Standard Model of Particle Physics

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1. Relativistic Quantum Mechanics

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Klein-Gordon Equation

The existence of plane waves

$$\phi(\underline{r},t) \propto \exp(i\underline{k}\cdot\underline{r} - i\omega t)$$

satisfying de Broglie & Einstein relations

$$\underline{p} = \hbar\underline{k} \quad , \quad E = \hbar\omega$$

implies the quantum operator interpretation

$$\underline{p} \to -i\hbar\underline{\nabla} \quad , \quad E \to i\hbar\frac{\partial}{\partial t} \quad .$$

Then the relativistic energy-momentum equation

$$E^2 = p^2c^2 + m^2c^4$$

implies the Klein-Gordon equation

$$-\hbar^2\frac{\partial^2\phi}{\partial t^2} = -\hbar^2c^2\underline{\nabla}^2\phi + m^2c^4\phi$$

i.e. in covariant notation (see handout)

$$\boxed{\left[\partial_\mu\partial^\mu + \left(\frac{mc}{\hbar}\right)^2\right]\phi = 0}$$

where

$$\partial_\mu\partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \underline{\nabla}^2$$

* KG wave function ϕ is a Lorentz-invariant (scalar) function,

$$\phi'(\underbrace{\underline{r}'}_{\text{transformed coordinates}},t) = \phi(\underline{r},t)$$

Hence it must represent a spin zero particle ($\rightarrow$ no orientation)

* Since $|\phi|^2$ is also invariant, this cannot represent a probability density.

A density ρ transforms as a time-like (0^{th}) component of a 4-vector

— due to Lorentz contraction of volume element.

* Correct definition of density follows from the conservation (continuity) equation

$$\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot \underline{J}$$

corresponding current vector

i.e.

$$\partial_\mu J^\mu = 0$$

where $J^\mu = (c\rho, \underline{J})$ is the 4-current.

* We can obtain an equation of this form from the KG equations for ϕ and ϕ^*,

$$i\hbar\left(\phi^*\frac{\partial^2\phi}{\partial t^2}-\phi\frac{\partial^2\phi^*}{\partial t^2}\right)=i\hbar c^2\left(\phi^*\underline{\nabla}^2\phi-\phi\underline{\nabla}^2\phi^*\right)$$

$$\rightarrow \quad i\hbar\frac{\partial}{\partial t}\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right)=i\hbar c^2\,\underline{\nabla}\cdot\left(\phi^*\underline{\nabla}\phi-\phi\underline{\nabla}\phi^*\right)$$

Hence

$$\boxed{\begin{aligned}\rho &= i\hbar\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right)\\ \underline{j} &= -i\hbar c^2\left(\phi^*\underline{\nabla}\phi-\phi\underline{\nabla}\phi^*\right)\end{aligned}}$$

i.e. $\quad j^\mu = i\hbar c^2\left(\phi^*\partial^\mu\phi-\phi\,\partial^\mu\phi^*\right)$

* Normalization is such that energy eigenstate

$\phi=\Phi(\underline{r})\,e^{-iEt/\hbar}$ has $\quad \rho = 2E\,|\Phi|^2$

i.e. $|\Phi|=1 \Rightarrow$ $2E$ particles per unit vol.

(relativistic normalization)

* Compare with Schrödinger current

$$\underline{j}^S = -\frac{i\hbar}{2m}\left(\phi^*\underline{\nabla}\phi-\phi\underline{\nabla}\phi^*\right)=\frac{1}{2mc^2}\,\underline{j}^{KG}$$

$\Rightarrow \dfrac{E}{mc^2}$ particles per unit vol

Problems with Klein-Gordon Equation

1) Density ρ is not necessarily positive (unlike $|\phi|^2$) $\Rightarrow$ equation rejected initially

2) Equation is second order in t

   $\Rightarrow$ need to know both ϕ and $\frac{\partial \phi}{\partial t}$ at $t=0$ in order to solve for ϕ at $t>0$

   $\Rightarrow$ extra degree of freedom, not present in Schrödinger equation

3) The equation on which it is based ($E^2 = p^2c^2 + m^2c^4$) has both positive and negative solutions for E.

Actually these problems are all related

* $\phi = \phi_0 \, e^{\mp iEt/\hbar} \quad \Longrightarrow \quad \rho = \pm 2E|\phi_0|^2$

* $\phi = \phi_+ \, e^{-iEt/\hbar} + \phi_- \, e^{+iEt/\hbar}$

$$\Rightarrow \left\{ \begin{aligned} \phi(t=0) &= \phi_+ + \phi_- \\ \frac{i\hbar}{E} \frac{\partial \phi}{\partial t}\Big|_{t=0} &= \phi_+ - \phi_- \end{aligned} \right\}$$

both needed to fix ϕ_+ and ϕ_-

Electromagnetic Waves

* In units where $\varepsilon_0 = \mu_0 = c = 1$ ('Heaviside-Lorentz')

Maxwell's equations are

$$\underline{\nabla} \cdot \underline{E} = \rho_{em} , \qquad \underline{\nabla} \times \underline{E} = - \frac{\partial \underline{B}}{\partial t}$$

$$\underline{\nabla} \cdot \underline{B} = 0 , \qquad \underline{\nabla} \times \underline{B} = \underline{J}_{em} + \frac{\partial \underline{E}}{\partial t}$$

where $(\rho_{em}, \underline{J}_{em}) = J^\mu_{em}$ is the e.m. 4-current

* In terms of the scalar & vector potentials $V, \underline{A}$

$$\underline{E} = - \partial \underline{A}/\partial t - \underline{\nabla} V , \qquad \underline{B} = \underline{\nabla} \times \underline{A}$$

So we find

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \nabla^2 \underline{A}$$

$$= \underline{J}_{em} - \partial^2 \underline{A}/\partial t^2 - \underline{\nabla} \frac{\partial V}{\partial t}$$

* In terms of the 4-potential $A^\mu = (V, \underline{A})$

$$(\partial_\nu \partial^\nu) A^\mu - \partial^\mu (\partial_\nu A^\nu) = J^\mu_{em}$$

i.e.

$$\boxed{\begin{aligned} \partial_\nu F^{\nu\mu} &= J^\mu_{em} \\ F^{\nu\mu} = \partial^\nu A^\mu - \partial^\mu A^\nu &= - F^{\mu\nu} \end{aligned}}$$

* $\underline{E}$ and $\underline{B}$, and hence Maxwell's equations are invariant w.r.t. gauge transformations

↳ more later!

$$A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu \chi$$

where $\chi(\underline{r},t)$ is a scalar function.

