Relativistic quantum mechanics: Dirac - Equation

Schrödinger Eq based on classical energy-momentum relation is not Lorentz invariant and not suitable to describe relativistic particles.

Klein-Gordon Eq: (historically 1st attempt to use a relativistic relation)

Using the relativistic energy-momentum relation one can obtain a wave equation which is quadratic in time/space derivatives.

$$E^2 = \vec{p}^2 + m^2$$

$$\Big\downarrow \quad E \to i\frac{\partial}{\partial t} \quad \vec{p} \to -i\vec{\nabla}$$

$$-\frac{\partial^2}{\partial t^2}\phi = -\vec{\nabla}^2\phi + m^2\phi$$

$$\to \quad \underbrace{\frac{\partial^2}{\partial t^2}\phi - \vec{\nabla}^2\phi}\ + m^2\phi = 0$$

$$(\underbrace{\Box^2}_{=\partial_\nu\partial^\nu} + m^2)\,\phi = 0$$

For a free particle w/ energy E and momentum $\vec{p}$ one can make a plane wave ansatz for $\phi(x)$

$$\phi = N\cdot e^{-ik\cdot x} = N e^{-ip\cdot x} = N\cdot e^{i(\vec{p}\vec{x} - Et)}$$

Inserting ϕ into the Klein-Gordon Eq one finds for k_0 (or E)

$$k_0 = \pm\sqrt{\vec{k}^2 + m^2} \quad \text{or for } E = \pm\sqrt{\vec{p}^2 + m^2}$$

→ One finds solutions w/ pos./neg k_0 (or respectively E):

Using $\omega_0 = +\sqrt{k^2+m^2}$: 2 solutions are:

$$\phi_1 = e^{i(\omega_0 t - \vec{k}\vec{x})} = e^{ik\cdot x}$$

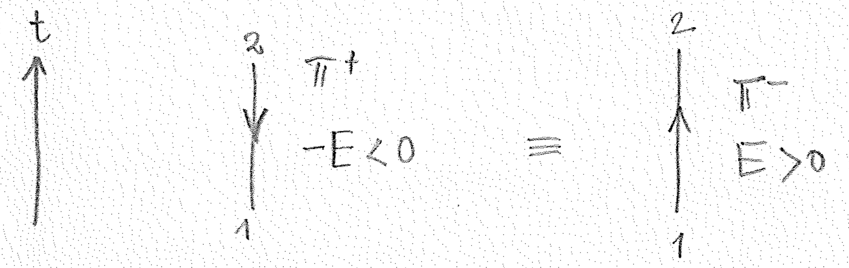
$$\phi_2 = e^{-i(\omega_0 t - \vec{k}\vec{x})} = e^{-ik\cdot x}$$

How to interprete solutions with neg. energies (needed to describe e.g. steady waves) ?

- <u>QFT</u> : Kl.-G. Eq used to describe scalar particles.
  However $\phi(x)$ are field operators : they create (pos. E_{k_0}) or annihilate (neg. E_{k_0}) particles with momentum k,

- <u>Historically</u> : people ~~who~~ tried to come-up with an interpretation for classical fields describing particles:
  Feyman - Stückelberg interpetation

Particle „current" (e.g. π^+) with $\underset{(neg. E)}{-E}$, $\vec{p}$ „propagates backward" in time, can be interpeted as the anti-particle (π^-) with $\underset{(pos\,E)}{E}$ and $\vec{p}$ propagating fw in time.



Simple reason : $e^{-i(-E)(-t)} = e^{-iEt}$

- i.e. to be valid the Kl. G. Eq requires existence of antiparticles.!

- Also beside particle/antiparticle, there is no additional degree of freedom to describe the spin:
  → only valid <u>for scalar (spin 0)</u> particles

Klein-Gordon Eq describes scalar particles

Dirac equation:

Dirac was looking for a covariant wave equation that is first order in time to avoid the above problems with „neg. energies".

$$H\psi = (\beta m + \vec{\alpha}\vec{p})\psi$$

$$\Rightarrow \quad i\frac{\partial}{\partial t}\psi = (\beta\cdot m - i\vec{\alpha}\vec{\nabla})\psi$$

If one requires that the field (here: classical field) ψ also satisfies the Kl-G.-Eq one obtains relations β and α_i must full

$$\left.\begin{aligned} &\beta^2 = \alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1 \\ &\beta\alpha_i + \alpha_i\beta = 0 \\ &\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \quad i \neq j \end{aligned}\right\}$$

obtained by squaring Dirac-Eq and requiring that ψ fulfills Kl.G.Eq

→ it turns out that β and α_i must be matrices, at least 4x4.

