

A few underlying concepts

Units in particle physics

When quoting single particles mass, momentum and energy, SI units are not very practical. Instead we use GeV (MeV) as the base unit:

Mass	GeV/c^2	
Momentum	GeV/c	
Energy	GeV	
Length	$(\text{GeV}/\hbar c)^{-1}$	\longrightarrow 0.197 fm
Time	$(\text{GeV}/\hbar)^{-1}$	\longrightarrow $0.658 \cdot 10^{-24}$ s
Cross section	$(\text{GeV}/\hbar c)^{-2}$	\longrightarrow $0.389 \cdot \text{mb}$ $0.389 \cdot 10^{-27} \text{cm}^2$

$$\hbar c = 0.197 \text{ GeV} \cdot \text{fm}$$

$$\hbar = 0.658 \text{ GeV} \cdot 10^{-24} \text{ s}$$

$$(\hbar c)^2 = 0.389 \text{ GeV}^2 \cdot \text{mbarn} \quad 1 \text{ barn} = 1 \text{ b} = 10^{-28} \text{ m}^2$$

Natural units

$$\hbar = c = 1 \quad \text{in addition:} \quad \epsilon_0 = \mu_0 = 1$$

$$\text{Simplification of formulae:} \quad E^2 = p^2 + m^2$$

All components of a 4-vector have the same dimension / unit.

All units can be now expressed in GeV. To calculate a result in SI units, need to multiply result with powers of $\hbar c$, \hbar , or c

$$\text{Definition of } \alpha: \quad \alpha = \frac{e^2}{4\pi} = \frac{1}{137} \quad \left(\text{In SI units: } \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137} \right)$$

E.g.: Lifetime of a particle τ .

To determine the lifetime of a particle the total decay widths Γ (=sum of partial decay widths Γ_i) is calculated. The dimension of the decay widths is energy.

With $\hbar = 1$ the relation between lifetime and Γ simplifies to:

$$\tau = \frac{1}{\Gamma}$$

Lifetime in units 1/GeV can be easily converted into seconds by multiplying with $\hbar = 0.658 \text{ GeV} \cdot 10^{-24} \text{ s}$

Relativistic Kinematics

Special relativity:

Particles in particle physics are often relativistic: Need relativistic description of the kinematics and a relativistic formulation of quantum mechanics.

4-vectors:

contravariant form: $(x^\mu) = (x^0, x^1, x^2, x^3) = (t, \vec{x}) = x$ (time, space)
 $(p^\mu) = (p^0, p^1, p^2, p^3) = (E, \vec{p}) = p$ (energy, momentum)

covariant form: $(x_\mu) = (x^0, -x^1, -x^2, -x^3) = (t, -\vec{x})$
 $(p_\mu) = (p^0, -p^1, -p^2, -p^3) = (E, -\vec{p})$

$$x_\mu = g_{\mu\nu} x^\nu \quad \text{and} \quad x^\mu = g^{\mu\nu} x_\nu \quad \text{with} \quad g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

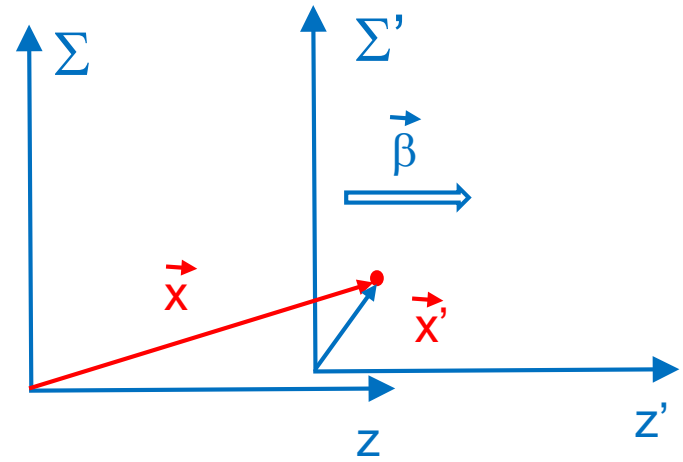
metric tensor

Scalar product: $a \cdot b = a_\mu b^\mu = g_{\mu\nu} a^\nu b^\mu$

Lorentz transformations:

