3 Electroweak theory

The fundamental structure describing the interactions between the elementary particles of the Standard Model is gauge interaction. We have already looked at QED, the interaction of leptons and quarks with a photon, leading to interactions between fermion currents. We need to have a closer look at this gauge symmetry structure, before we can describe for instance the weak interaction.

3.1 QED gauge invariance

Even though we already know how to use Feynman rule to compute QED scattering amplitudes, and we even have an idea how these Feynman rules are related to the underlying quantum field theory, let us have another look at the QED Lagrangian. We already know that the Lagrangian in Eq.(1.6) describes the quantum version of the photons from electrodynamics,

$$\mathscr{L}_{\text{photon}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \text{with} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \; , \tag{3.1}$$

The field strength is already symmetric under the local gauge transformation with a space-time-dependent parameter $\alpha(x)$,

$$\begin{aligned} A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha \qquad \Rightarrow \qquad F_{\mu\nu} &\to \partial_\mu \left(A_\nu - \frac{1}{e}\partial_\nu \alpha \right) - \partial_\nu \left(A_\mu - \frac{1}{e}\partial_\mu \alpha \right) \\ &= F_{\mu\nu} - \frac{1}{e}\left(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\right)\alpha = F_{\mu\nu} \; . \end{aligned} \tag{3.2}$$

We know this shift symmetry from electrodynamics, but it is not clear what α really means. This becomes clear when we introduce a generic fermion spinor ψ as in Eq.(1.24),

$$\begin{aligned} \mathscr{L}_{\text{fermion-photon}} &= \overline{\psi}\left(i\not{D} - m\mathbb{1}\right)\psi \\ &\equiv \overline{\psi}\left(i(\not\partial + ieq\not{A}) - m\mathbb{1}\right)\psi \; , \end{aligned} \tag{3.3}$$

where q is the fermion charge in units of the electron charge. We can rotate the fermion field by the angle $\alpha(x)$,

$$\begin{aligned} \psi &\to e^{iq\alpha}\psi \\ \overline{\psi} &\to e^{-iq\alpha}\overline{\psi} \; , \end{aligned} \tag{3.4}$$

such that the fermion mass term is symmetric in itself, which means QED has no problem with massive fermions. The kinetic term with the covariant derivative transforms into

$$\begin{aligned} i\overline{\psi}\not{D}\psi \equiv i\overline{\psi}\gamma^\mu(\partial_\mu + ieqA_\mu)\psi \to i\overline{\psi}e^{-iq\alpha}\gamma^\mu &\left(\partial_\mu + ieq\left(A_\mu - \frac{1}{e}\partial_\mu\alpha\right)\right)e^{iq\alpha}\psi \\ &= i\overline{\psi}\gamma^\mu e^{-iq\alpha}\left(\partial_\mu e^{iq\alpha} + e^{iq\alpha} ieq\left(A_\mu - \frac{1}{e}\partial_\mu\alpha\right)\right)\psi \\ &= i\overline{\psi}\gamma^\mu e^{-iq\alpha}\left(e^{iq\alpha} iq\partial_\mu\alpha + e^{iq\alpha}\partial_\mu + e^{iq\alpha} ieq\left(A_\mu - \frac{1}{e}\partial_\mu\alpha\right)\right)\psi \\ &= i\overline{\psi}\gamma^\mu\left(iq\partial_\mu\alpha + \partial_\mu + ieqA_\mu - iq\partial_\mu\alpha\right)\psi \\ &= i\overline{\psi}\gamma^\mu\left(\partial_\mu + ieqA_\mu\right)\psi \\ &\equiv i\overline{\psi}\not{D}\psi \; . \end{aligned} \tag{3.5}$$

The combined QED Lagrangian is invariant under a local rotation of the fermion field(s), if we also shift the photon field the way we know it from electrodynamics. Already there, this shift was referred to as a gauge transformation. In combination, we say that the QED interaction between photons and fermions is defined by a local $U(1)$ gauge transformation of both fields.

To understand why the covariant derivative is useful, we look at the first and last lines of Eq.(3.5). We immediately see that the gauge transformation of the covariant derivative is

$$\not{D} \to e^{iq\alpha} \not{D} e^{-iq\alpha} \; . \tag{3.6}$$

We can also replace the definition of the field strength in terms of the gauge field by a definition in terms of the covariant derivative, for example acting on a test function $f(x)$ with $[A, f] = 0$,

$$\begin{aligned} F_{\mu\nu}\, f &= \frac{1}{ieq}\, [D_\mu, D_\nu]\, f \\ &= \frac{1}{ieq}\, (\partial_\mu + ieqA_\mu)(\partial_\nu + ieqA_\nu)\, f - \frac{1}{ieq}\, (\partial_\nu + ieqA_\nu)(\partial_\mu + ieqA_\mu)\, f \\ &= (\partial_\mu A_\nu) f + A_\nu(\partial_\mu f) + A_\mu(\partial_\nu f) - (\partial_\nu A_\mu) f - A_\mu(\partial_\nu f) - A_\nu(\partial_\mu f) \\ &= ((\partial_\mu A_\nu) f - (\partial_\nu A_\mu))\, f \; . \end{aligned} \tag{3.7}$$

In this form the partial derivative acts only on the gauge field, so unlike the first line the definition in the last line is not an operator equation. In this derivation we assume that the gauge field commutes, which we call abelian,

$$[A_\mu, A_\nu] = 0 \; . \tag{3.8}$$

3.2 Massive gauge bosons

One of the shortcomings of QED is that it only defines long-range interactions. From the work of Hideki Yukawa in 1935 we know that the mass of the exchange particles changes the form of the interaction potential in Fourier space,

$$\begin{aligned} V(r) &= -\frac{e^2}{r} && \text{massless particle exchange} \\ V(r) &= -g^2\, \frac{e^{-mr}}{r} && \text{massive particle exchange with } m. \end{aligned} \tag{3.9}$$

Yukawa did not actually talk about the weak nuclear force at the quark level. His model was based on fundamental protons and neutrons, and his exchange particles were pions. But his argument applies perfectly to the electroweak Fermi interaction between quarks. This leads to the challenge of formally including a photon mass in the gauge-invariant QED Lagrangian.