β and α_i are the historical notations. Today we use $\gamma^0, \gamma^1 \ldots \gamma^3$ instead:

$$\gamma^0 = \beta$$
$$\gamma^i = \beta\alpha_i \quad i = 1,2,3$$

and one can easily verify that the γ-matrices fulfill the following relations:

$$\gamma^{0\,2} = \mathbb{1}$$
$$(\gamma^k)^2 = -\mathbb{1} \quad k = 1 \ldots 3$$
$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$\left.\begin{cases} \gamma^{0\dagger} = \gamma^0 \\ \gamma^{k\dagger} = \gamma^0\gamma^k\gamma^0 = -\gamma^k \quad k = 1,2,3 \end{cases}\right\} \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

→ see text books

Although one often writes $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$, γ^μ is <u>not</u> a 4-vector (all components of γ^μ are constant matrices and invariant under Lorentz trf.) → not a 4-vector!!

Using the matrices γ^μ one can rewrite the Dirac Eq in „covariant form":

$$\underbrace{\underbrace{\left(i\gamma^\mu\partial_\mu\right.}_{\left(\ddots\right)^\mu\cdot\partial_\mu} - \underbrace{\left.m\mathbb{1}\right)}_{\left(\begin{smallmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{smallmatrix}\right)}}\psi = 0 \qquad \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$

[If one writes out all components:
$$\sum_{i=1}^{4}\left(\sum_{\mu=0}^{3}\left[i\left(\gamma^\mu\right)_{ik}\partial_\mu - m\delta_{ik}\right]\cdot\psi_k\right)$$]

the Lorentz invariance suggested by this form of writing the Dirac Eq in a „covariant" form is not at all obvious, but can be shown.

[often also using p_μ:
$$\left(\gamma^\mu p_\mu - m\right)\psi = 0$$]

Solutions of the Dirac Eq are spinors - if one chooses the 4x4 matrix representation, spinors are 4-dim column vectors $\psi = \begin{pmatrix}\psi_1\\ \vdots\\ \psi_4\end{pmatrix}$. Each of the 4-components (by construction) also fulfills the Kl-G. Eq.

Defining the <u>adjoint spinor</u> $\bar\psi = \psi^\dagger\gamma^0$ one can derive the Dirac Eq for the adjoint spinors $\bar\psi$:

$$\bar\psi\left(i\gamma^\mu\partial_\mu \,\underline{\underline{+}}\, m\right) = 0$$

[Multiplying Dirac Eq with $\bar\psi\cdot|$ and the adjoint with $|\cdot\psi$ and adding both one obtains a continuity equation for a current j^μ

$$\bar\psi\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0$$
$$+\ \bar\psi\left(i\gamma^\mu\partial_\mu + m\right)\psi = 0$$
$$\overline{\bar\psi\left(i\gamma^\mu\partial_\mu\right)\psi \quad = 0}$$

$\Rightarrow$ or $\partial_\mu\left(\bar\psi\gamma^\mu\psi\right) = 0$ w/ current $j^\mu = (\rho, \vec{j}) = \bar\psi\gamma^\mu\psi$]

Using the 4-momentum operator one can also rewrite the Dirac Eq in terms of the momentum operator:

$$(\gamma^\mu p_\mu - m)\psi = 0$$

$$\bar\psi(\gamma^\mu p_\mu + m) = 0$$

To solve the Dirac Eq, often a specific representation of the γ-matrices is used: Dirac-Pauli representation

w/ Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad i = 1 \dots 3$$

In this representation it is straightforward to show that

$$(\gamma^0)^2 = \mathbb{1} \quad (\gamma^i)^2 = -\mathbb{1} \quad \gamma^{0\dagger} = \gamma^0 \quad (\gamma^i)^\dagger = -\gamma^k$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

Remark: In the theory part of the lecture no explicit representation of γ-matrices is used. Instead only the properties of the γ^μ (Clifford Algebra) are exploited.

Here we want to use an explicit representation to discuss explicit solutions $\rightarrow$ helps in the understanding

Solutions of Dirac - Eq

Reminder: we are discussing classical fields, but the solutions also apply for field operators in QFT. → Peskin

Ansatz: plane wave describing free particle

$$\psi = u(E,\vec{p})\, e^{i(\vec{p}\vec{x} - Et)}$$

4-component spinors $u = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_4 \end{pmatrix}$

If the wave function satisfies Dirac Eq one finds that the spinor must satisfy the equation:

$$\left(\gamma^\nu p_\nu - m\mathbb{1}\right) u = 0 \qquad (*)$$

Before discussing the general solution it is instructive to look at the solution for a particle at rest, i.e. $\vec{p} = 0$:

$(*) \underset{\vec{p}=0}{\Rightarrow}$ $$\left(\gamma^0 E - m\mathbb{1}\right) u = 0$$

using $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ $$\begin{pmatrix} E\cdot\mathbb{1} & 0 \\ 0 & -E\mathbb{1} \end{pmatrix} u = \begin{pmatrix} m\cdot\mathbb{1} & 0 \\ 0 & m\mathbb{1} \end{pmatrix} u$$

⇒ There are 4 orthogonal solutions:

1) $E = m$ $\quad u_1 = N\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad u_2 = N\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

2) $E = -m$ $\quad u_3 = N\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u_4 = N\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

where N is a normalization for the wave function.

