This is the maximum value of m_H consistent with perturbative unitarity for $WW \to WW$ scattering. Replacing the Higgs mass by the self coupling we can formulate the same constraint as $\lambda < 2\pi$. If we include higher order effects as well as higher dimensional operators we get $m_H \lesssim 1$ TeV, in line with the measured Higgs mass.

In many discussions of unitarity and the Higgs sector we explain the role of the Higgs boson in the unitarization of WW scattering as a cancellation of the leading divergences through virtual Higgs exchange. Clearly, this is not what we see in our argument. Nevertheless, both answers are correct, because the separation of a gauge–invariant transition amplitude into Feynman diagrams is gauge dependent. Our Higgs–Goldstone gauge assumes the existence of a Higgs boson when we include the coupling strength m_H in the Feynman rules for the Goldstones. In the unitary gauge, where the gauge bosons are massive and the Goldstones are eaten, the Higgs is an additional state which we can remove at the expense of ruining renormalizability. At tree level this gauge gives us a cancellation of the leading divergences from gauge boson exchange with the help of the Higgs diagrams. Because the transition amplitude is gauge invariant the limit on the Higgs mass will be identical.

We note that our argument only works if the WWH coupling is exactly what it should be. Looking at processes like $WW \to f\bar{f}$ or $WW \to WWH$ or $WW \to HHH$ we can fix all Higgs couplings, including g_{Hff}, g_{HHH}, g_{HHHH}, using exactly the same argument.

6.6 Renormalization group analysis

The unitarity condition derived above is just one of many theory constraints which we can derive as self consistency conditions on a Higgs boson in the Standard Model. We can derive two additional theoretical constraints from the renormalization group equation of the Higgs potential, specifically from the renormalization scale dependence of the self coupling $\lambda(Q^2)$. From Sec. 4.5 we know that such a scale dependence arises automatically when we encounter ultraviolet divergences and absorb the $1/\epsilon$ poles into a minimal counter term. In the case of a running quartic Higgs coupling λ the one-loop s, t and u-channel diagrams only depending on λ itself are



Skipping the calculation we quote the complete renormalization group equation including diagrams with the Higgs boson, the top quark and the weak gauge bosons inside the loops

$$\boxed{\frac{d\,\lambda}{d\,\log Q^2} = \frac{1}{16\pi^2}\left[12\lambda^2 + 6\lambda y_t^2 - 3y_t^4 - \frac{3}{2}\lambda\left(3g^2 + g'^2\right) + \frac{3}{16}\left(2g^4 + (g^2+g'^2)^2\right)\right]}\ , \tag{6.51}$$

with $y_t = \sqrt{2}m_t/v$. This running of the Higgs self-coupling is the same effects as the running of the strong coupling in Eq.(4.54).

The first regime we study is where the Higgs self coupling λ becomes large. In this regime the leading term in Eq.(6.51) reads

$$\frac{d\,\lambda}{d\,\log Q^2} = \frac{1}{16\pi^2}\,12\lambda^2 + \mathcal{O}(\lambda) = \frac{3}{4\pi^2}\lambda^2 + \mathcal{O}(\lambda)\ . \tag{6.52}$$

Because of the positive sign on the right hand side the quartic coupling will grow and eventually diverge for large Q^2. Such a divergence should not appear in a physical model and will give us a constraint on the maximum value of λ allowed. We can solve Eq.(6.52) by replacing $\lambda = x^{-1}$

$$\begin{aligned}
\frac{d}{d\log Q^2}\frac{1}{x} = -\frac{1}{x^2}\frac{d\,x}{d\log Q^2} &\stackrel{!}{=} \frac{3}{4\pi^2}\frac{1}{x^2} \\
\Leftrightarrow \qquad \frac{d\,x}{d\log Q^2} = -\frac{3}{4\pi^2} \qquad &\Leftrightarrow \qquad x(Q^2) = -\frac{3}{4\pi^2}\log Q^2 + C\,. \qquad (6.53)
\end{aligned}$$

The boundary condition $\lambda(Q^2 = v^2) = \lambda_0$ fixes the integration constant C

$$\begin{aligned}
x_0 = \frac{1}{\lambda_0} = -\frac{3}{4\pi^2}\log v^2 + C \quad &\Leftrightarrow \quad C = x_0 + \frac{3}{4\pi^2}\log v^2 \\
\Rightarrow \quad x(Q^2) = -\frac{3}{4\pi^2}\log Q^2 + x_0 + \frac{3}{4\pi^2}\log v^2 &= -\frac{3}{4\pi^2}\log\frac{Q^2}{v^2} + x_0 \\
\Leftrightarrow \quad \boxed{\lambda(Q^2) = \left[-\frac{3}{4\pi^2}\log\frac{Q^2}{v^2} + \frac{1}{\lambda_0}\right]^{-1} = \lambda_0\left[1 - \frac{3}{4\pi^2}\lambda_0\log\frac{Q^2}{v^2}\right]^{-1}}\,. \qquad &(6.54)
\end{aligned}$$

We start from scales $Q \sim v$ where the expression in brackets is close to one. Moving towards larger scales the denominator becomes smaller until λ hits a pole at the critical value Q_{pole}

$$\begin{aligned}
1 - \frac{3}{4\pi^2}\lambda_0\log\frac{Q^2_{\text{pole}}}{v^2} \stackrel{!}{=} 0 \qquad &\Leftrightarrow \qquad \frac{3}{4\pi^2}\lambda_0\log\frac{Q^2_{\text{pole}}}{v^2} = 1 \\
&\Leftrightarrow \qquad Q_{\text{pole}} = v\ \exp\frac{2\pi^2}{3\lambda_0} = v\ \exp\frac{4\pi^2 v^2}{3m_H^2} \qquad (6.55)
\end{aligned}$$

As for QCD, this is called a Landau pole. It give the maximum scale beyond which we cannot rely on perturbative theory. In the upper line of Fig. 3 we show Q_{pole} versus the Higgs mass, approximately computed in Eq.(6.55). It gives the maximum scale where our theory is valid, so we have to reside below the upper line in Fig. 3. Large cutoff values around the Planck scale 10^{19} GeV correspond to $m_H \lesssim 180$ GeV, again in good agreement with the observed Higgs mass around 125 GeV.