The first step towards defining a massive version of QED is to include a photon mass in the kinematic Lagrangian of Eq.(3.1). We immediately see that just adding a photon mass term

$$\frac{1}{2} m^2 A^2 \to \frac{1}{2} m^2 \left(A_\mu - \frac{1}{e} \partial_\mu \alpha \right)^2 \tag{3.10}$$

is not allowed by the gauge symmetry. The key idea is to add an innocent looking real scalar field without a mass and without a coupling to the photon, but with a scalar–photon mixing term and a well-chosen gauge transformation. The result is called the Boulware–Gilbert model or Stückelberg mass generation,

$$\begin{aligned} \mathscr{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} e^2 f^2 A_\mu^2 + \frac{1}{2} \left(\partial_\mu \phi\right)^2 - ef A_\mu \partial^\mu \phi \\ &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} e^2 f^2 \left(A_\mu - \frac{1}{ef} \partial_\mu \phi \right)^2 \; , \end{aligned} \tag{3.11}$$

where f is a common mass scale for the photon mass and the mixing. It ensures that all terms in the Lagrangian have mass dimension four — remembering that bosonic fields like A_μ and ϕ have mass dimension one. If we define the massive photon field as

$$\tilde{A}_\mu = A_\mu - \frac{1}{ef}\partial_\mu \phi \tag{3.12}$$

the field strength does not change,

$$\begin{aligned} F_{\mu\nu}\Big|_{\tilde{A}} &= \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = \partial_\mu \left(A_\nu - \frac{1}{ef}\partial_\nu \phi \right) - \partial_\nu \left(A_\mu - \frac{1}{ef}\partial_\mu \phi \right) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}\Big|_A , \end{aligned} \tag{3.13}$$

and we can write the Lagrangian of Eq.(3.11) as

$$\begin{aligned} \mathscr{L} &= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} e^2 f^2 \tilde{A}_\mu^2 \\ &= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} m_A^2 \tilde{A}_\mu^2 \qquad \text{with} \qquad m_A = ef \; . \end{aligned} \tag{3.14}$$

For a gauge invariant theory, a suitable gauge transformation of the scalar field has to cancel the contribution of the explicit mass term in Eq.(3.11). The simple choice

$$\phi \longrightarrow \phi - f\alpha \; . \tag{3.15}$$

indeed gives us

$$A_\mu - \frac{1}{ef}\partial_\mu \phi \to A_\mu - \frac{1}{e}\partial\alpha - \frac{1}{ef}\partial_\mu \phi + \frac{1}{ef}\partial_\mu (f\alpha) = A_\mu - \frac{1}{ef}\partial_\mu \phi \tag{3.16}$$

This Lagrangian describes a massive photon field $\tilde{A}_\mu$, which has absorbed the real scalar ϕ as its additional longitudinal component. This is because a massless gauge boson A_μ has only two on–shell degrees of freedom, the left-handed and right-handed polarizations, while the massive $\tilde{A}_\mu$ has an additional longitudinal polarization. To describe it, the massive photon $\tilde{A}$ has 'eaten' the real scalar field ϕ.

What kind of properties does this field ϕ need to have, so that we can use it to provide a photon mass? From the gauge transformation we immediately see that any additional purely scalar term in the Lagrangian, like a scalar potential $V(\phi)$, needs to be symmetric under the shift $\phi \to \phi - f\alpha$, so it does not spoil gauge invariance. This means that we cannot write down polynomial terms ϕ^n, like a mass, a self coupling, or an interaction term ϕAA. Only derivative interactions proportional to $\partial\phi$ attached to gauge-invariant currents are allowed. For them, we the shift by α turns into a total derivative in the Lagrangian.

This example illustrates a few vital properties of Nambu–Goldstone bosons (NGB). Such massless physical states appear in many areas of physics and are described by Goldstone's theorem. It applies to global continuous symmetries of the Lagrangian which are violated by a non–symmetric vacuum state, a mechanism called spontaneous symmetry breaking. Based on Lorentz invariance and states with a positively definite norm we can then prove: *If a global symmetry group is spontaneously broken into a group of lower rank, its broken generators correspond to physical Goldstone modes. These scalar fields transform non–linearly under the larger and linearly under the smaller group. This way they are massless and cannot form a potential, because the non–linear transformation only allows derivative terms in the Lagrangian.*

For our massive QED case we are breaking the $U(1)$ gauge symmetry, which naively introduces a massless scalar degree of freedom ϕ. Following Eq.(3.12) it provides the missing longitudinal polarization for the massive photon. This combines two problems into one solution — we can break the gauge symmetry without creating unobserved massless particles, and our massive photon gets an additional degree of freedom. We will use the same trick for the Higgs later.