* Therefore we can always choose A^μ such that $\partial_\mu A^\mu = 0$ (Lorenz gauge)

[If $\partial_\mu A^\mu = f \neq 0$, change to $A'^\mu = A^\mu + \partial^\mu \chi$ where $\partial_\mu \partial^\mu \chi = -f$.]

* Then in free space ($J^\mu = 0$)

$$\boxed{\partial_\nu \partial^\nu A^\mu = 0}$$

→ wave equation ≡ massless KG equation for each component of A^μ

A^μ = 'wave function' of (massless) photon

A^μ is a 4-vector ⇒ photon has spin 1

* Plane wave solutions

$$A^\mu = \varepsilon^\mu e^{i\underline{k}\cdot\underline{r} - i\omega t} \equiv \varepsilon^\mu e^{-ik\cdot x}$$

ε^μ = polarization 4-vector

$k\cdot x = k^\mu x_\mu$, $k^\mu = (\omega, \underline{k})$ = wave 4-vector

* From wave equation

$k\cdot k = 0 \Rightarrow \omega^2 = \underline{k}^2$ $(E^2 = p^2c^2)$ massless photons

* From Lorentz gauge condition

$\varepsilon\cdot k = 0 \Rightarrow \varepsilon^0 = \underline{\varepsilon}\cdot\underline{k}/\omega$

* Polarization 4-vector $\varepsilon'^\mu = \varepsilon^\mu + a k^\mu$

is equivalent to ε^μ for any constant a.

Hence we can always choose $\varepsilon^0 = 0$.

Then Lorentz condition becomes

transversity condition : $\underline{\varepsilon}\cdot\underline{k} = 0$

* E.g. for $\underline{k}$ along z-axis we can express

ε^μ in terms of plane polarization states

$\varepsilon^\mu_x = (0,1,0,0)$ $\varepsilon^\mu_y = (0,0,1,0)$

or circular polarization states $\varepsilon^\mu_\pm = \frac{1}{\sqrt{2}}(0,1,\pm i,0)$

N.B. only 2 polarizations for real photons

Electromagnetic Interactions

As in classical (& n.r. quantum) physics, we introduce e.m. interactions via the 'minimal' substitution in the equations of motion (see later)

charge on particle

$$E \to E - eV \quad , \qquad \underline{p} \to \underline{p} - e\underline{A}$$

i.e.

$$p^\mu \to p^\mu - eA^\mu \qquad (p^\mu \to i\partial^\mu)$$

$\hbar = 1$

$$\boxed{\partial^\mu \to \partial^\mu + ieA^\mu}$$

* The Klein-Gordon equation becomes

$$(\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu)\phi + m^2\phi = 0$$

i.e.

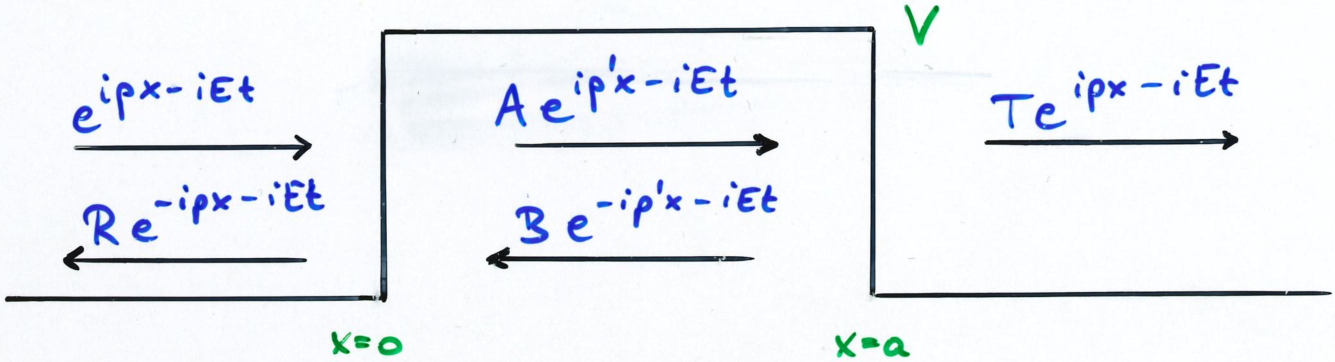
$$(\partial_\mu \partial^\mu + m^2)\phi = -ie\left[\partial_\mu(A^\mu\phi) + A_\mu(\partial^\mu\phi)\right] + e^2 A_\mu A^\mu \phi$$

* The conserved current is now ($\hbar = c = 1$)

$$J^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) - 2eA^\mu \phi^* \phi$$

Klein Paradox

Consider KG plane waves incident on electrostatic barrier, height V, width a



* KG equation for $x<0$, $x>a$ gives

$$E^2 = p^2 + m^2 \quad \Rightarrow \quad p = +\sqrt{E^2 - m^2}$$

B.C.

* In $0<x<a$, $A^\mu = (V, \underline{0})$

$$(E - eV)^2 = p'^2 + m^2$$

$$\Rightarrow \quad p' = +\sqrt{(E - eV - m)(E - eV + m)}$$

Matching ϕ and $\partial\phi/\partial x$ (current) at $x=0,a$ gives (as for Schrödinger equation)

$$|T|^2 = \left| \cos p'a - \frac{i}{2}\left(\frac{p}{p'} + \frac{p'}{p}\right) \sin p'a \right|^{-2}$$

Consider behaviour as V is increased :

1) $eV < E - m$: p' is real, $|T| \leq 1$

($|T| = 1$ when $p'a = n\pi$)

2) $E - m < eV < E + m$: p' is imaginary, $|T| < 1$

transmission by tunnelling

3) $eV > E + m$: p' is real again!

$|T| = 1$ when $p'a = n\pi$!?