This limit is often referred to as the triviality bound, which at first glance is precisely not what this theory is — trivial or non–interacting. The name originates from the fact that if we want our Higgs potential to be perturbative at all scales, the coupling λ can only be zero everywhere. Any finite coupling will hit a Landau pole at some scale. Such a theory with zero interaction is called trivial.

Now, we go back to the full renormalization group equation of Eq.(6.51) and ask a different question: if the Higgs coupling λ runs, how long will $\lambda > 0$ ensure that our Higgs potential is bounded from below? This is called the stability bound. In Eq.(6.51) there are two terms with a negative sign and which can drive λ through zero. In the small-λ regime we encounter the competing terms

$$\begin{aligned}
\frac{d\,\lambda}{d\log Q^2} &= \frac{1}{16\pi^2}\left[-3\frac{4m_t^4}{v^4} + \frac{3}{16}\left(2g^4 + \left(g^2 + g'^2\right)^2\right) + \mathcal{O}(\lambda)\right] \\
\Leftrightarrow \quad \lambda(Q^2) &\simeq \lambda(v^2) + \frac{1}{16\pi^2}\left[-\frac{12m_t^4}{v^4} + \frac{3}{16}\left(2g^4 + \left(g^2 + g'^2\right)^2\right)\right]\ \log\frac{Q^2}{v^2}\,. \qquad (6.56)
\end{aligned}$$

As before, the boundary condition $\lambda(v^2) = m_H^2/(2v^2)$ is the starting point from which the top Yukawa coupling drives λ through zero. The condition for a zero Higgs self coupling $\lambda(Q^2_{\text{stable}}) = 0$ is

$$\begin{aligned}
\lambda(v^2) &\stackrel{!}{=} \frac{1}{16\pi^2}\left[\frac{12m_t^4}{v^4} - \frac{3}{16}\left(2g^4 + \left(g^2 + g'^2\right)^2\right)\right]\ \log\frac{Q^2_{\text{stable}}}{v^2} \\
&= \frac{1}{16\pi^2}\,[2.4 - 0.1]\ \log\frac{Q^2_{\text{stable}}}{v^2} \\
&\approx \frac{1}{7\pi^2}\ \log\frac{Q^2_{\text{stable}}}{v^2} \qquad (6.57)
\end{aligned}$$

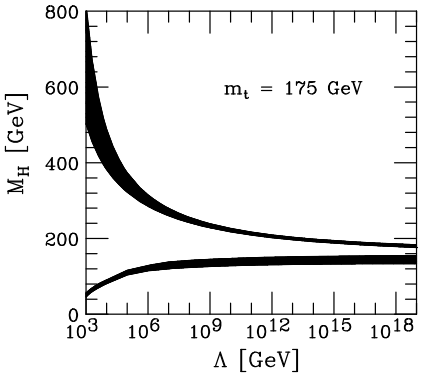


Figure 3: Triviality or Landau pole (upper) and stability bounds (lower) for the Standard Model Higgs boson in the $m_H - Q$ plane.

where for the numerical estimate we assume a running mass $m_t = 163$ GeV. We can translate this limit on $\lambda(v^2)$ into a limit on the Higgs mass,

$$\begin{aligned} m_H &= \left[2\lambda(v^2)v^2\right]^{1/2} \\ &= \left[\frac{2v^2}{7\pi^2}\log\frac{Q^2_{\text{stable}}}{v^2}\right]^{1/2} \approx \begin{cases} 70 \text{ GeV} & \text{for} Q_{\text{stable}} = 10^3 \text{ GeV} \\ 130 \text{ GeV} & \text{for} Q_{\text{stable}} = 10^{16} \text{ GeV} \end{cases} , \end{aligned} \tag{6.58}$$

adding some literature values. Only for energy scales below Q_{stable} or the corresponding m_H values the Higgs potential is bounded from below and our vacuum stable. In Figure 3 we show the bound on the Higgs mass as a function of Q_{stable} as the lower curve, above which our consistent theory has to reside. Towards larger $Q_{\text{stable}} \gg v$ the critical Higgs mass will increase, but for a numerical evaluation we need to know the running couplings. Our measured Higgs mass is right at the edge of a UV-complete Standard Model and closer inspection shows that our vacuum really seems meta-stable.

6.7 UV-complete vs effective theories

Our derivation of the Mexican hat or the renormalization group running assumed a Standard Model Higgs. This means that all particles couple to the Higgs proportionally to their masses, including the Higgs to itself. When we compute quantum corrections to Higgs signals with shifted Higgs couplings, our theory is no longer a valid quantum field theory. This makes is hard to define a measurement of the Higgs couplings as part of the LHC precision program.

One way out is higher-dimensional operators. Supplementing the SM-Lagragian with dimension-6 operators provides us with 2499 new parameters to measure in the SMEFT framework. All corresponding higher-dimension Wilson coefficients should vanish in the Standard Model. Strictly speaking, measuring all dimension-6 Wilson coefficients is not sufficient to test all LHC data, because for instance triple and quartic gauge couplings are still gauge-related at this level (unlike the triple and quartic Higgs self-coupling). Splitting anomalous gauge couplings requires an unrealistic number of dimension-8 operators.

The fundamental assumption of an effective theory is that heavy unknown particles are integrated out, which means we turn their propagators into higher-dimensional operators. Which operators are induced by a given theory depends on the symmetry assumptions, but there will never be just one dimension-6 operators. By definition, we should not consider one Wilson coefficient at a time. With the SMEFT framework LHC physics has arrived where flavor physics and hadron physics have been for a long time. We live on a stack of effective field theories, matched to each other while integrating out the heaviest particles. With a fundamental Higgs, LHC can still hope to be special and renormalizable without a UV-cutoff, but that is something we need to prove. This means we should always keep in

mind that in spite of using SMEFT we have a bias that at some point our effective theory should describe a UV-complete theory, where heavy particles are well-defined but decoupled. We always expect our SMEFT results to be matched to a well-defined theory.