3.3 Fermion doublets

The structural element of the Fermi theory and also of the Standard Model is the $SU(2)$ doublet structure of paired fermions, like protons and neutrons. If a common gauge transformation should link the two double components, an obvious choice is to replace the local $U(1)$ gauge invariance by a local $SU(2)$ gauge invariance. We remind ourselves that an $SU(2)$ transformation is very similar to an $O(3)$ rotation, and the generators of the $SU(2)$ transformation are the Pauli matrices. This means that a representation of $SU(2)$ transformations is given by

$$U = e^{i\alpha_a \tau_a/2} \qquad \text{with} \qquad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.17}$$

They satisfy the relation

$$\tau_a \tau_b = \delta_{ab} + i\epsilon_{abc}\tau_c \qquad \Leftrightarrow \qquad [\tau_a, \tau_b] = 2i\epsilon_{abc}\tau_c \; . \tag{3.18}$$

This means that we can write a commutation of two objects with $SU(2)$ indices as an object with one $SU(2)$ index. For later use we also need a sum rule for the Pauli matrices $\tau_{1,2,3}$,

$$\sum_{a,b} \tau_a \tau_b = \sum_{a,b} (\delta_{ab} + i\epsilon_{abc}\tau_c) = \sum \delta_{ab} + i \sum_{a \neq b} \epsilon_{abc}\tau_c = \sum \delta_{ab} + i \sum_{a<b} (\epsilon_{abc} + \epsilon_{bac})\, \tau_c = \sum \delta_{ab} \; . \tag{3.19}$$

The basis of three Pauli matrices we can write in terms of $\tau_{1,2,3}$ or in terms of $\tau_{+,-,3}$ with

$$\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.20}$$

From the QED Lagrangian we know how to describe fermion masses and interactions with gauge bosons in a dimension-4 Lagrangian. Before we move on, let us briefly look at the dimensionality of the Lagrangian, so we can use it to structure the weak Lagrangian.

The dimensionality is crucial for the renormalizability or fundamental nature of a Lagrangian. The definition of the transition in terms of the interaction Hamiltonian in Eq.(1.29),

$$\mathcal{S} \sim \exp\left[-i \int dt \mathcal{H}(t) \right] = \exp\left[-i \int d^4x \, \left(\pi \dot{\phi} - \mathscr{L} \right) \right] \; , \tag{3.21}$$

suggests that the Hamiltonian should have the unit energy or momentum, and the Lagrangian should have mass dimension four. The elements of the Lagrangian have the mass dimensions.

$$\begin{array}{lr} \text{scalar field} & [\phi] = M \\ \text{explicit mass} & [m] = M \\ \text{gauge field} & [A_\mu] = M \\ \text{space derivative} & [\partial_\mu] = M \\ \text{field strength} & [F_{\mu\nu}] = M^2 \\ \text{fermion spinor} & [\psi] = M^{3/2} \; . \end{array} \tag{3.22}$$

If we want our theory to be renormalizable and valid to arbitrarily large scales, the Lagrangian cannot include inverse masses, because in that case a momentum or energy in the numerator would eventually lead to an exploding ratio in the ultraviolet. This means all terms in a fundamental Lagragian should have mass dimension two to four, complemented by explicit masses. This is the way we will organize the weak Lagrangian.

As a starting point, the interaction of fermions, our case quarks, with gauge bosons is most easily written in terms of covariant derivatives. Just like the kinetic term for the gauge bosons, they have mass dimension four,

$$\mathscr{L}_{D4} = \overline{Q}_L i \not{D} Q_L + \overline{Q}_R i \not{D} Q_R - \frac{1}{4} F_{a,\mu\nu} F_a^{\mu\nu} \ldots \tag{3.23}$$

From Eq.(3.7) we know that the covariant derivatives can be used to describe the field strengths. However, going from the abelian $U(1)$ transformation to the non-abelian $SU(2)$ transformation the condition $[A_\mu, A_\nu] = 0$ is not true anymore. The good news is that the definition of the field strength in terms of the covariant derivative still holds for the non-ablelian case,

$$\begin{aligned} F_{a,\mu\nu} &\equiv \frac{1}{ieq}[D_\mu, D_\nu]_a \\ &= \partial_\mu A_{a,\nu} - \partial_\nu A_{a,\mu} + ieq[A_\mu, A_\nu]_a \ , \end{aligned} \tag{3.24}$$

now with the $SU(2)$-index a. The gauge invariance of the field strength follows conveniently from Eq.(3.7).

3.4 Weak gauge bosons

The main theme of the electroweak theory is that it combines our known QED with the $U(1)$ gauge transformation of the physical photon and the fermion singlet with the $SU(2)$ gauge transformations of the gauge bosons and fermion doublets. In this combination there appears a mixing which we can describe in two different ways:

1. The neutral gauge bosons can mix, so the observed mass eigenstates are the photon A_μ and the massive Z_μ, but they are related to the interaction eigenstate W^3_μ and a massless B_μ. We can think of the B_μ as the photon of a proto-QED before we combine QED with the weak symmetry to the actual QED. This description leads to the weak mixing angle.
2. The $U(1)$ rotation of the fermion fields and the neutral $SU(2)$ rotation via τ_3 can be combined to individual $U(1)$ rotations of the left-handed and right-handed fermion spinors. This desciption leads to the Gell-Mann–Nishijima formula.

Starting with the mixing from interaction eigenstates to mass eigenstates, we assume that the two ingredients to the neutral electroweak interactions are a massless proto-photon B_μ and the neutral electroweak gauge boson field W^3_μ. Both are neutral particles with the same quantum numbers, so they can mix to the mass eigenstates A_μ and Z_μ,

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} c_w & s_w \\ -s_w & c_w \end{pmatrix} \begin{pmatrix} B_\mu \\ W^3_\mu \end{pmatrix} \qquad \text{with} \qquad s_w \equiv \sin\theta_w \quad c_w \equiv \cos\theta_w \ . \tag{3.25}$$

The photon describes the $U(1)$ charge transformation and couples to electric charge. The field B_μ couples to the so-called $U(1)$ hypercharge y. The angle θ_w is the weak mixing angle or Weinberg angle. Both B_μ and A_μ are massless, because the determinant of the mass matrix keeps its zero eigenvalue after rotations.