For each of the 2 energie values $E = \pm m$ always 2 solutions describe an additional degree of freedom: <u>spin</u> ↑ or ↓

General „free particle" solution:

w/ $\psi(x) = u(p)\, e^{-ip\cdot x}$ one finds for the spinor u:

$$\left(\gamma^0 E - \gamma^i p_i\right) u - m u = 0$$
$$(\gamma^\nu p_\nu - m)\, u = 0$$

Use ansatz $u = \begin{pmatrix} u_a \\ u_b \end{pmatrix}$ w/ $u_{a,b}$ are 2-comp. spinors.

$$\Rightarrow \begin{pmatrix} (E-m)\,\mathbb{1} & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\vec{p} & -(E+m)\,\mathbb{1} \end{pmatrix} \begin{pmatrix} u_a \\ u_b \end{pmatrix} = 0$$

$\Rightarrow$ leads to 2 coupled eq for u_a, u_b:

(*) $u_a = \frac{\vec{\sigma}\vec{p}}{E-m}\, u_b$ and $u_b = \frac{\vec{\sigma}\vec{p}}{E+m}\, u_a$ (**)

„neg. E" „pos. E"

Use for u_a the simplest orthogonal choice: $u_a^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_a^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

one finds from (**):

$$u_b^1 = \frac{1}{E+m} \begin{pmatrix} p_z \\ p_x + i p_y \end{pmatrix}$$

$$u_b^2 = \frac{1}{E+m} \begin{pmatrix} p_x - i p_y \\ -p_z \end{pmatrix}$$

and therefore one obtains the 2 solutions for the spinor:

$$u_1 = \begin{pmatrix} u_a^1 \\ u_b^1 \end{pmatrix} = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \end{pmatrix} \qquad u_2 = \begin{pmatrix} u_a^2 \\ u_b^2 \end{pmatrix} = N_1 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - i p_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

The 2 other solutions u_3 and u_4 can be found using Eq (*)

$$u_3 = N_2 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + i p_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \qquad u_4 = N_2 \begin{pmatrix} \frac{p_x - i p_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

The full wave functions are thus given by

$$\psi_i = u_i(E,\vec{p})\, e^{i(px-Et)}$$

In the limit $\vec{p}=0$ we obtain the spinor from before ($\vec{p}=0$)

Therefore: u_1, u_2 correspond to the positive E : $E=+\sqrt{p^2+m^2}$

u_3, u_4 correspond to negative E : $E=-\sqrt{p^2+m^2}$

Following the Feynman Stückelberg interpretation
the negative energy spinors u_3 and u_4 can be written
in terms of physical "positive energy" antiparticle spinors
v_1 and v_2 :

$$\underbrace{\begin{aligned}\psi_1 &= v_1(E,\vec{p})\, e^{-i(\vec{p}x-Et)}\\ \psi_2 &= v_2(E,\vec{p})\, e^{-i(\vec{p}x-Et)}\end{aligned}}_{\text{physical energy}} \quad \underbrace{\begin{aligned}&= u_4(-E,-\vec{p})\, e^{i(-px-(-E)t)}\\ &= u_4(-E,-\vec{p})\, e^{i(-\vec{p}x-(-E)t)}\end{aligned}}_{\text{unphysical solutions}}$$

Using $\psi_{1,2}$ one obtains the Dirac Eq for the antiparticle spinors

$$(\gamma^\mu p_\mu + m)\, v = 0$$

from which one obtains in the same way:

$$v_1 = N_1 \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = N_1 \begin{pmatrix} \frac{+p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

We have the following 4 solutions of the Dirac Eq:

$$\left.\begin{aligned}\psi_{1,2}(x) &= u_{1,2}\, e^{+i(px-Et)} && \text{for the particles}\\ \psi_{3,4}(x) &= v_{1,2}\, e^{-i(px-Et)} && \text{for the antiparticles}\end{aligned}\right\} \; N_i = \sqrt{E+m} \text{ resulting from } u^2 \text{ or } v^2$$

They describe particle/antiparticles w/ spin 1/2.