7 Dark matter

7.1 Thermal history

Dark matter is the leading signal for physics beyond the Standard Model, in the sense that only the neutrino is a valid particle to describe the observed dark matter density in the Universe. In cosmology, a mass density ρ_j is usually given relative to the critical density separating an expanding from a collapsing Universe with $\Lambda = 0$,

$$\Omega_j = \frac{\rho_j}{\rho_c} \qquad \text{with} \qquad \rho_c = 3M_{\rm Pl}^2\, H_0^2 \qquad\qquad (j = m, b, \chi, ...) \; . \tag{7.1}$$

Here, we use the index χ for the unknown dark matter candidate. We will see that dark matter has to be treated in presence of an expansion of the Universe, described by the measured Hubble constant H_0. Two objects anywhere in the Universe move away from each other with a velocity proportional to their current distance r. The proportionality constant is defined through Hubble's law

$$H_0 := \frac{\dot r}{r} \tag{7.2}$$

Because H_0 is not at all a number of order one we replace it with the dimensionless ratio

$$h := \frac{H_0}{100\; \frac{\rm km}{\rm s\; Mpc}} \; . \tag{7.3}$$

Note that H_0 is defined at the current point in time. From the global analysis of the Planck satellite combined with large scale structure observations we know

$$\begin{aligned} \Omega_\chi h^2 &= 0.1198 \pm 0.0015 \\ \Omega_b h^2 &= 0.02225 \pm 0.00016 \\ \Omega_\Lambda &= 0.6844 \pm 0.0091 \\ H_0 &= 67.27 \pm 0.66\, \frac{\rm km}{\rm Mpc\; s} \qquad \Leftrightarrow \qquad h = 0.67 \; . \end{aligned} \tag{7.4}$$

In general, we describe the expanding (flat) universe using a time-dependent scale factor $a(t)$ in the metric,

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right) \qquad \text{with} \qquad k = 0 \; . \tag{7.5}$$

Looking at objects with no relative motion except for the expanding Universe, we can re-write Hubble's law as

$$\begin{aligned} r'(t) = a(t)\, r \qquad &\Leftrightarrow \qquad \dot r'(t) = \dot a(t)\, r \stackrel{!}{=} H(t) r'(t) = H(t) a(t)\, r \\ &\Leftrightarrow \qquad H(t) = \frac{\dot a(t)}{a(t)} \; . \end{aligned} \tag{7.6}$$

From this relation we can learn a basic concept of cosmology: we can describe the evolution of the universe in terms of (i) time t, which is fundamental, but hard to directly observe; (ii) the Hubble 'constant' $H(t)$ describing the expansion of the Universe; (iii) the scale factor $a(t)$ entering the distance metric; or (iv) the temperature $T(t)$. If we insert the relativistic matter density into the Friedmann equation we find

$$H^2 = \left(\frac{1}{2t} \right)^2 = \left(\frac{\pi \sqrt{g_{\rm eff}}}{\sqrt{90}}\, \frac{T^2}{M_{\rm Pl}} \right)^2 \; . \tag{7.7}$$

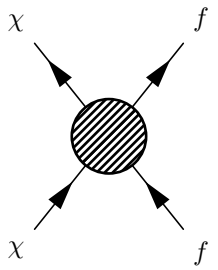
This relation links time, temperature, and Hubble constant as three possible scales in the evolution of our Universe in the relativistic regime. To allow for different states with different temperatures we define g_{eff} as a weighted sum with the specific temperatures of each component,

$$g_{\text{eff}}(T) = \sum_{\text{bosons}} g_b \, \frac{T_b^4}{T^4} + \sum_{\text{fermions}} \frac{7}{8} \, g_f \, \frac{T_f^4}{T^4} \, . \tag{7.8}$$

When we reduce the temperature, this number of active degrees of freedom changes whenever a particle species vanishes at the respective threshold. Today's value is around $g_{\text{eff}}(T_0) = 3.6$.

7.2 WIMP miracle

The assumption behind thermal or weakly interacting massive particle (WIMP) dark matter is that it is produced out of the thermal bath. After that, the dark matter particle χ is kept in thermal equilibrium with Standard Model fermions f through a generic 4-point interaction:



Such a Feynman diagram or amplitude mediates three different processes:

- left-to-right we can compute dark matter annihilation, $\chi\bar{\chi} \to f\bar{f}$;
- bottom-to-top it describes dark matter scattering of visible matter $\chi f \to \chi f$;
- right-to-left it describes dark matter pair-production, $f\bar{f} \to \chi\bar{\chi}$.

This strong link between very different observables makes dark matter so interesting for particle physicists, including the possibility of global analyses for any model that can predict this amplitude.

Strictly speaking, we want to estimate the relic density solving the Boltzmann equation for the number density as a function of time or temperature. However, as a short cut we assume that dark matter is in thermal equilibrium until the expansion of the Universe leads to a thermal decoupling and freeze-out with a specific relic density. The so-called interaction rate corresponding to the above dark matter interaction process is

$$\Gamma(T) := \sigma_{\chi\chi} \, v(T) \, n_\chi(T) \, . \tag{7.9}$$

It combines the cross section, the relevant number density, and the velocity. At the point of decoupling, given by T_{dec}, it just compensates the increasing scale factor from the expanding Universe,

$$\boxed{\Gamma(T_{\text{dec}}) \stackrel{!}{=} H(T_{\text{dec}})} \, . \tag{7.10}$$

For non-relativistic decoupling we first estimate the velocity as

$$\frac{m_\chi}{2} v^2 = T \qquad \Leftrightarrow \qquad v = \sqrt{\frac{2T}{m_\chi}} \, , \tag{7.11}$$

Next, the temperature-dependent number density can be computed using a unified picture for relativistic and non-relativistic particles, as well as radiation. Seetting $k_B = 1$ we can integrate the Fermi or Bose-Einstein distribution over the momentum sphere

$$\begin{aligned} n(T) &= g \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{E/T} \pm 1} && \text{for fermions/bosons} \\ &= g\, 4\pi \int_m^\infty \frac{EdE}{(2\pi)^3} \frac{\sqrt{E^2-m^2}}{e^{E/T} \pm 1} && \text{using } E^2 = p^2 + m^2 \text{ and } pdp = EdE \\ &= \begin{cases} g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} & \text{non-relativistic states } T \ll m \\ \frac{\zeta_3}{\pi^2}\, gT^3 & \text{relativistic bosons } T \gg m \\ \frac{3}{4}\frac{\zeta_3}{\pi^2}\, gT^3 & \text{relativistic fermions } T \gg m. \end{cases} \end{aligned} \tag{7.12}$$