Unlike QED, the weak interaction knows about the chirality of the fermion fields, so we have to distinguish $\not{D}_L$ and $\not{D}_R$. In the interaction basis the covariant derivatives include the massless B_μ and the three massive $W_{a,\mu}$,

$$\begin{aligned} D_{L\mu} &= \partial_\mu + ig' \frac{y}{2} B_\mu + ig \sum_{a=1,2,3} W_{a,\mu} \frac{\tau_a}{2} \\ D_{R\mu} &= D_{L\mu} \Big|_{\tau_{1,2,3}=0} \ . \end{aligned} \tag{3.26}$$

This definition implies that our $SU(2)$ gauge transformation only act on the left-handed doublets, so we refer to it as $SU(2)_L$. The right-handed fields can be written as ntuples, but they do not have a doublet structure under $SU(2)$. The effect of this is structure is that the massive charged W-boson only couples doublet like (u_L, d_L).

In the mass basis for the neutral states the covariant derivative from Eq.(3.26) has to read

$$D_{L\mu} = \partial_\mu + ieqA_\mu + ig_Z \left(-q s_w^2 \mathbb{1} + \frac{\tau_3}{2} \right) Z_\mu + ig \left(\frac{\tau_1}{2} W^1_\mu + \frac{\tau_2}{2} W^2_\mu \right) \tag{3.27}$$

We omit the relations between the couplings e and g_Z to g' and g and the weak mixing angle. The relation of y to q and the τ_3 eigenvalues will be discussed later.

The mass basis of the charged weak bosons is most conveniently written in terms of $\tau_\pm$. To switch bases we only have to make sure we keep the standard normalization of all fields,

$$\begin{aligned}
\sqrt{2}\left(\tau_+ W_\mu^+ + \tau_- W_\mu^-\right) &= \sqrt{2}\begin{pmatrix} 0 & W_\mu^+ \\ 0 & 0 \end{pmatrix} + \sqrt{2}\begin{pmatrix} 0 & 0 \\ W_\mu^- & 0 \end{pmatrix} \\
&\stackrel{!}{=} \tau_1 W_\mu^1 + \tau_2 W_\mu^2 = \begin{pmatrix} 0 & W_\mu^1 \\ W_\mu^1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iW_\mu^2 \\ iW_\mu^2 & 0 \end{pmatrix} \\
\Leftrightarrow \qquad & W_\mu^+ = \frac{1}{\sqrt{2}}\left(W_\mu^1 - iW_\mu^2\right) \qquad\qquad W_\mu^- = \frac{1}{\sqrt{2}}\left(W_\mu^1 + iW_\mu^2\right) \\
\Rightarrow \qquad & D_{L\mu} = \partial_\mu + ieqA_\mu + ig_Z\left(-qs_w^2 + \frac{\tau_3}{2}\right)Z_\mu + i\frac{g}{\sqrt{2}}\left(\tau_+ W_\mu^+ + \tau_- W_\mu^-\right)
\end{aligned} \tag{3.28}$$

To check this mass basis, we look at the masses of the gauge bosons, which appear as dimension-2 mass terms in the electroweak Lagrangian. Using the above relation, we write them in terms of the charged W-fields,

$$\mathscr{L}_{D2} = \frac{m_W^2}{2}\left(W^{1,\mu}W_\mu^1 + W^{2,\mu}W_\mu^2\right) + \frac{m_Z^2}{2}Z^\mu Z_\mu = m_W^2 W^{+,\mu}W_\mu^- + \frac{m_Z^2}{2}Z^\mu Z_\mu \; . \tag{3.29}$$

Now both mass terms are proportional to the field combination $\phi^*\phi = |\phi|^2$, as we know it from scalars. The relative factor two in front of the W mass appears because the Z field is neutral and the W field is charged, again the same as for neutral and charged scalars.

Finally, we look at the Lagrangian terms describing the fermion masses with mass dimension three. From Eq.(1.50) we know that it requires a combination of the left-handed doublet Q_L and the right-handed singlet fields Q_R,

$$\mathscr{L}_{D3} = -\overline{Q}_L m_Q Q_R + ... \tag{3.30}$$

This form will require a doublet structure of the Higgs–Goldstone fields, which we will discuss next term. For now we ignore this complication. Moreover, these mass terms can be matrices in generation space, which implies that we might have to rotate the fermion fields from an interaction basis into the mass basis, where these mass matrices are diagonal. Flavor physics dealing with such 3×3 mass matrices is its own field of physics. We will also omit this complication for now.

The problem with the mass term in Eq.(3.30) is that they are not invariant under the $SU(2)_L$ gauge transformation $U(x)$, which only transforms the left-handed fermion fields

$$Q_L \stackrel{U}{\to} UQ_L \qquad\qquad Q_R \stackrel{U}{\to} Q_R \; . \tag{3.31}$$

Obviously, there is no way we can make left–right mixing fermion mass terms in Eq.(3.30) invariant under a left-handed $SU(2)_L$ gauge transformation,

$$\overline{Q}_L m_Q Q_R \stackrel{U}{\to} \overline{Q}_L U^{-1} m_Q Q_R \neq \overline{Q}_L m_Q Q_R \; . \tag{3.32}$$

To see what we need to add to make fermion masses consistent with the electroweak gauge symmetry, we need to combine the local $U(1)$ transformations with the neutral component of the $SU(2)_L$ transformation

$$V = e^{i\beta\tau_3/2} \; . \tag{3.33}$$

From our QED calculation we know that the gauge transformation of the fermion fields is related to the form of the covariant derivative. Equation (3.28) then tell us how to combine V with the $U(1)$ charge transformation to a $U(1)$

hypercharge transformation. This means we need to evaluate

$$\begin{aligned}
\exp(i\beta q)\; V^\dagger &= \exp(i\beta q)\;\exp\left(-\frac{i}{2}\beta\tau_3\right) && \text{with}\quad V = U(x)\Big|_{\tau_3} = \exp\left(\frac{i}{2}\beta\tau_3\right) \\
&= \exp\left(i\beta\frac{y\mathbb{1}+\tau_3}{2}\right)\;\exp\left(-\frac{i}{2}\beta\tau_3\right) && \text{with}\quad q\mathbb{1} \equiv \frac{y\mathbb{1}+\tau_3}{2} \qquad y_Q = \frac{1}{3} \quad y_L = -1 \\
&= \exp\left(i\frac{\beta}{2}y\mathbb{1}\right)\; . && (3.34)
\end{aligned}$$