System Σ' moves with speed $\vec{\beta}$ relative to Σ :

($\vec{\beta}$ is the velocity in units of c)



$$x' = x$$

$$y' = y$$

$$z' = \gamma(z - \beta t)$$

$$t' = \gamma(t - \beta z)$$

or using
4-vectors

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & & & -\beta\gamma \\ & 1 & & \\ & & 1 & \\ -\beta\gamma & & & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$$x^\mu = (\Lambda)^\mu_\nu x^\nu$$

For the covariant form one finds:

$$(x'_\mu) = \Lambda^{-1} (x_\mu) \quad \Lambda^{-1} \text{ same form as } \Lambda \text{ but } -\beta\gamma \rightarrow +\beta\gamma$$

One can easily show that scalar products $a_\mu b^\mu$ are Lorentz invariant ($\Lambda \Lambda^{-1} = 1$)

4-derivatives:

$$\frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad \xrightarrow{\text{Lorentz Transformation}}$$

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial z'} = \left(\frac{\partial z}{\partial z'} \right) \frac{\partial}{\partial z} + \left(\frac{\partial t}{\partial z'} \right) \frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial t'} = \left(\frac{\partial z}{\partial t'} \right) \frac{\partial}{\partial z} + \left(\frac{\partial t}{\partial t'} \right) \frac{\partial}{\partial t}$$

one finds:

$$\begin{pmatrix} \partial / \partial t' \\ \partial / \partial x' \\ \partial / \partial y' \\ \partial / \partial z' \end{pmatrix} = \begin{pmatrix} \gamma & & & +\beta\gamma \\ & 1 & & \\ & & 1 & \\ +\beta\gamma & & & \gamma \end{pmatrix} \begin{pmatrix} \partial / \partial t \\ \partial / \partial x \\ \partial / \partial y \\ \partial / \partial z \end{pmatrix}$$

i.e. the 4-derivative transforms like a covariant vector. Therefore $\frac{\partial}{\partial x^\mu}$

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$\partial^\mu = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right)$$

$$\implies \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

Scalar products of 4-momenta and kinematics.

Lorentz invariance of scalar products is very useful if the kinematics of scattering or decay processes is discussed. One can change to the most appropriate coordinate system (most often CMS) to calculate the kinematic quantities.

Example: 4-momentum p^μ of a particle with rest mass m

$$p_\mu p^\mu = E^2 - \vec{p}^2 = m^2$$

where the last equality can easily be calculated in the particle's rest frame where $E=m$ and $\vec{p}=0$.

Example: Lab-system $e^+, E_1 = 2\text{GeV} \xrightarrow{\vec{p}_1} \xleftarrow{\vec{p}_2} e^-, E_2 = 8\text{GeV}$

Kinematics in CMS:

$$\xrightarrow{\vec{p}_1^*, E_1^*} \xleftarrow{\vec{p}_2^*, E_2^*} \quad \text{with} \quad \vec{p}_1^* = -\vec{p}_2^*$$

$$E_1^* = E_2^* = \frac{E_{\text{CMS}}}{2}$$

$$(p_1^\mu + p_2^\mu)^2 = \underbrace{(E_1 + E_2, \vec{p}_1 + \vec{p}_2)^2}_{\substack{\text{if electron mass is neglected} \\ |\vec{p}_i| = E_i}} = (p_1^{*,\mu} + p_2^{*,\mu})^2 = \underbrace{(E_1^* + E_2^*, \vec{p}_1^* + \vec{p}_2^*)^2}_{= (E_1^* + E_2^*, 0)^2 = E_{\text{CMS}}^2}$$