The Riemann zeta function has the value $\zeta_3 = 1.2$. If we set the number of relevant degrees of freedom of the dark matter agent to $g = 2$, corresponding to a complex scalar or a Majorana fermion, the condition for non-relativistic dark matter freeze-out at $T_{\rm dec}$ is

$$\begin{aligned} \Gamma(T_{\rm dec}) = \sigma_{\chi\chi} \sqrt{\frac{2T_{\rm dec}}{m_\chi}}\, g \left(\frac{m_\chi T_{\rm dec}}{2\pi}\right)^{3/2} e^{-m_\chi/T_{\rm dec}} &\stackrel{!}{=} H(T_{\rm dec}) = \frac{\pi}{3\sqrt{10}\, M_{\rm Pl}} \sqrt{g_{\rm eff}(T_{\rm dec})}\, T_{\rm dec}^2 \\ \Leftrightarrow \qquad \sigma_{\chi\chi}\, \frac{m_\chi T_{\rm dec}^2}{\pi^{3/2}}\, e^{-x_{\rm dec}} &= \frac{\pi}{3\sqrt{10}\, M_{\rm Pl}} \sqrt{g_{\rm eff}(T_{\rm dec})} T_{\rm dec}^2 \qquad \text{with} \qquad x := \frac{m_\chi}{T} \\ \Leftrightarrow \qquad e^{-x_{\rm dec}} &= \frac{\pi^{5/2}}{3\sqrt{10}} \frac{\sqrt{g_{\rm eff}(T_{\rm dec})}}{m_\chi M_{\rm Pl}\, \sigma_{\chi\chi}} \\ &= 1.8\, \frac{\sqrt{g_{\rm eff}(T_{\rm dec})}}{m_\chi M_{\rm Pl}\, \sigma_{\chi\chi}}\,. \end{aligned} \tag{7.13}$$

In this calculation the explicit temperature dependence drops out, so the result can be considered an equation for $x_{\rm dec}$. If we want to include the temperature dependence of $g_{\rm eff}$ we cannot solve this equation in a closed form.

Finally, we assume a generic, low-energy electroweak annihilation cross section inspired by an s-channel process, where we use the Z-mass and Z-coupling to write

$$\sigma_{\chi\chi}(T \ll m_\chi) = \frac{\pi\alpha^2 m_\chi^2}{c_w^4 m_Z^4}\,. \tag{7.14}$$

This formula combines the dark matter mass m_χ with a weak interaction represented by a $1/m_Z$ suppression, implicitly assuming $m_\chi \ll m_Z$. Equation (7.13) then becomes

$$e^{-x_{\rm dec}} = \frac{\pi\sqrt{\pi}}{3\sqrt{10}\,\alpha^2}\, \frac{c_w^4 m_Z^4}{m_\chi^3 M_{\rm Pl}}\, \sqrt{g_{\rm eff}(T_{\rm dec})}\,. \tag{7.15}$$

The complete Standard Model gives $g_{\rm eff} = 106.75$. In the slightly lower range $T_{\rm dec} = 5 \ldots 80$ GeV the weak bosons and the top quark decouple, and Eq.(7.8) gives the slightly reduced value

$$g_{\rm eff}(T_{\rm dec}) = (8\times 2 + 2) + \frac{7}{8}\,(5\times 3\times 2\times 2 + 3\times 2\times 2 + 3\times 2) = 18 + \frac{7}{8}\,78 = 86.25\,. \tag{7.16}$$

Combining all prefactors we find the the range

$$e^{-x_{\rm dec}} \approx 6\cdot 10^5\, \frac{m_Z^4}{m_\chi^3 M_{\rm Pl}} = \begin{cases} 2\cdot 10^{-9} & \Leftrightarrow \quad x_{\rm dec} \approx 20 & (m_\chi = 10 \text{ GeV}) \\ 6\cdot 10^{-11} & \Leftrightarrow \quad x_{\rm dec} \approx 23 & (m_\chi = 30 \text{ GeV}) \\ 8\cdot 10^{-12} & \Leftrightarrow \quad x_{\rm dec} \approx 26 & (m_\chi = 60 \text{ GeV})\,. \end{cases} \tag{7.17}$$

As a benchmark, we will use $m_\chi = 30$ GeV and $x_{\text{dec}} \approx 23$. Because of the leading exponential dependence, we expect this result for x_{dec} to be insensitive to our detailed assumptions.

With this x_{dec}-value we can compute the non-relativistic number density at the point of decoupling from Eq.(7.12),

$$\begin{aligned} n_\chi(T_{\text{dec}}) &= g \left(\frac{m_\chi T_{\text{dec}}}{2\pi}\right)^{3/2} e^{-x_{\text{dec}}} \\ &= \frac{\pi}{3\sqrt{20}\, M_{\text{Pl}}} \sqrt{\frac{m_\chi}{T_{\text{dec}}}}\, \sqrt{g_{\text{eff}}(T_{\text{dec}})}\, T_{\text{dec}}^2\, \frac{c_w^4 m_Z^4}{\pi \alpha^2 m_\chi^2} \\ &\approx 10^3\ \frac{m_Z^4}{M_{\text{Pl}}}\ \left(\frac{T_{\text{dec}}}{m_\chi}\right)^{3/2} \approx \frac{10^3}{x_{\text{dec}}^{3/2}}\ \frac{m_Z^4}{M_{\text{Pl}}}\ . \end{aligned} \tag{7.18}$$

From the time of decoupling we still have to evolve the density to the current time or temperature T_0. We start with the fact that once a particle has decoupled, its number density simply drops like $1/a^3$

$$\rho_\chi(T_0) = m_\chi\ n_\chi(T_0) = m_\chi\ n_\chi(T_{\text{dec}})\ \left(\frac{a(T_{\text{dec}})}{a(T_0)}\right)^3\ . \tag{7.19}$$