The relation between the charge q, the hypercharge y, and the isospin τ_3 is the same as in Eq.(3.28), called the Gell-Mann–Nishijima formula. The indices Q and L denote quark and lepton doublets. In this notation we do not distinguish $U(1)$ rotations by a real angle and $SU(2)$ rotations proportional to a unit matrix. This is reflected in the Gell-Mann–Nishijima formula, where τ_3 has to be replaced by its eigenvalue ± 1 for up–type and down–type fermions to relate charge and hypercharge.

In analogy to Eq.(3.31) left-handed and right-handed fermion spinors transform under the combination of V and the two $U(1)$, now denoted as V, as

$$\begin{aligned}
Q_L &\xrightarrow{V} \exp\left(i\beta q_Q\right) V^\dagger Q_L = \exp\left(i\frac{\beta}{2}y_Q\mathbb{1}\right) Q_L \\
Q_R &\xrightarrow{V} \exp\left(i\beta q_Q\right) Q_R\; . \qquad (3.35)
\end{aligned}$$

The right-handed fermions only see the electric charge.

3.5 Sigma model

One way of solving this problem with weak gauge invariance of fermion mass terms is to introduce an additional field $\Sigma(x)$. This field will a similar role as the real scalar field we used for the photon mass generation. Its physical properties will become clear piece by piece from the way it appears in the Lagrangian and from the required gauge invariance.

First, we introduce Σ into the fermion mass term. This will tell us what it takes to make this mass term gauge invariant under the weak transformations defined in Eqs.(3.31) and (3.35)

$$\begin{aligned}
\overline{Q}_L\Sigma m_Q Q_R &\xrightarrow{U} \overline{Q}_L U^{-1}\Sigma^{(U)} m_Q Q_R \overset{!}{=} \overline{Q}_L\Sigma m_Q Q_R \\
\Leftrightarrow &\qquad \Sigma \to \Sigma^{(U)} = U\Sigma\; . \qquad (3.36)
\end{aligned}$$

and

$$\begin{aligned}
\overline{Q}_L\Sigma m_Q Q_R &\xrightarrow{V} \overline{Q}_L \exp\left(-i\frac{\beta}{2}y\mathbb{1}\right)\Sigma^{(V)} m_Q \exp\left(i\beta q\right) Q_R \\
&= \overline{Q}_L\Sigma^{(V)}\exp\left(-i\frac{\beta}{2}y\mathbb{1}\right)\;\exp\left(i\beta q\right) m_Q Q_R && \exp\left(i\frac{\beta}{2}y\mathbb{1}\right)\ \text{always commuting} \\
&= \overline{Q}_L\Sigma^{(V)} V m_Q Q_R \\
&\overset{!}{=} \overline{Q}_L\Sigma\, m_Q Q_R \qquad \Leftrightarrow \qquad \Sigma \to \Sigma^{(V)} = \Sigma V^\dagger\; . && (3.37)
\end{aligned}$$

Combining both gives us the needed transformation property

$$\Sigma \to U\Sigma V^\dagger\; . \qquad (3.38)$$

We see that Σ is a 2×2 matrix with mass dimension zero. The fermion mass Lagrangian is gauge invariant without specifying anything about the relation of Σ with propagating or physical fields

$$\mathscr{L}_{D3} = -\overline{Q}_L\Sigma m_Q Q_R - \overline{L}_L\Sigma m_L L_R + \text{h.c.} + \ldots \qquad (3.39)$$

In a second step, we use Σ to introduce gauge boson masses. We recall the covariant derivative from Eq.(3.26),

$$D_{L\mu} = \partial_\mu + ig' \frac{y}{2} B_\mu + igW_{a,\mu} \frac{\tau_a}{2} \; . \tag{3.40}$$

We first choose the form of the covariant derivative acting on Σ,

$$\begin{aligned} D_\mu \Sigma &= \partial_\mu \Sigma + ig' \Sigma B_\mu \frac{y}{2}\Big|_{q=0} + igW_{a,\mu} \frac{\tau_a}{2} \Sigma \\ &= \partial_\mu \Sigma - ig' \Sigma B_\mu \frac{\tau_3}{2} + igW_{a,\mu} \frac{\tau_a}{2} \Sigma \; , \end{aligned} \tag{3.41}$$

With the abbreviations

$$V_\mu \equiv \Sigma (D_\mu \Sigma)^\dagger \qquad \text{and} \qquad T \equiv \Sigma \tau_3 \Sigma^\dagger \; , \tag{3.42}$$

we will show in Sec. 3.6 that we can write the gauge boson mass Lagrangian as

$$\mathscr{L}_{D2} = -\frac{v^2}{4} \; \text{Tr}[V_\mu V^\mu] - \Delta\rho \frac{v^2}{8} \; \text{Tr}[T V_\mu] \; \text{Tr}[T V^\mu] \; . \tag{3.43}$$

The trace acts on the 2×2 $SU(2)$ matrices. The parameter $\Delta\rho$ is conventional and will be the focus of Section 3.8.