* Note that the density inside the barrier is

$$\rho' = 2(E - eV)|\phi|^2$$

$$< 0 \quad \text{when} \quad eV > E$$

$$< -2m|\phi|^2 \quad \text{when} \quad eV > E + m$$

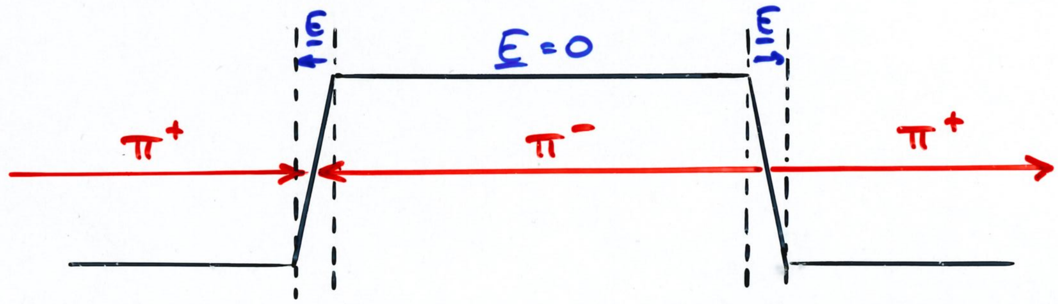
* Meanwhile, the current inside remains positive,

$$J_x' = 2p \qquad \text{(by current conservation)}$$

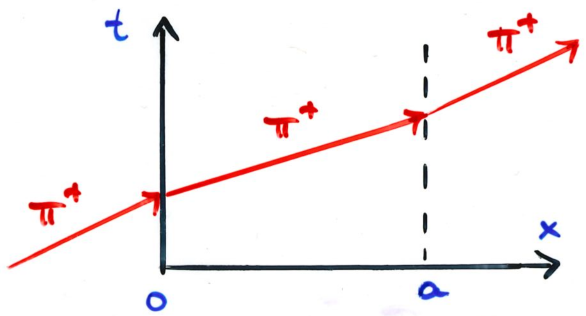
Hence when $eV > E$ there is a negative density flowing from right to left, giving a positive current. We interpret this as a flow of antiparticles :

$$J^\mu_{em} = e J^\mu \qquad \text{always}$$

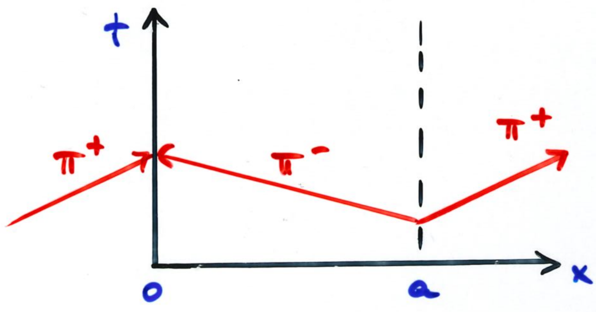
* When $eV > E+m$ and $|T| = 1$, <u>antiparticles</u> created at the back of the barrier ($x=a$) travel to $x=0$ and annihilate the incident particles.
  At the same time, <u>particles</u> created at $x=a$ travel to $x>a$, replacing the incident beam.



* Antiparticles are trapped inside barrier, but field is zero there, so there can be perfect transmission for any thickness.
* Antiparticles are like particles propagating <u>backwards in time</u>.



$eV < E-m$



$eV > E+m$

Charge Conjugation

* If ϕ is a negative-energy plane-wave sol'n of the KG equation, with momentum $\underline{p}$

$$\phi = e^{i\underline{p}\cdot\underline{r} + iEt} \qquad (E>0)$$

then $$\phi^* = e^{-i\underline{p}\cdot\underline{r} - iEt}$$

i.e. ϕ^* is a positive-energy wave with momentum $-\underline{p}$. Furthermore, in e.m. fields ϕ^* behaves as a particle of charge $-e$:

$$(\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu)\phi + m^2\phi = 0$$

$$\Rightarrow \quad (\partial_\mu - ieA_\mu)(\partial^\mu - ieA^\mu)\phi^* + m^2\phi^* = 0$$

* Thus if ϕ is a negative-energy solution we take it to represent an <u>antiparticle</u> with wave function ϕ^* (and hence positive energy, opposite charge & momentum).

* Correspondingly, KG equation is invariant w.r.t. $\phi \to \phi^*$, $e \to -e$.

This is called <u>charge conjugation</u>, $\mathcal{C}$.

<u>N.B.</u> $\quad J^\mu \xrightarrow{\mathcal{C}} -J^\mu$ as expected.

Electromagnetic Scattering

We assume (for the moment) the same formula as in NRQM for the scattering amplitude in terms of the first-order perturbation due to e.m. field:

$$A_{fi} = -i\int \phi_f^* \{ie[\partial_\mu(A^\mu\phi_i) + A_\mu(\partial^\mu\phi_i)]\} d^4x$$

by parts →
$$= e\int A_\mu [\phi_f^*(\partial^\mu\phi_i) - (\partial^\mu\phi_f^*)\phi_i] d^4x$$

$$= -ie\int A_\mu J^\mu_{fi} d^4x$$

where $J^\mu_{fi} = i[\phi_f^*(\partial^\mu\phi_i) - (\partial^\mu\phi_f^*)\phi_i]$

is generalization of J^μ to $\phi_f \neq \phi_i$ (transition current). Note that to get A_{fi} to order e^1 we only need J^μ_{fi} to order e^0. Similarly, for A_μ we use the free-field form

$$A_\mu = \varepsilon_\mu e^{-ik\cdot x}$$

* For plane waves $\phi_{f,i} = e^{-ip_{f,i}\cdot x}$

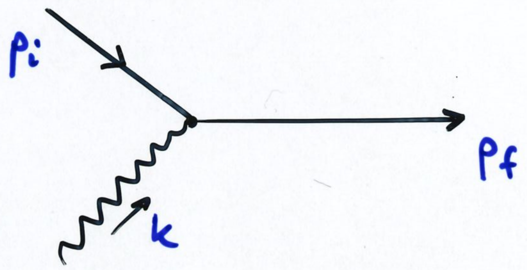
$$J^\mu_{fi} = (p_i + p_f)^\mu e^{i(p_f - p_i)\cdot x}$$

$$\Longrightarrow A_{fi} = -ie\,\varepsilon_\mu (p_i + p_f)^\mu \int e^{i(p_f - p_i - k)\cdot x} d^4x$$

$$= -ie(2\pi)^4\, \varepsilon\cdot(p_i + p_f)\, \delta^4(p_f - p_i - k)$$

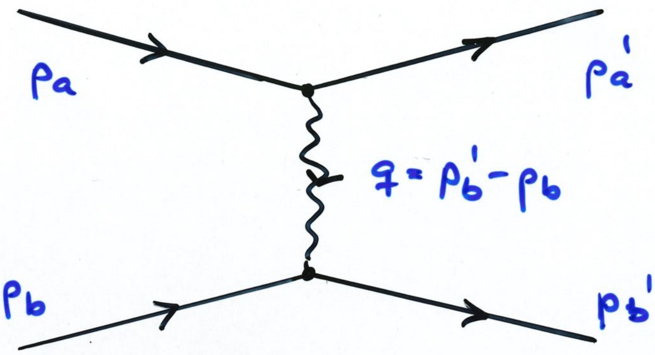
$$\boxed{A_{fi} = -ie\,(2\pi)^4\,\varepsilon\cdot(p_i+p_f)\;\delta^4(p_f-p_i-k)}$$