To translate this into a temperature dependence, we need to quote a thermodynamic result, namely that

$$a(T) \times T \approx \text{const} \tag{7.20}$$

When we take into account the active degrees of freedom and their individual temperature dependence the relation is more precisely

$$\left(\frac{a(T_{\text{dec}})T_{\text{dec}}}{a(T_0)T_0}\right)^3 = \frac{g_{\text{eff}}(T_0)}{g_{\text{eff}}(T_{\text{dec}})} \approx \frac{3.6}{100} = \frac{1}{28}\ , \tag{7.21}$$

again for $T_{\text{dec}} > 5$ GeV and depending slightly on the number of neutrinos we take into account. This gives us the current non-relativistic density

$$\begin{aligned} \rho_\chi(T_0) &= m_\chi\ \left(\frac{a(T_{\text{dec}})T_{\text{dec}}}{a(T_0)T_0}\right)^3\ \frac{T_0^3}{T_{\text{dec}}^3}\, n_\chi(T_{\text{dec}}) \\ &= \frac{x_{\text{dec}}}{28}\ T_0^3\, \frac{n_\chi(T_{\text{dec}})}{T_{\text{dec}}^2} \\ &= T_0^3\, \frac{n_\chi(T_{\text{dec}}) x_{\text{dec}}^3}{28 m_\chi^2} \approx 3 \cdot 10^3\ \frac{m_Z^4}{m_\chi^2 M_{\text{Pl}}}\ T_0^3\ . \end{aligned} \tag{7.22}$$

We can translate this into the dimensionless dark matter density,

$$\begin{aligned} \Omega_\chi h^2 &= \frac{\rho_\chi(T_0) h^2}{3 M_{\text{Pl}}^2 H_0^2} \\ &\approx 3 \cdot 10^3\ \frac{m_Z^4}{m_\chi^2 M_{\text{Pl}}}\ \frac{(2.4 \cdot 10^{-4})^3}{(2.5 \cdot 10^{-3})^4}\ \frac{h^2}{\text{eV}} \\ &\approx 3 \cdot 10^3\ \frac{7 \cdot 10^7}{2 \cdot 10^{18}}\ \frac{\text{GeV}^3}{m_\chi^2}\ \frac{1}{5}\ \frac{10^9}{\text{GeV}} \approx 20\ \frac{\text{GeV}^2}{m_\chi^2} \qquad \Leftrightarrow \qquad \boxed{\Omega_\chi h^2 \approx 0.12\ \left(\frac{13\ \text{GeV}}{m_\chi}\right)^2}\ . \end{aligned} \tag{7.23}$$

This agrees with the result from PLANCK, an outcome usually referred to as the WIMP miracle: if we assume an dark matter agent with an electroweak-scale mass and an annihilation process mediated by the weak interaction, the predicted relic density comes out exactly as measured.

Let us recapitulate where the WIMP mass dependence of Eq.(7.23) comes from: first, the annihilation cross section in Eq.(7.14) is assumed to be mediated by electroweak interactions and includes a dependence on m_χ. Our original

assumption $m_\chi \ll m_W$ is not perfectly fulfilled, but also not completely wrong. Second, the WIMP mass enters the relation between the number and energy density, but some of this dependence is absorbed into the value $x_{\text{dec}} = 23$, which means that the decoupling of the non-relativistic WIMPs is supposed to happen at a very low temperature of $T_{\text{dec}} \approx m_\chi/23$. In addition, some of our assumption in this originally multi-scale calculation are not super-convincing. We need to check Eq.(7.23) with an alternative estimate.

7.3 Boltzmann equation

A more principled way to compute the dark matter relic density of a WIMP is using the Boltzmann equation. It describes the change of a number density $n(t)$ with time. The first effect included in the equation is the increasing scale factor $a(t)$ with conserved dark matter particle number. It even occurs in full equilibrium,

$$0 = \frac{d}{dt}\,\left[n(t)a(t)^3\right] = \dot{n}(t)a(t)^3 + 3n(t)a(t)^2\dot{a}(t) \qquad \Leftrightarrow \qquad \dot{n}(t) + 3H(t)n(t) = 0\,. \tag{7.24}$$

At some point, the underlying assumption of thermal equilibrium breaks down. For the number of WIMPs the relevant process is not a process which guarantees thermal equilibrium with other states, but the explicit pair production or pair annihilation via a weakly interacting process

$$\chi\chi \leftrightarrow f\bar{f}\,, \tag{7.25}$$

with any available pair of light fermions in the final state. The depletion rate from this WIMP pair annihilation process is given by the corresponding rate $\sigma_{\chi\chi}\,v\,n_\chi^2$. It describes the probability of the WIMP annihilation process in Eq.(7.25) to happen, given the WIMP density and their velocity. The thermal averaging replaces the global value of $\sigma_{\chi\chi}\,v$, as it gets added to the equilibrium Boltzmann equation Eq.(7.24),

$$\dot{n}(t) + 3H(t)n(t) = -\langle\sigma_{\chi\chi}\,v\rangle\,\left(n(t)^2 - n_{\text{eq}}(t)^2\right)\,. \tag{7.26}$$

The time dependence of n induced by the annihilation process is proportional to n^2 because of the two WIMPs in the initial state of the annihilation process. The form of the equation guarantees that for $n = n_{\text{eq}}$ the only change in density occurs from the expanding Universe. The approximate result from the Boltzmann approach is

$$\Omega_\chi h^2 \approx 0.12\,\frac{x_{\text{dec}}}{23}\,\frac{\sqrt{g_{\text{eff}}}}{10}\,\frac{1.7\cdot 10^{-9}\,\text{GeV}^{-2}}{\langle\sigma_{\chi\chi}\,v\rangle}\,. \tag{7.27}$$

Given this result we can insert the WIMP annihilation rate from Eq.(7.14) and Eq.(7.11),

$$\begin{aligned}
&\langle\sigma_{\chi\chi}\,v\rangle = \sigma_{\chi\chi}\,v + \mathcal{O}(v^2) \approx \sqrt{\frac{2}{x}}\,\frac{\pi\alpha^2 m_\chi^2}{c_w^4 m_Z^4} \\
\Rightarrow\quad &\Omega_\chi h^2 = 0.12\,\frac{x_{\text{dec}}}{23}\,\frac{\sqrt{g_{\text{eff}}}}{10}\,\frac{c_w^4 m_Z^4\sqrt{x}}{\sqrt{2}\pi\alpha^2 m_\chi^2}\,\frac{1.7\cdot 10^{-9}}{\text{GeV}^2} = 0.12\,\left(\frac{x_{\text{dec}}}{23}\right)^{3/2}\,\frac{\sqrt{g_{\text{eff}}}}{10}\,\left(\frac{35\,\text{GeV}}{m_\chi}\right)^2\,.
\end{aligned} \tag{7.28}$$

Compared to Eq.(7.23) these numbers indeed make sense for a weakly interacting particle with a weak-scale mass.