Before we compute the weak boson masses, we see which gauge invariant terms of mass dimension four we can write down using Σ. Our first attempt for a building block

$$\Sigma^\dagger \Sigma \stackrel{U,V}{\to} (U\Sigma V^\dagger)^\dagger (U \Sigma V^\dagger) = V \Sigma^\dagger U^\dagger U \Sigma V^\dagger = V \Sigma^\dagger \Sigma V^\dagger \neq \Sigma^\dagger \Sigma \tag{3.44}$$

is forbidden by invariance under Eq.(3.38). However, a circular trace $\text{Tr}(\Sigma^\dagger\Sigma) \to \text{Tr}(V\Sigma^\dagger\Sigma V^\dagger) = \text{Tr}(\Sigma^\dagger\Sigma)$ allows for the additional potential terms, meaning terms with no derivatives

$$\mathscr{L}_\Sigma = -\frac{\mu^2 v^2}{4} \text{Tr}(\Sigma^\dagger \Sigma) - \frac{\lambda v^4}{16} \left(\text{Tr}(\Sigma^\dagger \Sigma) \right)^2 + \cdots \; , \tag{3.45}$$

with properly chosen prefactors μ, v, λ. This finalizes our construction of the weak Lagrangian organized by mass dimension,

$$\mathscr{L} = \mathscr{L}_{D2} + \mathscr{L}_{D3} + \mathscr{L}_{D4} + \mathscr{L}_\Sigma \; . \tag{3.46}$$

3.6 Weak boson masses

To check that Eq.(3.43) gives the correct masses in the Standard Model we assume that Σ acquires a vacuum expectation value. The simplest way to achieve this and obtain the correct fermion masses is to just write

$$\Sigma(x) = \mathbb{1} \; . \tag{3.47}$$

This choice is called unitary gauge. It looks like a dirty trick to first introduce $\Sigma(x) = \mathbb{1}$ and then use this field for a gauge invariant implementation of gauge boson masses. Clearly, a constant does not exhibit the correct transformation property under the U and V symmetries, but we can always work in a specific gauge and only later check the physical predictions for gauge invariance.

We now check $\mathscr{L}_{D2}$ as written in Eq.(3.43) for the correct gauge boson masses. Using the covariant derivative from Eq.(3.41) acting on a now constant field we can compute V_μ in unitary gauge

$$\begin{aligned} V_\mu = \Sigma (D_\mu \Sigma)^\dagger &= \mathbb{1} (D_\mu \Sigma)^\dagger \\ &= -igW_\mu^+ \frac{\tau_+}{\sqrt{2}} - igW_\mu^- \frac{\tau_-}{\sqrt{2}} - igW_\mu^3 \frac{\tau_3}{2} + ig' B_\mu \frac{\tau_3}{2} \\ &= -i \frac{g}{\sqrt{2}} \left(W_\mu^+ \tau_+ + W_\mu^- \tau_- \right) - ig_Z Z_\mu \frac{\tau_3}{2} \; , \end{aligned} \tag{3.48}$$

with $Z_\mu = c_w W_\mu^3 - s_w B_\mu$ and the two coupling constants

$$g_Z = \frac{g}{c_w} \qquad \text{and} \qquad g' = \frac{g s_w}{c_w} \; . \tag{3.49}$$

This gives us the first of the two terms in $\mathscr{L}_{D2}$ using $\tau_\pm^2 = 0$ and $\mathrm{Tr}(\tau_3 \tau_\pm) = 0$,

$$\begin{aligned} \mathrm{Tr}[V_\mu V^\mu] &= -2\,\frac{g^2}{2} W_\mu^+ W^{-\mu}\,\mathrm{Tr}(\tau_+\tau_-) - \frac{g_Z^2}{4} Z_\mu Z^\mu\,\mathrm{Tr}(\tau_3^2) \\ &= -g^2 W_\mu^+ W^{-\mu} - \frac{g_Z^2}{2} Z_\mu Z^\mu \; , \end{aligned} \tag{3.50}$$

In the second step we use $\mathrm{Tr}(\tau_\pm \tau_\mp) = 1$, and $\mathrm{Tr}(\tau_3^2) = \mathrm{Tr}\,\mathbb{1} = 2$. The mass term proportional to $\Delta\rho$ also simplifies in unitary gauge

$$\begin{aligned} & T = \Sigma \tau_3 \Sigma^\dagger = \tau_3 \\ \Rightarrow \qquad & \mathrm{Tr}(T V_\mu) = \mathrm{Tr}\left(-i g_Z Z_\mu \frac{\tau_3^2}{2}\right) = -i g_Z Z_\mu \\ \Rightarrow \quad & \mathrm{Tr}(T V_\mu)\ \mathrm{Tr}(T V^\mu) = -g_Z^2 Z_\mu Z^\mu \; . \end{aligned} \tag{3.51}$$

Combining both terms with the prefactor in Eq.(3.43) yields the complete gauge boson mass term

$$\begin{aligned} \mathscr{L}_{D2} &= -\frac{v^2}{4}\left(-g^2 W_\mu^+ W^{-\mu} - \frac{g_Z^2}{2} Z_\mu Z^\mu\right) - \Delta\rho \frac{v^2}{8}\left(-g_Z^2 Z_\mu Z^\mu\right) \\ &= \frac{v^2 g^2}{4} W_\mu^+ W^{-\mu} + \frac{v^2 g_Z^2}{8}\left(1 + \Delta\rho\right) Z_\mu Z^\mu \; . \end{aligned} \tag{3.52}$$

Identifying the masses with the form given in Eq.(3.29) and assuming universality of neutral and charged current interactions ($\Delta\rho = 0$) we find

$$\begin{aligned} m_W &= \frac{g v}{2} \\ m_Z &= \sqrt{1 + \Delta\rho}\ \frac{g_Z v}{2} \stackrel{\Delta\rho=0}{=} \frac{g_Z v}{2} = \frac{g v}{2 c_w} \; . \end{aligned} \tag{3.53}$$

A possible additional and unwanted Z-mass contribution $\Delta\rho$ will come back in Sec. 3.8. From the known gauge boson masses ($m_W \sim 80$ GeV) and weak coupling ($g \sim 0.7$) we find $v \sim 246$ GeV.