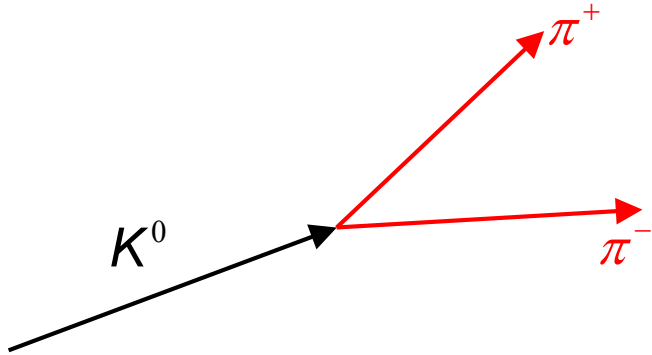
$$= \left[(2 + 8)^2 - (8 - 2)^2 \right] \text{GeV}^2$$

$$= 64 \text{ GeV}^2$$

$$E_{\text{CMS}} = 8 \text{ GeV}$$

$$|\vec{p}_1^*| = |\vec{p}_2^*| = 4 \text{ GeV}$$

Example: Particle decay and invariant mass



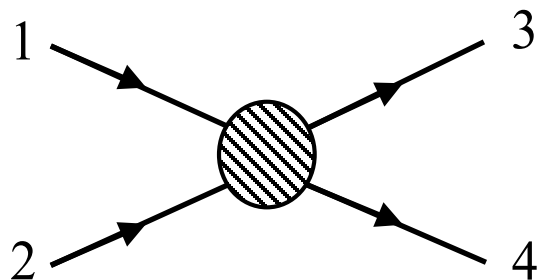
$$m_K^2 = p_K^2 = \underbrace{(p_K^\mu)^2}_{\text{Rest frame of mother particle K}} = \underbrace{(p_{\pi 1}^\mu + p_{\pi 2}^\mu)^2}_{\text{Laboratory frame}} = (p_{\pi 1} + p_{\pi 2})^2$$

Rest frame of
mother particle K

Laboratory frame

Mandelstam variables

Cross sections and other Lorentz invariant observables are often expressed in an Lorentz invariant form using scalar products of 4-momenta: **Mandelstam variables** (square of 4-momenta)



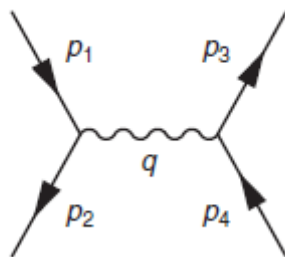
$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$

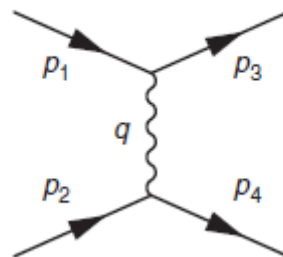
$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

Meaning of s , t , u becomes clear if you look at the q^2 of different scattering processes:



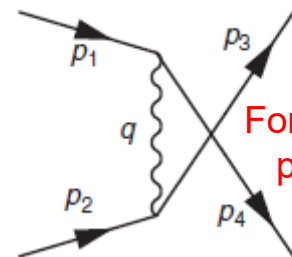
$$q^2 = s = E_{CMS}^2$$

s-channel



$$q^2 = t$$

t-channel



$$q^2 = u$$

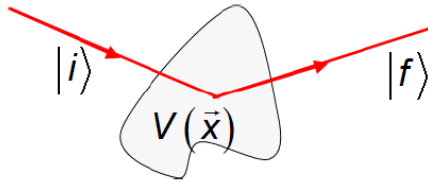
u-channel

For identical particles

Non-relativistic perturbation theory

Transition amplitude in quantum mechanics:

$V(\vec{x})$ small perturbation



Hamilton-Operator:

$$\hat{H} = \hat{H}_0 + \hat{H}_{WW} = \frac{\hat{p}^2}{2m} + V(\vec{x})$$

$|i\rangle, |f\rangle$ are solutions of the unperturbed free particle Hamiltonian \hat{H}_0

$$\psi_j = N \exp(i\vec{k}_j\vec{x} - i\omega_j t) \text{ with } \vec{k}_j = \frac{\vec{p}_j}{\hbar} \text{ and } \omega_j = \frac{E_j}{\hbar}$$