7.4 Axions

To motivate the existence of new light particles beyond the Standard Model, we turn to QCD and CP-violation. We know from Sec. 3.2 that breaking a global symmetry gives rise to Nambu-Goldstone bosons. If the global symmetry group is compact or has hermitian generators and unitary representations, these Nambu-Goldstone bosons are CP-odd. We illustrate their behavior in a little more detail for a complex scalar field that transforms under a $U(1)$ rotation, as discussed in Sec. 3.1,

$$\phi \to \phi\,e^{ia/f_a}\,. \tag{7.29}$$

If the complex scalar couples to chiral fermions ψ_L and ψ_R, charged under $U(1)$, the Lagrangian includes the terms

$$\mathscr{L} \supset i\overline{\psi}_L \gamma^\mu \partial_\mu \psi_L + i\overline{\psi}_R \gamma^\mu \partial_\mu \psi_R - y\,\phi\,\overline{\psi}_R \psi_L + \text{h.c.} \tag{7.30}$$

Just like the Standard Model Higgs, a VEV $\langle\phi\rangle = f_a$ breaks the $U(1)$-symmetry spontaneously. We can rewrite the Yukawa coupling so that after the rotation the phase is absorbed in the definition of the fermion fields,

$$\begin{aligned} y\,\langle\phi\rangle\,\overline{\psi}_R \psi_L \to y\,f_a\,\overline{\psi}_R e^{ia/f_a}\,\psi_L &\equiv y\,f_a\,\overline{\psi}'_R \psi'_L \qquad \text{with} \quad \psi'_{R,L} = e^{\mp ia/(2f_a)}\psi_{R,L} \\ &\equiv m_\psi\,\overline{\psi}'_R \psi'_L \qquad \text{with} \qquad m_\psi = y f_a\ . \end{aligned} \tag{7.31}$$

In this rotated basis the kinetic terms read

$$\begin{aligned} i\overline{\psi}_L \gamma^\mu \partial_\mu \psi_L + i\overline{\psi}_R \gamma^\mu \partial_\mu \psi_R &= i\,\overline{\psi}'_L e^{-ia/(2f_a)} \gamma^\mu \partial_\mu e^{ia/(2f_a)} \psi'_L + i\,\overline{\psi}'_R e^{ia/(2f_a)} \gamma^\mu \partial_\mu e^{-ia/(2f_a)} \psi'_R \\ &= i\,\overline{\psi}'_L \gamma^\mu \left(\partial_\mu + i\frac{(\partial_\mu a)}{2f_a}\right) \psi'_L + i\,\overline{\psi}'_R \gamma^\mu \left(\partial_\mu - i\frac{(\partial_\mu a)}{2f_a}\right) \psi'_R + \mathcal{O}(f_a^{-2}) \\ &= i\,\overline{\psi}\gamma^\mu \partial_\mu \psi + \frac{(\partial_\mu a)}{2f_a}\,\overline{\psi}\,\gamma^\mu\gamma_5\,\psi + \mathcal{O}(f_a^{-2}) \qquad \text{with} \quad \psi \equiv (\psi'_L, \psi'_R)\ , \end{aligned} \tag{7.32}$$

now in terms of the 4-component spinor. The derivative coupling and the CP-odd nature of the new particle a are evident. Completing Eq.(7.32), the general EFT Lagrangian for such a CP-odd pseudo-scalar reads

$$\mathscr{L} \supset \frac{1}{2}(\partial_\mu a)\,(\partial^\mu a) + \frac{a}{f_a}\frac{\alpha_s}{8\pi} G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu} + c_\gamma \frac{a}{f_a}\frac{\alpha}{8\pi} F_{\mu\nu}\widetilde{F}^{\mu\nu} + \frac{(\partial_\mu a)}{2f_a} \sum_\psi c_\psi\ \overline{\psi}\gamma^\mu\gamma_5\psi\ . \tag{7.33}$$

Here $\widetilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\tau}F^{\rho\tau}/2$ and correspondingly $\widetilde{G}_{\mu\nu}$ are the dual field-strength tensors.

This setup has come to fame as a possible solution to the strong CP-problem, where in QCD, the dimension-4 operator

$$\frac{\theta_{\text{QCD}}}{8\pi} G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu} \tag{7.34}$$

respects the $SU(3)$ gauge symmetry, but would induce observable CP-violation. The non-observation of a neutron dipole moment translates into very strong constraints $\theta_{\text{QCD}} < 10^{-10}$. We add it the operator Eq.(7.33) and obtain

$$\mathscr{L} \supset \frac{1}{2}(\partial_\mu a)\,(\partial^\mu a) + \frac{\alpha_s}{8\pi}\left(\frac{a}{f_a} - \theta_{\text{QCD}}\right)\ G^a_{\mu\nu} \widetilde{G}^{a\,\mu\nu} + c_\gamma \frac{a}{f_a}\frac{\alpha}{8\pi} F_{\mu\nu}\widetilde{F}^{\mu\nu} + \frac{(\partial_\mu a)}{2f_a} \sum_\psi c_\psi\ \overline{\psi}\gamma^\mu\gamma_5\psi\ . \tag{7.35}$$

In the presence of a light pseudoscalar with a VEV, the choice $\langle a\rangle/f_a = \theta_{\text{QCD}}$ eliminates the gluonic CP-violation. Because of its axial couplings to matter fields, the field a is called axion.

The axion would be a bad dark matter candidate if it were truly massless. On the one hand, a mass term breaks the shift symmetry and should vanish for a true Nambu-Goldstone boson. On the other hand, quark masses arising from the Higgs VEV and not related to the axion VEV break the shift invariance of the Lagrangian explicitly. They turn the axion into a pseudo Nambu-Goldstone boson, similar to the pions in QCD. Correspondingly, we write the axion mass term as part of a chiral Lagrangian where the propagating fields are hadrons instead of quarks

$$m_a^2 = \frac{m_u m_d}{(m_u + m_d)^2}\frac{m_\pi^2 f_\pi^2}{f_a^2} \approx \frac{m_\pi^2 f_\pi^2}{2f_a^2}\ , \tag{7.36}$$

where $f_\pi \approx m_\pi \approx 140$ MeV are the pion decay constant and mass. The original axion proposal was with $f_a \sim v$ and correspondingly $m_a \sim 10$ keV. With the couplings also fixed by f_a, such a particle was excluded very fast by searches for rare kaon decays, like for instance $K^+ \to \pi^+ a$. However, for the QCD axion we only know the relation between m_a and f_a.