3.7 Weak boson propagators

Finally, let us at least mention different gauge choices and the appearance of Goldstone modes. If we break the full electroweak gauge symmetry $SU(2)_L \times U(1)_Y \to U(1)_Q$ we expect three Goldstone bosons which become part of the weak gauge bosons and promote those from massless gauge bosons (with two degrees of freedom each) to massive gauge bosons (with three degrees of freedom each). This is the point of view of the unitary gauge, in which we never see Goldstone modes.

In the general renormalizable R_ξ gauge we can actually see the Goldstone modes in the gauge boson propagators

$$\begin{aligned} \Delta_{VV}^{\mu\nu}(q) &= \frac{-i}{q^2 - m_V^2 + i\epsilon}\left[g^{\mu\nu} + (\xi - 1)\frac{q^\mu q^\nu}{q^2 - \xi m_V^2}\right] \\ &= \begin{cases} \dfrac{-i}{q^2 - m_V^2 + i\epsilon}\left[g^{\mu\nu} - \dfrac{q^\mu q^\nu}{m_V^2}\right] & \text{unitary gauge } \xi \to \infty \\ \dfrac{-i}{q^2 - m_V^2 + i\epsilon}\, g^{\mu\nu} & \text{Feynman gauge } \xi = 1 \\ \dfrac{-i}{q^2 - m_V^2 + i\epsilon}\left[g^{\mu\nu} - \dfrac{q^\mu q^\nu}{q^2}\right] & \text{Landau gauge } \xi = 0 \; . \end{cases} \end{aligned} \tag{3.54}$$

If these gauge choices are physically equivalent, something has to compensate for the fact that in Feynman gauge the whole Goldstone term vanishes and the polarization sum looks like a massless gauge boson, while in unitary gauge we can see the effect of these modes. This is done by the Goldstone propagator

$$\Delta_{VV}(q^2) = \frac{-i}{q^2 - \xi m_V^2 + i\epsilon} \; , \tag{3.55}$$

The Goldstone mass $\sqrt{\xi} m_V$ depends on the gauge: in unitary gauge the infinitely heavy Goldstones do not propagate ($\Delta_{VV}(q^2) \to 0$), while in Feynman gauge and in Landau gauge we have to include them as particles. From this form we can guess that they will indeed cancel the second term of the gauge boson propagators.

These different gauges have different Feynman rules and Green's functions, even a different particle content. For a given problem one or the other might be the most efficient to use in computations or proofs. For example, the proof of renormalizability was first formulated in unitary gauge. Loop calculations might be most efficient in Feynman gauge, because of the simplified propagator structure, while many QCD processes benefit from an explicit projection on the physical external gluons. Tree level helicity amplitudes are usually computed in unitary gauge, etc...

3.8 Custodial symmetry

Analyzing the appearance of $\Delta\rho$ in Eq.(3.43) and Eq.(3.53) we will see that not only higher energies, but also higher precision leads to a breakdown of the effective sigma model. The general gauge-symmetric Lagrangian for the gauge boson masses in Eq.(3.43) involves both terms, where $\text{Tr}[V_\mu V^\mu]$ gives m_W and m_Z proportional to $g \equiv g_W$ and g_Z, while $(\text{Tr}[TV_\mu])^2$ only contributes to m_Z.

The the two gauge boson masses can be expressed in terms of the weak mixing angle θ_w, assumping that that G_F or g universally govern charged-current and neutral-current interactions. At tree level this experimentally very well tested relation corresponds to $\Delta\rho = 0$ or

$$\frac{m_W^2}{m_Z^2} = \frac{g^2}{g_Z^2} = c_w^2 \; . \tag{3.56}$$

We can introduce a free parameter ρ, which breaks this relation

$$\begin{aligned} g_Z^2 &\to g_Z^2 \; \rho \\ m_Z &\to m_Z \; \sqrt{\rho} = m_Z \; \sqrt{1 + \Delta\rho} \; , \end{aligned} \tag{3.57}$$

It corresponds the theoretically derived $\Delta\rho$. In experimental reality, we need a reason to ensure $\Delta\rho = 0$, and the $SU(2)_L \times U(1)_Y$ gauge symmetry unfortunately does not do the job.

In the Standard Model $\rho = 1$ is actually violated at the one-loop level. This means we are looking for an approximate symmetry of the Standard Model. What we can hope for is that this symmetry is at least a good symmetry in the $SU(2)_L$ gauge sector and slightly broken elsewhere. One possibility is to replace $SU(2)_L \times U(1)_Y$ symmetry with a larger $SU(2)_L \times SU(2)_R$ symmetry, which could even be global,

$$\begin{aligned} &\Sigma \to U \Sigma V^\dagger \qquad\qquad U \in SU(2)_L \qquad\qquad V \in SU(2)_R \\ &\text{Tr}(\Sigma^\dagger \Sigma) \to \; \text{Tr}\left(V \Sigma^\dagger U^\dagger U \Sigma V^\dagger \right) = \text{Tr}(\Sigma^\dagger \Sigma) \; . \end{aligned} \tag{3.58}$$

In this setup, the three components of W^μ form a triplet under $SU(2)_L$ and a singlet under $SU(2)_R$, so $\rho = 1$.

In the gauge boson and fermion mass terms computed in unitary gauge the Σ field becomes identical to its vacuum expectation value $\mathbb{1}$. The two $SU(2)$ transformations act on the vacuum expectation value as

$$\langle \Sigma \rangle \to \langle U \Sigma V^\dagger \rangle = \langle U \mathbb{1} V^\dagger \rangle = U V^\dagger \stackrel{!}{=} \mathbb{1} \; . \tag{3.59}$$

The symmetry requirement can only be satisfied if $U = V$, which means that the vacuum expectation value for Σ breaks $SU(2)_L \times SU(2)_R$ to the diagonal or custodial subgroup $SU(2)_{L+R}$.