This corresponds to the **Feynman rules** for the diagram



* An overall factor of
  $(2\pi)^4\;\delta^4(p_f-p_i-k)$ → momentum conservation *)
* ε_μ for an external photon line
* $-ie\,(p_i+p_f)^\mu$ for a vertex involving a spin-0 boson of charge e

*) NB 4-momentum cannot be conserved in this process for free particles! But we shall see it can occur as part of a more complicated process, e.g. particle-particle scattering by photon exchange.



* We shall consider process as scattering of a in e.m. field of b (also spin-0)

$$A_{fi} = -ie_a \int A_\mu \, J^\mu_{a'a} \, d^4x$$

* Then (in Lorentz gauge) A^μ satisfies

$$\partial_\nu \partial^\nu A^\mu = e_b \, J^\mu_{b'b}$$

NB We assume the correct source current is

$$J^\mu_{b'b} = (p_b + p_b')^\mu \, e^{i(p_b' - p_b)\cdot x}$$

* Solution is then

$$A^\mu = -\frac{1}{q^2} \, e_b \, (p_b + p_b')^\mu \, e^{iq\cdot x}$$

where $\quad q = p_b' - p_b \,, \qquad q^2 = q\cdot q$

$$\Rightarrow A_{fi} = \frac{ie_a e_b}{q^2} (p_a + p_a')\cdot(p_b + p_b') \int e^{i(p_a' + p_b' - p_a - p_b)} d^4x$$

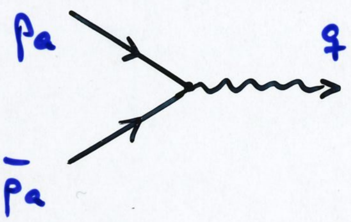
$$= \left[-ie\,(p_a + p_a')^\mu\right] \underbrace{\left[\frac{-ig_{\mu\nu}}{q^2}\right]}_{\text{photon propagator factor}} \left[-ie\,(p_b + p_b')^\nu\right] \times (2\pi)^4 \delta^4(p_a' + p_b' - p_a - p_b)$$

NB: symmetry in a, b

* Thus we have the additional Feynman rule:

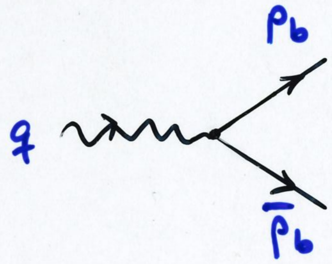
$$\frac{-i g_{\mu\nu}}{q^2} \quad \text{for an virtual photon line}$$

* In processes involving antiparticles, remember we use particles with opposite energy and momentum: $p^\mu = -\bar{p}^\mu$



$$\text{"}p_i\text{"} = p_a\,, \quad \text{"}p_f\text{"} = -\bar{p}_a\,, \quad \text{"}k\text{"} = -q$$

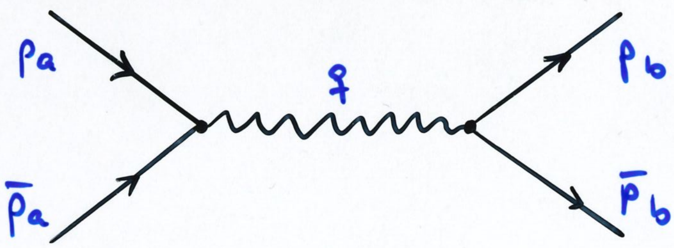
$$A_{fi} = -i e_a (2\pi)^4 \varepsilon \cdot (p_a - \bar{p}_a)\, \delta^4(q - p_a - \bar{p}_a)$$



$$\text{"}p_i\text{"} = -\bar{p}_b\,, \quad \text{"}p_f\text{"} = p_b\,, \quad \text{"}k\text{"} = q$$

$$A_{fi} = -i e_b (2\pi)^4 \varepsilon \cdot (p_b - \bar{p}_b)\, \delta^4(p_b + \bar{p}_b - q)$$

* Annihilation process



$$A_{fi} = i \frac{e_a e_b}{q^2} (p_a - \bar{p}_a) \cdot (p_b - \bar{p}_b) (2\pi)^4 \delta(p_b + \bar{p}_b - p_a - \bar{p}_a)$$

$$q = p_a + \bar{p}_a = p_b + \bar{p}_b$$

* Since we have already normalized to $2E$ particles per unit volume, we have

$$A_{fi} = M_{fi} (2\pi)^4 \delta^4 (\Sigma p_f - \Sigma p_i)$$

where M_{fi} is the invariant matrix element (see handout)

* Thus for e.g. annihilation process

$$M_{fi} = \frac{i e_a e_b}{q^2} (p_a - \bar{p}_a) \cdot (p_b - \bar{p}_b)$$

* In terms of the Mandelstam variables

$$s = (p_a + \bar{p}_a)^2 = q^2$$
$$t = (p_b - p_a)^2 = (\bar{p}_a - \bar{p}_b)^2$$
$$u = (p_a - \bar{p}_b)^2 = (\bar{p}_a - p_b)^2$$

We get $\quad M_{fi} = \frac{i e_a e_b}{s} (u - t)$

and hence the invariant differential cross section is

$$\boxed{\frac{d\sigma}{dt} = \frac{e_a^2 e_b^2 (u-t)^2}{64 \pi s^3 (p_a^*)^2}}$$

where p_a^* = c.m. momentum of a

$$= \sqrt{s/4 - m_a^2}$$

Dirac Equation

* Historically, Dirac (1928) was looking for a covariant wave equation that was first order in time, to avoid the above "problems" of the Klein-Gordon equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \beta m c^2 \psi - i\hbar c \underline{\alpha} \cdot \underline{\nabla} \psi \equiv H_{Dirac} \psi$$