7.5 Axion dark matter

If a new light axion exists, could it be dark matter? From our WIMP discussion we know that we would require a non-thermal production process. We consider a toy model with a spatially homogeneous but time-dependent complex scalar field $\phi(t)$ in a potential $V(\phi)$. Its Taylor expansion is dominated by a quadratic mass term m_ϕ. If we account for the metric g, describing the expanding Universe, the Lagrangian for a complex scalar field reads just like in Eq.(6.8)

$$\frac{1}{\sqrt{|g|}}\,\mathscr{L} = (\partial^\mu\phi^*)(\partial_\mu\phi) - V(\phi) = (\partial^\mu\phi^*)(\partial_\mu\phi) - m_\phi^2\,\phi^*\phi\,. \tag{7.37}$$

The equation of motion for a spatially homogeneous field $\phi(t)$ is

$$\begin{aligned} 0 = \partial_t\left(\frac{\partial\mathscr{L}}{\partial(\partial_t\phi^*)}\right) - \frac{\partial\mathscr{L}}{\partial\phi^*} &= \partial_t\left(\sqrt{|g|}\,\partial_t\phi\right) + \sqrt{|g|}\,m_\phi^2\phi \\ &= (\partial_t\sqrt{|g|})\,(\partial_t\phi) + \sqrt{|g|}\,\partial_t^2\phi + \sqrt{|g|}\,m_\phi^2\phi \\ &= \sqrt{|g|}\left[\frac{(\partial_t\sqrt{|g|})}{\sqrt{|g|}}\,(\partial_t\phi) + \partial_t^2\phi + m_\phi^2\phi\right]\,. \end{aligned} \tag{7.38}$$

The definition of the scale factor a in (7.5) gives us for the determinant of the metric in flat space

$$\begin{aligned} |g| = a^6 \qquad \Rightarrow \qquad 0 &= \frac{(\partial_t a^3)}{a^3}\,(\partial_t\phi) + \partial_t^2\phi + m_\phi^2\phi \\ &= \frac{3\dot a}{a}\,\dot\phi + \ddot\phi + m_\phi^2\phi\,. \end{aligned} \tag{7.39}$$

Finally, we can use the definition of the Hubble constant in Eq.(7.6) to write

$$\boxed{\ddot\phi + 3H\dot\phi + m_\phi^2\phi = 0}\,. \tag{7.40}$$

We can solve this equation for the evolving Universe, described by a decreasing Hubble constant with increasing time, Eq.(7.7). In a regime with slowly changing H — an approximation we need to check later — we use the ansatz

$$\begin{aligned} \phi(t) = e^{i\omega t} \qquad &\Rightarrow \qquad \dot\phi(t) = i\omega\phi(t) \qquad \Rightarrow \qquad \ddot\phi(t) = -\omega^2\phi(t) \\ &\Rightarrow \qquad -\omega^2 + 3iH\omega + m_\phi^2 = 0 \\ &\Leftrightarrow \qquad \omega = \frac{3i}{2}H \pm \sqrt{-\frac{9}{4}H^2 + m_\phi^2}\,. \end{aligned} \tag{7.41}$$

This functional form defines three distinct regimes in the evolution of the Universe:

– In the early Universe, $H \gg m_\phi$, the two solutions are $\omega = \{0, 3iH\}$. The scalar field is a superposition of a constant mode and an exponentially decaying mode.

$$\phi(t) = \phi_1 + \phi_2\,e^{-3Ht} \overset{\text{time evolution}}{\longrightarrow} \phi_1\,. \tag{7.42}$$

It settles into the constant field value. This does not have to be the minimum of the potential, because due to the Hubble friction term in Eq.(7.40) there is no time for the field to evolve toward the proper minimum. This behavior gives the process its name, misalignment mechanism.

– For decreasing $H \sim m_\phi$ we see in Eq.(7.41) that a transition occurs when the square root vanishes at

$$H_{\text{prod}} \approx \frac{2}{3}m_\phi \qquad \Leftrightarrow \qquad \omega \approx \frac{3i}{2}H_{\text{prod}}\,. \tag{7.43}$$

The evolution of $\phi(t)$ switches from the exponential decay towards ϕ_1 to an oscillation mode. If we identify the oscillation modes of ϕ with a dark matter degree of freedom, this marks the time of dark matter production.

– Later, with $H \ll m_\phi$, we expand the complex eigen-frequency,

$$\omega = \frac{3i}{2}H \pm m_\phi\sqrt{1-\frac{9H^2}{4m_\phi^2}} \approx \frac{3i}{2}H \pm m_\phi\left(1-\frac{9H^2}{8m_\phi^2}\right) \approx \pm m_\phi + \frac{3i}{2}H$$
$$\Rightarrow \qquad \phi(t) \approx \phi_3\, e^{\pm i m_\phi t}\, e^{-3Ht/2}\,. \tag{7.44}$$

The leading time dependence of $\phi(t)$ is the oscillation. We identify its modes with the physical degrees of freedom of our light dark matter.

First, we determine the energy density of this dark matter field. According to the virial theorem, the total energy density stored in our oscillating field is twice the average potential energy

$$\rho(t) = 2V(t) = m_\phi^2|\phi(t)|^2 \to m_\phi^2\,|\phi_3|^2\, e^{-3Ht} \tag{7.45}$$

We can compare it to the variation of the scale factor for constant H given in Eq.(7.6),

$$H = \frac{\dot a(t)}{a(t)} \qquad \Rightarrow \qquad a(t) \propto e^{Ht} \qquad \Rightarrow \qquad \rho(t) \propto \frac{1}{a(t)^3} \tag{7.46}$$

The energy density is inversely proportional to the space volume element of the expanding Universe. This is exactly what we expect from a non-relativistic relic without any interaction or quantum effects.