Even beyond tree level the global $SU(2)_L \times SU(2)_R$ symmetry structure can protect the relation $\rho = 1$. If fermions reside in $SU(2)_L$ and $SU(2)_R$ doublets we cannot generate any difference between up–type and down–type fermions, which implies for instance $m_b = m_t$. The measured masses $m_t \gg m_b$ leads to $\rho \neq 1$, because self energy loops in the W propagator mix a the bottom and top quark, while the Z propagator includes pure bottom and top loops,

$$
\begin{aligned}
\Delta\rho \supset & \frac{3G_F}{8\sqrt{2}\pi^2}\left(m_t^2 + m_b^2 - 2\frac{m_t^2 m_b^2}{m_t^2 - m_b^2}\log\frac{m_t^2}{m_b^2}\right) \\
= & \frac{3G_F}{8\sqrt{2}\pi^2}\left(2m_b^2 + m_b^2\delta - 2m_b^2\frac{1+\delta}{\delta}\log(1+\delta)\right) \qquad \text{defining} \quad m_t^2 = m_b^2(1+\delta) \\
= & \frac{3G_F}{8\sqrt{2}\pi^2}\left(2m_b^2 + m_b^2\delta - 2m_b^2\left(\frac{1}{\delta}+1\right)\left(\delta - \frac{\delta^2}{2} + \frac{\delta^3}{3} + \mathcal{O}(\delta^4)\right)\right) \\
= & \frac{3G_F}{8\sqrt{2}\pi^2} m_b^2 \left(2 + \delta - 2 - 2\delta + \delta + \delta^2 - \frac{2}{3}\delta^2 + \mathcal{O}(\delta^3)\right) \\
= & \frac{3G_F}{8\sqrt{2}\pi^2} m_b^2 \left(\frac{1}{3}\delta^2 + \mathcal{O}(\delta^3)\right) \\
= & \frac{G_F m_W^2}{8\sqrt{2}\pi^2}\left(\frac{\left(m_t^2 - m_b^2\right)^2}{m_W^2 m_b^2} + \cdots\right) .
\end{aligned}
\tag{3.60}
$$

In the Taylor series above the assumption of δ being small is of course not realistic, but the result is nevertheless instructive: the shift vanishes very rapidly towards the symmetric limit $m_t \sim m_b$. For the realistic Standard Model mass ratios it becomes

$$
\Delta\rho \supset \frac{3G_F}{8\sqrt{2}\pi^2} m_t^2 \left(1 - 2\frac{m_b^2}{m_t^2}\log\frac{m_t^2}{m_b^2}\right) = \frac{3G_F m_W^2}{8\sqrt{2}\pi^2}\,\frac{m_t^2}{m_W^2}\,\left(1 + \mathcal{O}\left(\frac{m_b^2}{m_t^2}\right)\right) . \tag{3.61}
$$

A second contribution to the ρ parameter will arise from Higgs loops,

$$
\Delta\rho \supset -\frac{11 G_F m_Z^2 s_w^2}{24\sqrt{2}\pi^2}\,\log\frac{m_H^2}{m_Z^2} . \tag{3.62}
$$

We want to mention that is another parameterization of the same effect, the T parameter. It is part of an effective theory parameterization of deviations from the tree level relations between gauge boson masses, mixing angles, and neutral and charged current couplings,

$$
\{S, T, U\} \tag{3.63}
$$

Two of these so-called Peskin–Takeuchi parameters can be understood fairly easily: the S-parameter corresponds to a shift of the Z mass. The T parameter compares contributions to the W and Z masses. The third parameter U is less important for most models. Again, we quote the contributions from the heavy fermion doublet,

$$
\begin{aligned}
\Delta S = & \frac{N_c}{6\pi}\left(1 - 2Y\log\frac{m_t^2}{m_b^2}\right) \\
\Delta T = & \frac{N_c}{4\pi s_w^2 c_w^2 m_Z^2}\left(m_t^2 + m_b^2 - \frac{2m_t^2 m_b^2}{m_t^2 - m_b^2}\log\frac{m_t^2}{m_b^2}\right) ,
\end{aligned}
\tag{3.64}
$$

with $Y = 1/6$ for quarks and $Y = -1/2$ for leptons. While the parameter S has nothing to do with our custodial symmetry, ρ and $T \sim \Delta\rho/\alpha$ are closely linked. Their main difference is the reference point, where $\rho = 1$ refers to its tree level value and $T = 0$ is often chosen for some kind of light Higgs mass and including the Standard Model top-bottom corrections.

Typical experimental constraints form an ellipse in the S vs T plane along the diagonal. They are usually quoted as ΔT with respect to a reference Higgs mass. Compared to a 125 GeV Standard Model Higgs boson the measured values range around $T \sim 0.1$ and $S \sim 0.05$. Additional contributions $\Delta T \sim 0.1$ tend to be within the experimental errors, much larger contributions are in contradiction with experiment.

There are two reasons to discuss these loop contributions breaking the custodial symmetry in the Standard Model. First, $\Delta\rho$ is experimentally very strongly constrained by electroweak precision measurements, which means that alternative models for electroweak symmetry breaking usually include the same kind of approximate custodial symmetry by construction. Second, in the Standard Model we can measure the symmetry violations from the heavy quarks and from the Higgs sector shown in Eqs.(3.60) and (3.62) in electroweak precision measurements. Even though the Higgs contributions depend on the Higgs mass only logarithmically, we can then derive an upper bound on the Higgs mass of the order of $\mathcal{O}(200 \text{ GeV})$. Since the Higgs discovery , studying electroweak precision data given the measured Higgs mass is one of the most sensitive consistency tests of the Standard Model.