Standard Model of Particle Physics

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4. Renormalization

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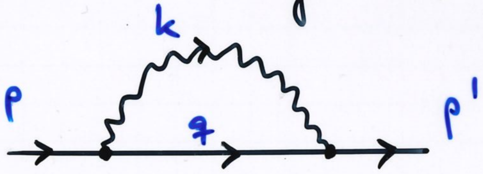
Ultraviolet Divergences

* In higher-order perturbation theory we encounter Feynman graphs with closed loops, associated with unconstrained momenta.

* For every such momentum k^μ, we have to integrate over all values, i.e.

$$\int \frac{d^4k}{(2\pi)^4}$$

E.g. "electron self-energy" in QED



$$A_{fi} = \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \bar{u}(p)\gamma^\mu \frac{(-ig_{\mu\nu})}{k^2} \frac{i(\not{q}+m)}{q^2-m^2} \gamma^\nu u(p') \times$$

$$\times (-ie)(2\pi)^4 \delta^4(p-q-k) \cdot (-ie)(2\pi)^4 \delta^4(q+k-p')$$

$$= -e^2 (2\pi)^4 \delta^4(p-p') \times$$

$$\times \bar{u}(p) \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\not{p}-\not{k}+m)\gamma_\mu}{k^2[(p-k)^2-m^2]} u(p)$$

* $\int^\infty d^4k \frac{k}{k^2(p-k)^2}$ is divergent!

* We say that

$$\int d^4k \; k^{D-4}$$

has 'superficial degree of divergence' = D

$$D = \begin{cases} 0 & \text{log - divergent} \\ 1 & \text{linearly divergent} \\ 2 & \text{qudratically divergent} \end{cases} \quad \text{etc.}$$

* The actual degree of divergence may be less, e.g. due to cancellations required by gauge invariance. For example: the electron self-energy is actually only log-divergent. Putting an upper cut-off Λ on the integral

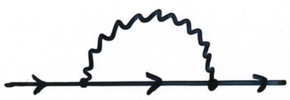
$$A_{fi} \sim -i\,(2\pi)^4\,\delta^4(p-p') \cdot \frac{3m\alpha}{2\pi}\,\ln\frac{\Lambda}{m} + \dots$$

* If the theory has only a finite set of (classes of) divergent (i.e. cut-off - dependent) graphs, their contributions can be absorbed into redefinitions of the coupling constant(s) and masses. This is called renormalization.

* For example, iteration of the electron self-energy leads to renormalization of the electron mass:

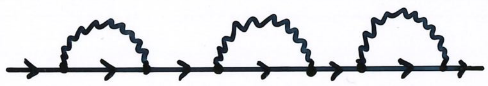
p 

$$\frac{i(\not{p}+m)}{p^2-m^2} \equiv \frac{i}{\not{p}-m}$$



$$\frac{i}{\not{p}-m}\, ie^2\Sigma\, \frac{i}{\not{p}-m}$$

with $\Sigma = -\frac{3m}{8\pi^2} \ln\frac{\Lambda}{m}$



$$\frac{i}{\not{p}-m} \sum_n \left[ie^2\Sigma \frac{i}{\not{p}-m}\right]^n =$$

$$= \frac{i}{\not{p}-m}\left[1 - ie^2\Sigma \frac{i}{\not{p}-m}\right]^{-1} = \frac{i}{\not{p}-m+e^2\Sigma}$$

* Hence $m \longrightarrow m + \delta m$ where

$$\delta m = \frac{3m\alpha}{2\pi} \ln\frac{\Lambda}{m} + \dots$$

The real, observed mass is $m + \delta m$. The "bare" mass m, i.e. the parameter in the Lagrangian, is not observable (and depends on Λ if we keep the observed mass fixed!).

Renormalizability

* How many classes of superficially divergent graphs are there in QED?

We have

$\int d^4k$ for every loop momentum (unconstrained momentum)

$\frac{i}{\not{k}-m}$ for every internal fermion line (electron)

$-\frac{ig^{\mu\nu}}{k^2}$ for every internal boson line (photon)

$$\Rightarrow \quad D = 4L - F_I - 2B_I$$

where

L = number of unconstrained momenta

F_I = number of internal fermion lines

B_I = number of internal boson lines

* But if V is the number of vertices

$$L = F_I + B_I - V + 1$$

* If vertex V involves F_V fermions and B_V bosons, we have

$$\sum_V F_V = 2F_I + F_E$$

$$\sum_V B_V = 2B_I + B_E$$

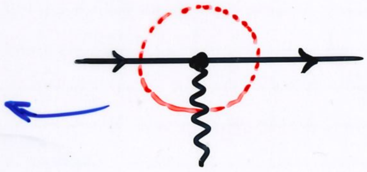
("law of conservation of fermion / boson ends")

where

F_E = number of external fermion lines

B_E = number of external boson lines

* In QED $F_V = 2$, $B_V = 1$



$$\Rightarrow \quad V = F_I + \tfrac{1}{2}F_E = 2B_I + B_E$$

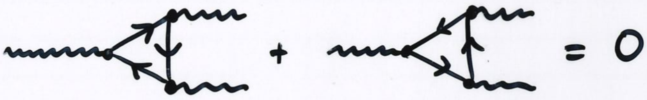
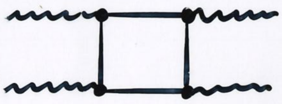
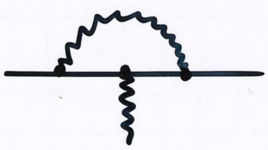
$$\Rightarrow \quad \boxed{D = 4 - \tfrac{3}{2}F_E - B_E} \quad \text{in QED}$$

Note that D is independent of L and V.