* We want ψ also to satisfy KG equation

$\Rightarrow$ $\beta, \alpha_x, \alpha_y, \alpha_z$ are matrices

(c.f. example sheet: 2 component KG equ)

$$-\frac{\partial^2 \psi}{\partial t^2} = \beta m i \frac{\partial \psi}{\partial t} + \underline{\alpha} \cdot \underline{\nabla} \frac{\partial \psi}{\partial t} \qquad (\hbar = c = 1)$$

$$= \beta^2 m^2 \psi - i m (\beta \underline{\alpha} + \underline{\alpha} \beta) \cdot \underline{\nabla} \psi - (\underline{\alpha} \cdot \underline{\nabla})^2 \psi$$

$$= m^2 \psi - \underline{\nabla}^2 \psi \qquad \text{(KG eq.)}$$

Hence $\beta^2 = \alpha_x^2 = \alpha_y^2 = \alpha_z^2 = 1$

and $\beta \alpha_j + \alpha_j \beta = \alpha_j \alpha_k + \alpha_k \alpha_j = 0$

for $j \neq k = x, y, z$

$\Rightarrow$ $\beta, \alpha_x, \alpha_y, \alpha_z$ are (at least) 4×4 matrices

* A suitable representation is

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right)$$

$$\alpha_j = \left(\begin{array}{c|c} 0 & \sigma_j \\ \hline \sigma_j & 0 \end{array}\right) \quad \text{with} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

* Then ψ is represented by a 4-component object called a <u>spinor</u> (<u>not</u> a 4-vector!)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

each component satisfies KG equ

* For a <u>particle</u> at rest

$$\psi = \phi \, e^{-imc^2 t/\hbar}$$

Dirac equation $\Rightarrow$ $\phi = \beta \phi$

hence $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ 0 \\ 0 \end{pmatrix}$ $\phi_{1,2}$ tell us <u>spin orientation</u>

* For <u>antiparticle</u> at rest

$$\psi = \phi \, e^{+imc^2 t/\hbar} \quad , \quad \phi = -\beta \phi$$

$$\phi = \begin{pmatrix} 0 \\ 0 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

$\phi_{3,4}$ tell spin orientation

Spin of Dirac Particles

How do we <u>prove</u> that Dirac equation corresponds to spin ½ ?

* Show that there exists an operator $\underline{S}$ such that

$$\underline{J} = \underline{L} + \underline{S} = \text{a constant of motion}$$

and $\underline{S}^2 = s(s+1) = \frac{3}{4} I$ ← unit operator $\quad (\hbar = 1)$

* Note first that $\underline{L} = \underline{r} \times \underline{p}$ is <u>not</u> a constant of the motion

$$H = \beta m + \underline{\alpha} \cdot \underline{p}$$

$$[L_z, H] = [x, H]\, p_y - [y, H]\, p_x$$

$$[x, p_x] = i \quad \Rightarrow \quad [x, H] = i\alpha_x$$

hence $[L_z, H] = i(\alpha_x p_y - \alpha_y p_x)$

& in general $\quad [\underline{L}, H] = i\, \underline{\alpha} \times \underline{p} \neq 0$

* Thus we need $[\underline{S}, H] = -i\, \underline{\alpha} \times \underline{p}$

$\Rightarrow$ define $\underline{S} = \frac{1}{2} \underline{\Sigma}$

$$\Sigma_j = \left(\begin{array}{c|c} \sigma_j & 0 \\ \hline 0 & \sigma_j \end{array} \right) \qquad (\rightarrow \underline{S} = -i \alpha_x \alpha_y \alpha_z \underline{\alpha})$$

then $\quad \underline{S}^2 = \frac{1}{4}\left(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2\right) = \frac{3}{4} I$

Magnetic Moment

In an electromagnetic field we make the usual minimal substitutions:

$$H \longrightarrow H - eV \quad , \qquad \underline{p} \longrightarrow \underline{p} - e\underline{A}$$

in the Dirac equation

$$\Rightarrow \quad H = \underline{\alpha} \cdot (\underline{p} - e\underline{A}) + \beta m + eV$$

* Note that we no longer get the KG equation when we "square":

$$(H - eV)^2 = \sum_{j,k} \alpha_j \alpha_k (p_j - eA_j)(p_k - eA_k) + m^2$$

$$= (\underline{p} - e\underline{A})^2 + m^2 - e \sum_{j \neq k} (\alpha_j \alpha_k p_j A_k + \alpha_j \alpha_k A_j p_k)$$

but for $j \neq k$, $\quad \alpha_j \alpha_k = i \varepsilon_{jke} \Sigma_e$

also $\quad p_j A_k = A_k p_j - i (\nabla_j A_k)$

and $\quad \varepsilon_{jke} \Sigma_e \nabla_j A_k = \underline{\Sigma} \cdot (\underline{\nabla} \times \underline{A}) = \underline{\Sigma} \cdot \underline{B}$

$$\Rightarrow \quad (H - eV)^2 = (\underline{p} - e\underline{A})^2 + m^2 - e\, \underline{\Sigma} \cdot \underline{B}$$

$$H - eV \underset{\substack{\uparrow \\ m \text{ large}}}{\simeq} m + \frac{1}{2m} (\underline{p} - e\underline{A})^2 - \underbrace{\frac{e}{2m} \underline{\Sigma} \cdot \underline{B}}_{?} + \ldots$$

* magnetic moment

$$\underline{\mu} = \frac{e}{m} \underline{S} = g_e \left(\frac{e}{2m} \right) \underline{S}$$

where $\quad g_e = 2 \qquad$ (exp: $2.0023193\ldots$)

Dirac Density and Current

$$i \frac{\partial \psi}{\partial t} = \beta m \psi - i \underline{\alpha} \cdot \underline{\nabla} \psi$$

$$\frac{\partial \psi}{\partial t} = -i m \beta \psi - \underline{\alpha} \cdot (\underline{\nabla} \psi)$$

* Transpose and complex conjugate:

$$\frac{\partial \psi^{+}}{\partial t} = + i m \psi^{+} \beta - (\underline{\nabla} \psi^{+}) \cdot \underline{\alpha}$$

NB $\beta, \underline{\alpha}$ are hermitian

Hence $$\frac{\partial}{\partial t} (\psi^{+} \psi) = - \underline{\nabla} (\psi^{+} \underline{\alpha} \psi)$$

* Thus we can take

$$\rho = \psi^{+} \psi \equiv |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$$

$$\underline{j} = \psi^{+} \underline{\alpha} \psi$$

NB: Density ρ is positive definite!