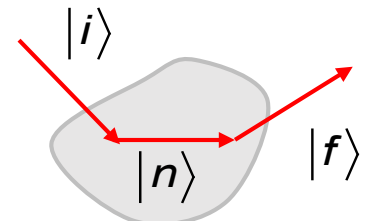
$N = \text{normalization}$

Transition amplitude (describing the transition probability):

$$\mathcal{A}(i \rightarrow f) = \mathcal{A}_{fi} = \langle f | \hat{H}_{WW} | i \rangle = \int \psi_f^* V(\vec{x}) \psi_i d\vec{x}$$

Considering higher orders:

$$\mathcal{A}_{fi} \rightarrow \mathcal{A}_{fi} + \sum_{n \neq i, f} \mathcal{A}_{fn} \frac{1}{E_i - E_n + i\epsilon} \mathcal{A}_{ni} + \dots$$



Fermi's Golden Rule

Transition rate from initial state $|i\rangle$ to final state $|f\rangle$

$$\omega_{fi} = 2\pi |\mathcal{A}_{fi}|^2 \rho(E_f)$$

With $\rho(E_f)$ density of final states at energy E_f : $\rho(E_f) = \left. \frac{dn}{dE} \right|_{E_f}$

State density for a free particle in a cubic box with $V=L^3$

Assuming periodic boundary conditions one find for one dimension (x):

In 1-D: $k_x = \frac{2\pi}{L} \cdot n, \quad n = 1, 2, 3 \dots$

Wave number:

$$dk_x = dp_x = \frac{2\pi}{L} dn \quad \frac{dn}{dp_x} = \frac{L}{2\pi} \quad \left(= \frac{L}{2\pi\hbar} \right)$$

In 3 dimension: $\frac{dn}{d^3 p} = \frac{L^3}{(2\pi)^3} = \frac{V}{(2\pi)^3}$

$$\rho(E_f) = \left. \frac{dn}{dE} \right|_{E_f} = \left. \frac{dn}{d^3 p} \right|_{E_f} \left. \frac{d^3 p}{dE} \right|_{E_f}$$

N-particle state density (phase space factor):

Only N-1 momenta / particles independent

$$\rho_N = \frac{dn_N}{dE} \quad \text{with} \quad dn_N = \frac{V^{N-1}}{(2\pi)^{3(N-1)}} \cdot d^3 p_1 d^3 p_2 \cdots d^3 p_{N-1}$$
$$= \frac{V^{N-1}}{(2\pi)^{3(N-1)}} \delta \left(\vec{P} - \sum_N \vec{p}_i \right) d^3 p_1 d^3 p_2 \cdots d^3 p_N$$
$$= (2\pi)^3 \delta \left(\vec{P} - \sum_N \vec{p}_i \right) \prod_{i=1}^N \frac{d^3 p_i}{(2\pi)^3}$$

Normalization volume
V often set to 1.

Lorentz invariance:

Above, neither phase space factor nor transition amplitude are Lorentz invariant.

For the amplitude the wave-functions are usually normalized to 1 particle in the normalization volume. Instead, to make the amplitude Lorentz invariant, one normalizes to $2E$ particles. For a process $a+b \rightarrow 1+2$ one finds the Lorentz invariant transition amplitude (matrix element):

$$\mathcal{M}_{fi} = \langle \psi'_a \psi'_b | \hat{H}_{WW} | \psi'_1 \psi'_2 \rangle = (2E_a \cdot 2E_b \cdot 2E_1 \cdot 2E_2)^{1/2} \mathcal{A}_{fi}$$

Lorentz invariant Phase space:

In the same way one needs to replace d^3p by $d^3p/2E$ to make the phase space factor Lorentz invariant.

Particle decay

In the following we discuss the two-body decay $\mathbf{a} \rightarrow \mathbf{1} + \mathbf{2}$ in a Lorentz invariant formulation (in lab frame, particle a is in rest).