* Thus there is only a finite number of classes of superficially divergent graphs in QED:

$$D = 4 - \tfrac{3}{2}F_E - B_E \geq 0$$

* In fact, there are only 5 classes of superficially divergent graphs in QED, of which only 3 are actually (log-) divergent.

F_E	B_E	D	diagrams	remarks
0	2	2		photon self-energy log-divergent ⇒ charge renorm'n
0	3	1		to all orders
0	4	0		light-by-light scattering actually convergent
2	0	1		electron self-energy log-divergent ⇒ mass and charge* renorm'n
2	1	0		vertex correction log-divergent ⇒ charge* renormalization

*) In QED, charge renormalizations from electron self-energy and vertex correction cancel, so it can be ascribed entirely to photon self-energy (= vacuum polarization).

Dimensions of Fields and Couplings

* In natural units we have only mass (equivalently, energy or momentum) dimensions

$$x = ct = \hbar c/E = \hbar/mc$$

$$\hbar = c = 1 \Rightarrow [L] = [T] = [E]^{-1} = [M]^{-1}$$

* Hence action (units $\hbar$) is dimensionless

$$S = \int \mathcal{L}\, d^4x \Rightarrow [\mathcal{L}] = [x]^{-4} = [M]^4$$

Further $$[\partial^\mu] = [p^\mu] = [M]$$

* From this we can deduce dimensions of fields and couplings:
  * $\mathcal{L}_{KG} = \partial^\mu \phi^+ \partial_\mu \phi - m^2 \phi^+ \phi \Rightarrow [\phi] = [M]$
  * $\mathcal{L}_D = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \Rightarrow [\psi] = [M]^{3/2}$
  * $\mathcal{L}_{em} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \Rightarrow [F^{\mu\nu}] = [M]^2$

    $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \Rightarrow [A^\mu] = [M]$
  * Higgs self-coupling $\lambda(\phi^+\phi)^2 \Rightarrow [\lambda] = [M]^0$
  * gauge couplings $D_\mu = \partial_\mu - \underset{(g)}{e} A_\mu \Rightarrow [e] = [g] = [M]^0$
  * Fermi $G_F \bar{\psi}\gamma^\mu \psi \bar{\psi}\gamma_\mu \psi \Rightarrow [G_F] = [M]^{-2}$
  * Yukawa $g_f \phi \bar{\psi}\psi \Rightarrow [g_f] = [M]^0$

* Thus in any theory we can associate dimension 4 with each vertex as follows:

$$4 = \frac{3}{2} F_v + B_v + P_v + g_v \,,$$

P_v: number of momentum factors

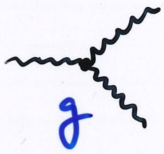
g_v: dimension of coupling

for example

Fermi

$$4 = \frac{3}{2}\cdot(4) + 0 + 0 + (-2)$$

3-gauge boson



$$4 = \frac{3}{2}\cdot(0) + 3 + 1 + 0$$

* Now we can derive superficial degree of divergence in any theory

$$D = 4L - F_I - 2B_I + \sum_v P_v$$

Recall

$$L = F_I + B_I - V + 1$$

$$\sum_v F_v = 2F_I + F_E$$

$$\sum_v B_v = 2B_I + B_E$$

$$D = 4 - 4V + 3F_I + 2B_I + \sum_v P_v$$

$$= 4 - 4V - \frac{3}{2}F_E - B_E + \sum_v \underbrace{\left(\frac{3}{2}F_v + B_v + P_v\right)}_{= 4 - g_v}$$

$$\Rightarrow \boxed{D = 4 - \frac{3}{2}F_E - B_E - \sum_v g_v}$$

* Standard Model couplings are all dimensionless, so $\sum_v g_v = 0$ and the situation is similar to QED:
  * Finite number of divergent sub-graphs ('primitive divergences')
  * Absorb cut-off dependence in bare parameters of Lagrangian

$\Rightarrow$ Theory is renormalizable.

N.B. Lots of work to prove this!
($\rightarrow$ 't Hooft, Veltman)

* Non-standard vertices have $g_v < 0$

$\Rightarrow$ D gets higher and higher in higher orders of perturbation theory.

$\Rightarrow$ theory becomes unrenormalizable

For example:

Higgs	$\lambda_6 (\phi^+\phi)^3$	$\Rightarrow$	$[\lambda_6] = [M]^{-2}$
Fermi	$G_F \bar{\psi}\psi\bar{\psi}\psi$	$\Rightarrow$	$[G_F] = [M]^{-2}$
	$\lambda_f \phi^+\phi\bar{\psi}\psi$	$\Rightarrow$	$[\lambda_f] = [M]^{-1}$

* Is it surprising that Nature provides only renormalizable interactions?
Maybe not, because

unrenormalizability
$\Leftrightarrow$ bad (divergent) high-energy behaviour

E.g. Fermi theory:

$$\sigma(\nu_e e) \sim |\times|^2 \sim G_F^2$$

$$[G_F] = [M]^{-2} \quad , \quad [\sigma] = [M]^{-2}$$

$$\Rightarrow \quad \sigma(\nu_e e) \sim G_F^2 E^2 \xrightarrow[E\to\infty]{} \infty$$

* Thus if we suppose there exists a finite theory at very high energies (GUT? SUSY? Strings?), all unrenormalizable interactions will have <u>shrunk</u> to negligible values in going from that scale to present energies.