→ This is what Dirac wanted, but it is really a problem – what about antiparticles ?!

* Answer will not come until we learn some quantum field theory.

Covariant Notation

Nobody uses $\underline{\alpha}$ and β any more.

* Instead we define

$$\gamma^0 = \beta, \qquad \gamma^j = \beta\alpha_j \qquad (j = 1,2,3) \rightarrow \{\gamma^\mu, \gamma^\nu\} = = 2g^{\mu\nu}$$

and

$$\bar{\psi} = \psi^\dagger\beta = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

(in usual representation)

Then

$$\rho = \psi^\dagger\psi = \psi^\dagger\beta^2\psi = \bar{\psi}\gamma^0\psi$$

$$\underline{J} = \psi^\dagger\underline{\alpha}\psi = \psi^\dagger\beta^2\underline{\alpha}\psi = \bar{\psi}\underline{\gamma}\psi$$

and J^μ is a 4-vector

$$\boxed{J^\mu = (\rho, \underline{J}) = \bar{\psi}\gamma^\mu\psi}$$

* We can show that

$$\bar{\psi}\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$$

transforms like a scalar (invariant) under Lorentz transformations

* Multiplying through by β, Dirac eqn. becomes

$$i\gamma^0 \frac{\partial\psi}{\partial t} = m\psi + i\gamma^j\nabla_j\psi$$

$$\Rightarrow \quad \boxed{\begin{aligned}(\gamma^\mu\partial_\mu + im)\psi &= 0\\ \text{or} \quad (\gamma^\mu p_\mu - m)\psi &= 0\end{aligned}}$$

Dirac equation

Free - Particle Spinors

A positive - energy plane wave

$$\psi = u(E, \underline{p})\, e^{i\underline{p}\cdot\underline{r} - iEt} \qquad (E>0)$$

satisfies $(\gamma^\mu p_\mu - m)\, u = 0$

With $u = \begin{pmatrix} \phi \\ \hdashline \chi \end{pmatrix} \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \\ \chi_1 \\ \chi_2 \end{pmatrix}$ this means

$$\left(\begin{array}{c:c} E-m & -\underline{\sigma}\cdot\underline{p} \\ \hdashline +\underline{\sigma}\cdot\underline{p} & -E-m \end{array}\right) \begin{pmatrix} \phi \\ \hdashline \chi \end{pmatrix} = 0$$

Thus $\chi = \dfrac{\underline{\sigma}\cdot\underline{p}}{E+m}\,\phi$.

Remember $\underline{S} = \frac{1}{2}\underline{\Sigma} = \frac{1}{2}\left(\begin{array}{c:c} \underline{\sigma} & 0 \\ \hdashline 0 & \underline{\sigma} \end{array}\right)$.

Hence

$\phi = N\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for spin up ($\parallel z$)

$= N\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for spin down

We have

$$\underline{\sigma}\cdot\underline{p}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}, \quad \underline{\sigma}\cdot\underline{p}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix},$$

thus

$$u^{\uparrow} = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}, \quad u^{\downarrow} = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

* Normalization is as usual

$$\rho = \psi^\dagger\psi = u^\dagger u = 2E \quad \text{particles/unit volume}$$

$$= N^2\left(1 + \frac{p_x^2 + p_y^2 + p_z^2}{(E+m)^2}\right)$$

$$= \frac{N^2}{(E+m)^2}\left[(E+m)^2 + (E-m)(E+m)\right]$$

Hence $\boxed{N = \sqrt{E+m}}$

* Notice that 'small' (3,4) components are $O(v/c)$ relative to 'large' (1,2) ones.

* For antiparticle of 4-momentum $(E, \underline{p})$ we need solution with $p^\mu \to (-E, -\underline{p})$:

$$\psi = v(E, \underline{p})\, e^{-i\underline{p}\cdot\underline{r} + iEt}$$

With the Dirac equation we find

$$\left(\begin{array}{c|c} -E-m & \underline{\sigma}\cdot\underline{p} \\ \hline -\underline{\sigma}\cdot\underline{p} & E-m \end{array}\right)\left(\begin{array}{c} \phi \\ \hline \chi \end{array}\right) = 0$$

$$\Rightarrow \quad \phi = \frac{\underline{\sigma}\cdot\underline{p}}{E+m}\chi$$

* Like 4-momentum, spin must be reversed

$$v^\uparrow = N\begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}, \qquad v^\downarrow = N\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

Charge Conjugation

As the KG equation, the Dirac equation has charge conjugation symmetry:
If ψ is a negative-energy solution, there is a transformation

$$\psi \xrightarrow{\mathcal{C}} \psi^c = C\psi^*$$

such that ψ^c is a positive-energy solution for charge $-e$. To find C:

$$\gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0$$

$$\rightarrow \quad \gamma^{*\mu}(\partial_\mu - ieA_\mu)\psi^* - im\psi^* = 0$$

$$\Rightarrow \quad -\underbrace{C\gamma^{*\mu}C^{-1}}(\partial_\mu - ieA_\mu)\psi^c + im\psi^c = 0$$

Hence we need $\quad C\gamma^{*\mu}C^{-1} = -\gamma^\mu$,

i.e. $\quad \gamma^\mu C = -C\gamma^{*\mu}$

* Since all γ^μ are real except γ^2 (which is pure imaginary) in our standard representation, we can take

$$C = i\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

<u>NB</u> Explicitly: $v^{\uparrow c} = u^\uparrow$, $\quad v^{\downarrow c} = -u^\downarrow$

Parity Invariance

Similarly if $\psi(\underline{r},t)$ is a solution of Dirac equation, there exists transformation

$$\psi(\underline{r},t) \xrightarrow{\mathcal{P}} \psi^{P}(\underline{r},t) = P\,\psi(-\underline{r},t)$$

such that ψ^P is also a solution.

$$\left(\gamma^0 \frac{\partial}{\partial t} - \underline{\gamma}\cdot\underline{\nabla} + im\right)\psi(\underline{r},t) = 0$$

$$\rightarrow \quad \left(\gamma^0 \frac{\partial}{\partial t} + \underline{\gamma}\cdot\underline{\nabla} + im\right)\psi(-\underline{r},t) = 0$$

$$\Rightarrow \left(\underbrace{P\gamma^0 P^{-1}}\frac{\partial}{\partial t} + \underbrace{P\underline{\gamma}P^{-1}}\cdot\underline{\nabla} + im\right)\psi^P(\underline{r},t) = 0$$

Hence we need $P\gamma^0P^{-1} = \gamma^0$, $P\underline{\gamma}P^{-1} = -\underline{\gamma}$,

i.e. $P\gamma^0 = \gamma^0 P$, $P\gamma^j = -\gamma^j P$ $(j=1,2,3)$.