The partial decay width (decay rate) $\Gamma(\mathbf{a} \rightarrow \mathbf{1} + \mathbf{2})$ for the given process is given by the transition rate ω_{fi} . Using the non-invariant expressions one finds:

$$\Gamma(a \rightarrow 1 + 2) = \omega_{fi} = (2\pi)^4 \int |\mathcal{A}_{fi}|^2 \delta(E_a - E_1 - E_2) \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3}$$

Use Lorentz invariant transition amplitude:

$$\Gamma = \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}_{fi}|^2 \delta(E_a - E_1 - E_2) \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) \underbrace{\frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2}}_{\text{Lorentz invariant phase space dLISP}}$$

For N particles: $dLISP = \prod_{i=1}^N \frac{d^3 p_i}{(2\pi)^3 2E_i}$

Lorentz invariant
phase space dLISP

If there is more than one decay channel of particle a:

$$\Gamma = \sum_i \Gamma_i$$

Branching ratio: $BR_i = \frac{\Gamma_i}{\Gamma}$ lifetime: $\tau = \frac{1}{\Gamma}$

Γ_i can be calculated.

Coming back to the 2-body decay of particle a in rest:

$$\Gamma = \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}_{fi}|^2 \delta(E_a - E_1 - E_2) \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2}$$

↑ Mass m_a ↑ Mass m_a $\vec{p}_a = 0, \vec{p}_1 = -\vec{p}_2$

$$\Gamma = \frac{1}{8\pi^2 m_a} \int |\mathcal{M}_{fi}|^2 \delta(m_a - E_1 - E_2) \delta^3(\vec{p}_1 + \vec{p}_2) \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2}$$

$$= \frac{1}{8\pi^2 m_a} \int |\mathcal{M}_{fi}|^2 \frac{1}{4E_1 E_2} \delta(m_a - E_1 - E_2) d^3 p_1 \Big|_{\vec{p}_1 = -\vec{p}_2}$$

$$= \frac{1}{8\pi^2 m_a} \int |\mathcal{M}_{fi}|^2 \delta\left(m_a - \sqrt{m_1^2 - p_1^2} - \sqrt{m_2^2 - p_1^2}\right) \frac{p_1^2}{4E_1 E_2} dp_1 d\Omega_1$$

With some calculation....

$$\Gamma = \frac{p_1}{32\pi^2 m_a^2} \int |\mathcal{M}_{fi}|^2 d\Omega_1$$

with
$$p_1 = \frac{1}{2m_a} \sqrt{[m_a^2 - (m_1 + m_2)^2][m_a^2 - (m_1 - m_2)^2]}$$

For equal particles 1=2:
$$p_1 = \frac{1}{2} \sqrt{m_a^2 - 4m_1^2}$$

$$p_1 = |\vec{p}_1|$$

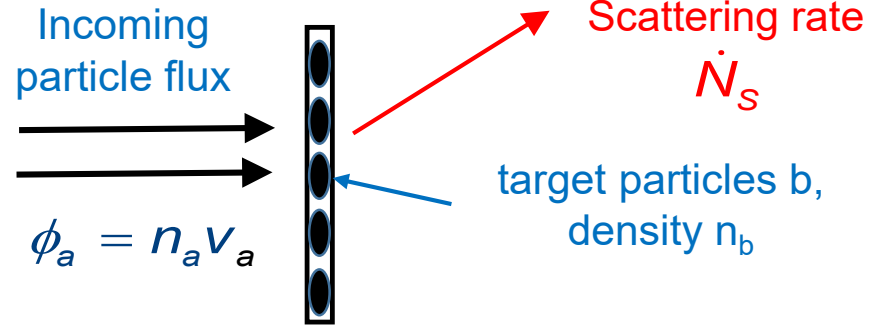
Decay width (rate) is proportional to $|\mathcal{M}_{fi}|^2$ (as expected) and the phase space dependence leaves to a proportionality to p_1

Cross section

Cross section =

$$\frac{\text{Scattering rate per target particle}}{\text{incident flux}}$$

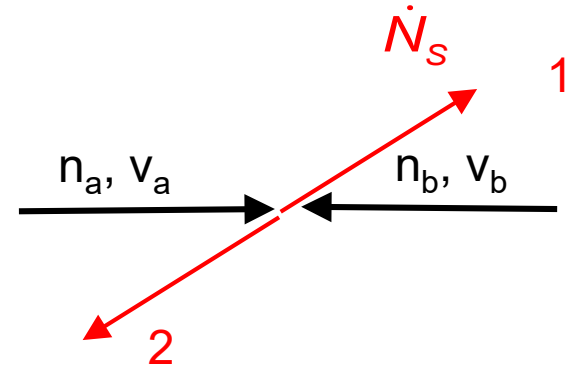
$$\sigma = \frac{\dot{N}_S}{\phi_a N_b}$$