Next, we can use Eq.(7.46) combined with the assumption of constant H to approximately relate the dark matter relic densities at the time of production with today, $a_0 = 1$,

$$\begin{aligned}
0.12 \approx \Omega_\chi h^2 = \frac{\rho_\chi}{\rho_c}\, h^2 &\overset{\text{Eq.(7.45)}}{=} \frac{m_\phi^2|\phi(t_0)|^2}{(2.5\cdot 10^{-3}\text{ eV})^4}\, h^2 \\
&\overset{\text{Eq.(7.46)}}{=} \frac{m_\phi^2|\phi(t_\text{prod})|^2}{(2.5\cdot 10^{-3}\text{ eV})^4}\, \frac{a(t_\text{prod})^3}{a_0^3}\, h^2 \\
&\overset{\text{Eq.(7.20)}}{=} \frac{m_\phi^2|\phi(t_\text{prod})|^2}{(2.5\cdot 10^{-3}\text{ eV})^4}\, \frac{T_0^3}{T_\text{prod}^3}\, h^2 \\
&\overset{\text{Eq.(7.7)}}{=} \frac{m_\phi^2|\phi(t_\text{prod})|^2}{(2.5\cdot 10^{-3}\text{ eV})^4}\, \frac{T_0^3}{(H_\text{prod} M_\text{Pl})^{3/2}}\, h^2 \\
&\overset{\text{Eq.(7.43)}}{=} \frac{m_\phi^2|\phi(t_\text{prod})|^2}{(2.5\cdot 10^{-3}\text{ eV})^4}\, \frac{T_0^3}{(m_\phi M_\text{Pl})^{3/2}}\, h^2\,.
\end{aligned} \tag{7.47}$$

Here, we ignore all order-one numbers. Taking the square root of this equation and using $T_0 = 2.4\cdot 10^{-4}$ eV gives us the relic density condition for dark matter produced by the misalignment mechanism

$$0.35 = \frac{m_\phi\phi(t_\text{prod})}{(2.5\cdot 10^{-3}\text{ eV})^2}\, \frac{T_0^{3/2}}{(m_\phi M_\text{Pl})^{3/4}}\, h \tag{7.48}$$
$$\Leftrightarrow \qquad m_\phi\phi(t_\text{prod}) \approx \frac{1/3}{2/3}\, \frac{(2.5\cdot 10^{-3}\text{ eV})^2}{(2.4\cdot 10^{-4}\text{ eV})^{3/2}}\, (m_\phi M_\text{Pl})^{3/4} \qquad \Leftrightarrow \qquad \boxed{m_\phi\phi(t_\text{prod}) \approx (m_\phi M_\text{Pl})^{3/4}\text{eV}^{1/2}}\,.$$

If the misalignment mechanism should be responsible for today's dark matter, inflation occurring after the field ϕ has picked its non-trivial starting value will have to give us the required spatial homogeneity.

Finally, we need to decide if the QCD axion from Sec. 7.4 is a valid dark matter candidate. Since the value of the axion field is not necessarily at the minimum of the potential at the time of the QCD phase transition, the axion can begin to oscillate around the minimum and the oscillation energy density contributes to the dark matter relic density. This is a special version of the misalignment mechanism. The observed relic density condition then becomes

$$m_a a(t_\text{prod}) \approx (m_a M_\text{Pl})^{3/4}\,\text{eV}^{1/2}\,. \tag{7.49}$$

If the maximum field value of the oscillation mode is given by $a(t_{\rm prod}) \approx f_a$, the relation to the pion tells us

$$m_a a(t_{\rm prod}) \approx m_a f_a \overset{(7.36)}{\approx} m_\pi f_\pi \qquad \Rightarrow \qquad \boxed{m_a \approx \frac{(m_\pi f_\pi)^{4/3}}{M_{\rm Pl}}\,{\rm eV}^{-2/3}}\,. \tag{7.50}$$

This dark matter condition and the QCD relation in Eq.(7.36) are both fulfilled for

$$m_a \approx 2\cdot 10^{-6}\,{\rm eV} \qquad \text{and} \qquad f_a \approx 10^{13}\,{\rm GeV}\,. \tag{7.51}$$

For heavier axions and smaller values of f_a, the axion can still constitute a part of the relic density. For example with a mass of $m_a = 6\cdot 10^{-5}$ eV and $f_a \approx 3\cdot 10^{11}$ GeV, axions make up one per-cent of the observed relic density.

Dark matter candidates with such low masses are hard to detect and we usually take advantage of their couplings to photons. In Eq.(7.33), there is no reason why the coupling c_γ needs to be there. However, from the perspective of the effective theory we expect all couplings which are allowed by the assumed symmetry structure to appear. Apart from finely tuned models the axion should couples to photons with an order-one coupling constant c_γ. This coupling that we rely on for axion detection also allows for the decay $a \to \gamma\gamma$. This looks dangerous for a dark matter candidate, to we can estimate the corresponding decay width,

$$\begin{aligned}\Gamma(a\to\gamma\gamma) &= \frac{\alpha^3}{256\,\pi^3}\frac{m_a^3}{f_a^2}|\,c_\gamma|^2 \\ &= \frac{1}{137^3}\frac{1}{256\,\pi^3}\frac{(6\cdot 10^{-6}\,{\rm eV})^3}{(10^{22}\,{\rm eV})^2}\,|c_\gamma|^2 \approx 1\cdot 10^{-70}\,{\rm eV}\,|c_\gamma|^2\end{aligned} \tag{7.52}$$

Assuming $c_\gamma = 1$ this corresponds to a lifetime of $\tau = 1/\Gamma \approx 2\cdot 10^{47}$ years, many orders of magnitude larger than the age of the universe.

While the axion is particularly interesting because it addresses the strong CP problem and dark matter at the same time, we can drop the relation to the CP problem and study axion-like particles (ALPs) as dark matter. For such a light pseudoscalar particle, Eq.(7.36) and (7.50) lose their QCD relation and are replaced by

$$m_\pi f_\pi \to \mu^2 \qquad \Rightarrow \qquad m_a = \frac{\mu^2}{f_a} \qquad \Rightarrow \qquad m_a = \frac{\mu^{8/3}}{M_{\rm Pl}}\,{\rm eV}^{-2/3}\,. \tag{7.53}$$

where μ is some kind of mass scale. In such models, the axion-like pseudoscalar can be very light. For example, for $\mu \approx 100$ eV, the axion mass is $m_a \approx 10^{-22}$ eV. For such a low mass and a typical velocity of $v \approx 100\,{\rm km/s}$, the de-Broglie wavelength is around 1 Kpc, the size of a galaxy. This type of dark matter is called fuzzy dark matter and can inherit the interesting properties of Bose-Einstein condensates or super-fluids.