This is satisfied by

$$P = \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

<u>NB</u> For particle at rest

$$\psi = u(m,0)\,e^{-iEt}, \qquad \psi^P = +\psi$$

but for antiparticle

$$\psi = v(m,0)\,e^{+iEt} \qquad \psi^P = -\psi$$

opposite intrinsic parity

* Notice that for KG equation the parity transformation is simply

$$\phi(\underline{r},t) \xrightarrow{P} \phi^P(\underline{r},t) = \phi(-\underline{r},t)$$

i.e. ϕ is a true scalar function, since

$$\left(\frac{\partial^2}{\partial t^2} - \underline{\nabla}^2 + m^2\right) \phi(\underline{r},t) = 0$$

$$\Rightarrow \quad \left(\frac{\partial^2}{\partial t^2} - \underline{\nabla}^2 + m^2\right) \phi(-\underline{r},t) = 0$$

* For the Dirac equation, the scalar is not ψ but

$$\Phi \equiv \bar{\psi}\psi = \psi^\dagger \gamma^0 \psi$$

Check: $\Phi(\underline{r},t) = \psi^\dagger(\underline{r},t)\, \gamma^0\, \psi(\underline{r},t)$

$$\rightarrow \Phi^P(\underline{r},t) = \psi^\dagger(-\underline{r},t)\, \gamma^{0\dagger} \gamma^0 \gamma^0\, \psi(-\underline{r},t)$$

$$= \psi^\dagger(-\underline{r},t)\, \gamma^0\, \psi(-\underline{r},t)$$

$$= \Phi(-\underline{r},t)$$

* Similarly, J^μ is a true vector:

$$J^\mu(\underline{r},t) = \psi^\dagger(\underline{r},t)\, \gamma^0 \gamma^\mu\, \psi(\underline{r},t)$$

$$J^{P\mu}(\underline{r},t) = \psi^\dagger(-\underline{r},t)\, \gamma^{0\dagger} \gamma^0 \gamma^\mu \gamma^0\, \psi(-\underline{r},t)$$

$$\gamma^{0\dagger}\gamma^0\gamma^\mu\gamma^0 = \gamma^\mu\gamma^0 = \begin{cases} \gamma^0\gamma^\mu & \text{for } \mu = 0 \\ -\gamma^0\gamma^\mu & \text{for } \mu = 1,2,3 \end{cases}$$

Hence

$$\left.\begin{aligned} J^{P0}(\underline{r},t) &= J^0(-\underline{r},t) \\ \underline{J}^P(\underline{r},t) &= -\underline{J}(-\underline{r},t) \end{aligned}\right\} \text{ as expected for a vector}$$

* Weak interactions involve the **axial current** $J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$

where
$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \left(\begin{array}{c|c} 0 & I \\ \hline I & 0 \end{array}\right)$$

in our standard representation.

Under parity transformations this is an **axial vector**:
$$J_A^{P\mu}(\underline{r},t) = \psi^\dagger(-\underline{r},t)\,\gamma^\mu\gamma^5\gamma^0\,\psi(-\underline{r},t).$$

Now $\gamma^5\gamma^0 = -\gamma^0\gamma^5$ (actually $\gamma^5\gamma^\mu = -\gamma^\mu\gamma_5$),

hence
$$J_A^{P0}(\underline{r},t) = -J_A^0(-\underline{r},t)$$
$$\underline{J}_A^P(\underline{r},t) = +\underline{J}_A(-\underline{r},t)$$

as expected for an axial vector.

* Similarly
$$\Phi_P = \bar{\psi}\gamma^5\psi$$

is a **pseudoscalar**:
$$\Phi_P^P(\underline{r},t) = \psi^\dagger(-\underline{r},t)\,\gamma^5\gamma^0\,\psi(-\underline{r},t)$$
$$= -\bar{\psi}(-\underline{r},t)\,\gamma^5\,\psi(-\underline{r},t)$$
$$= -\Phi^P(-\underline{r},t)$$

Massless Dirac Particles

For $m=0$ the positive-energy free particle solutions are

$$\psi = u(E,\underline{p})\, e^{i\underline{p}\cdot\underline{r} - iEt}$$

where $E = |\underline{p}|$ and so $u = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ gives

$$\left(\begin{array}{c|c} |\underline{p}| & -\underline{\sigma}\cdot\underline{p} \\ \hline \underline{\sigma}\cdot\underline{p} & -|\underline{p}| \end{array}\right) \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0$$

$\Rightarrow$ $\chi = \Lambda \phi$ where

$\Lambda = \underline{\sigma}\cdot\underline{p}/|\underline{p}|$ is the helicity operator:

'right-handed'

$$\Lambda = \begin{cases} +1 & \text{for spin aligned along momentum} \\ -1 & \text{for spin aligned opposite to momentum} \end{cases}$$

'left-handed'

* Note that

$$\gamma^5 \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \chi \\ \phi \end{pmatrix},$$

thus if ψ represents a massless particle

$$\gamma^5 \psi = \begin{pmatrix} \Lambda\phi \\ \phi \end{pmatrix} = \Lambda \psi \qquad (\Lambda^2 = 1)$$

* Hence γ^5 is the helicity operator for massless particles (minus helicity for massless antiparticles)

* Weak interactions are 'V − A',
i.e. they involve the current

$$(J^\mu - J^\mu_A)_{fi} = \bar{\psi}_f \gamma^\mu (1-\gamma^5) \psi_i .$$

If i is a massless particle, then $(1-\gamma^5)\psi_i$ vanishes for helicity $+1$, i.e. only left-handed states interact. The same applies to particle f, since

$$\bar{\psi}_f \gamma^\mu (1-\gamma^5)\psi_i = \psi_f^\dagger \gamma^0 (1+\gamma^5)\gamma^\mu \psi_i$$
$$= \left[(1-\gamma^5)\psi_f\right]^\dagger \gamma^0 \gamma^\mu \psi_i$$

* Thus, if neutrinos are massless, only left-handed neutrinos (right-handed antineutrinos) interact.