Scattering rate for scattering volume V :

$$\dot{N}_S = n_a (v_a + v_b) \cdot n_b V \cdot \sigma$$

$$\sigma = \frac{\dot{N}_S}{\phi_a \cdot n_b V} \text{ with } \phi_a = n_a (v_a + v_b)$$



With appropriate normalization of wave function $n_a = n_b = 1/V$

One can express the cross section through the transition rate (here use $V=1$):

$$\sigma = \frac{\omega_{fi} \cdot V}{(v_a + v_b)} = \frac{\omega_{fi}}{(v_a + v_b)}$$

Using Fermi's Golden Rule and the Lorentz invariant form of transition amplitude

$$\sigma = \frac{(2\pi)^4}{v_a + v_b} \int |\mathcal{A}_{fi}|^2 \delta(E_a + E_b - E_1 - E_2) \delta(\vec{p}_a + \vec{p}_b - \vec{p}_1 \vec{p}_2) \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3}$$

$$= \underbrace{\frac{(2\pi)^{-2}}{4E_a E_b (v_a + v_b)}} \int |\mathcal{M}_{fi}|^2 \delta(E_a + E_b - E_1 - E_2) \delta(\vec{p}_a + \vec{p}_b - \vec{p}_1 \vec{p}_2) \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2}$$

can be rewritten in Lorentz invariant form:

$$F = 4E_a E_b (v_a + v_b)$$

$$= 4(E_a |\vec{p}_b| + E_b |\vec{p}_a|)$$

$$F = 4 [(p_a \cdot p_b)^2 - m_a^2 m_b^2]^{\frac{1}{2}}$$

In the CMS frame:

$$F = 4\vec{p}_i \sqrt{s} \quad \leftarrow \vec{p}_a = -\vec{p}_b = \vec{p}_i \quad \text{und} \quad E_a + E_b = \sqrt{s}$$

Cross section im CMS:

$$\sigma = \frac{1}{(2\pi)^2} \frac{1}{4p_i \sqrt{s}} \int |\mathcal{M}_{fi}|^2 \delta(\sqrt{s} - E_1 - E_2) \delta(\vec{p}_1 + \vec{p}_2) \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2}$$

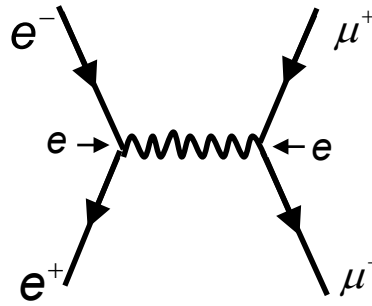
$$\vec{p}_1 = -\vec{p}_2 = \vec{p}_f$$

$$\sigma = \frac{1}{64\pi^2} \frac{p_f}{s p_i} \int |\mathcal{M}_{fi}|^2 d\Omega_f$$

Differential cross section:

$$\frac{d\sigma}{d\Omega_f} = \frac{1}{64\pi^2} \frac{p_f}{s p_i} |\mathcal{M}_{fi}|^2$$

Example: $e^+e^- \rightarrow \mu^+\mu^-$



The derivation of the Feynman rules and the calculation of the above process will be the topic of the first theory block. Here we just use the result.

If one uses unpolarized electron/positron beams one should average over the helicity states of the incoming electrons. Moreover, the muon polarization is in general not observed. For the cross section one needs to add all possible muon polarization states. This leads to an “average matrix element”

$$\overline{|\mathcal{M}_{fi}|^2} = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2} = 2e^4 \frac{t^2 + u^2}{s^2} = e^4 (1 + \cos \theta_f)$$

if one ignores the fermion masses

$$\frac{d\sigma}{d\Omega_f} = \frac{1}{64\pi^2} \frac{p_f}{s p_i} e^4 (1 + \cos \theta_f) = \frac{\alpha^2}{4s} (1 + \cos \theta_f) \Rightarrow \sigma = \frac{4\pi\alpha^2}{3s}$$