* In the Standard Model, right-handed neutrinos <u>do not exist</u>.

* This is consistent with relativity, because helicity is frame-independent for massless particles.

* If neutrinos have mass, both helicities must exist, but the right-handed states interact more weakly (as for electrons).

Electromagnetic Interactions

We already saw that in an e.m. field the Dirac Hamiltonian is

$$H = \underline{\alpha} \cdot (\underline{p} - e\underline{A}) + \beta m + eV$$
$$= H_0 + e\gamma^0\gamma^\mu A_\mu$$

where H_0 is the free-particle Hamiltonian.

* Hence first-order perturbation theory gives a transition amplitude

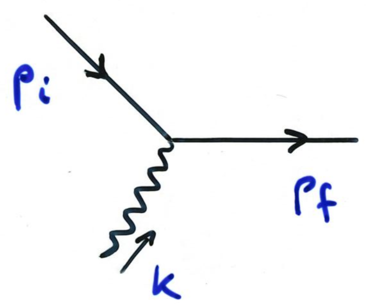
$$A_{fi} = i \int \psi_f^\dagger (e\gamma^0\gamma^\mu A_\mu)\, \psi_i \, d^4x$$
$$= -ie \int J^\mu_{fi} \, A_\mu \, d^4x$$

where

$$J^\mu_{fi} = \bar{\psi}_f \gamma^\mu \psi_i \, .$$

For plane waves $\psi_{f,i} = u_{f,i} \, e^{-ip_{f,i} \cdot x}$,

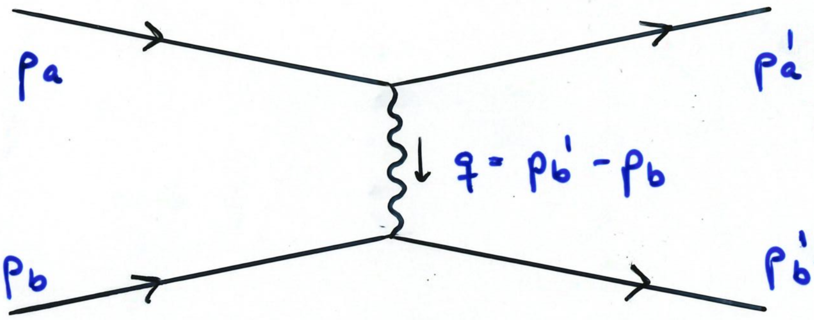
and so the only difference from the KG (spin zero) case is that we need a vertex factor of



$-ie\, \bar{u}_f \gamma^\mu u_i$ for spin ½

instead of

$-ie\,(p_f + p_i)^\mu$ for spin 0



The invariant matrix element is

$$M_{fi} = \frac{i e_a e_b}{q^2} \left(\bar{u}_{a'} \gamma^\mu u_a\right)\left(\bar{u}_{b'} \gamma_\mu u_b\right)$$

Hence
$$|M_{fi}|^2 = \frac{e_a^2 e_b^2}{t^2} L_a^{\mu\nu} L^b_{\mu\nu}$$

where
$$L_a^{\mu\nu} = \left(\bar{u}_{a'} \gamma^\mu u_a\right)\left(\bar{u}_{a'} \gamma^\nu u_a\right)^*$$
$$L^b_{\mu\nu} = \left(\bar{u}_{b'} \gamma_\mu u_b\right)\left(\bar{u}_{b'} \gamma_\nu u_b\right)^* .$$

For given spin states of a, b, a' and b', we can evaluate this explicitly using the above expressions for free-particle spinors. However, we often consider unpolarized scattering, when we have to average over initial spin states and sum over final. Then

$$L_a^{\mu\nu} = \frac{1}{2} \sum_{\text{spins}} \left(\bar{u}_{a'} \gamma^\mu u_a\right)\left(\bar{u}_{a'} \gamma^\nu u_a\right)^*$$

and similarly for $L^b_{\mu\nu}$.

Gamma Matrix Algebra

$$L_a^{\mu\nu} = \frac{1}{2} \sum_{\text{spins}} (\bar{u}_{a'} \gamma^\mu u_a)(\bar{u}_{a'} \gamma^\nu u_a)^*$$

can be expressed in terms of traces of products of γ - matrices, using

$$\sum_{\text{spins}} u\bar{u} = u^\uparrow \bar{u}^\uparrow + u^\downarrow \bar{u}^\downarrow = \gamma^\mu p_\mu + m$$

Check explicitly!

we have

$$(\bar{u}_{a'} \gamma^\nu u_a)^* = (u_{a'}^\dagger \gamma^0 \gamma^\nu u_a)^* = u_a^\dagger \gamma^{\nu\dagger} \gamma^0 u_{a'} = \bar{u}_a \gamma^\nu u_{a'}$$

since $\gamma^{\nu\dagger}\gamma^0 = \gamma^0 \gamma^\nu$.

* Thus

$$L_a^{\mu\nu} = \frac{1}{2} \sum_{a' \text{spins}} \bar{u}_{a'} \gamma^\mu (\not{p}_a + m_a) \gamma^\nu u_{a'}$$

where we use Feynman's notation $\not{p} = \gamma^\mu p_\mu$.

Now putting in Dirac matrix indices,

$$\bar{u}_\alpha \Gamma_{\alpha\beta} u_\beta = \mathrm{Tr}(u\bar{u}\Gamma),$$

so

$$L_a^{\mu\nu} = \frac{1}{2} \mathrm{Tr}\{(\not{p}_a' + m_a)\gamma^\mu (\not{p}_a + m_a)\gamma^\nu\}$$

$$\Rightarrow k_\mu k_\nu' L_a^{\mu\nu} = \frac{1}{2}\mathrm{Tr}\{(\not{p}_a' + m_a)\not{k}(\not{p}_a + m_a)\not{k}'\}$$

$$= \frac{1}{2}\mathrm{Tr}\{\not{p}_a' \not{k} \not{p}_a \not{k}'\} + \frac{1}{2} m_a^2 \mathrm{Tr}\{\not{k}\not{k}'\}$$

$$= 2\left(p_a'\cdot k \; p_a\cdot k' + p_a\cdot k \; p_a'\cdot k' - p_a\cdot p_a' \; k\cdot k' + m_a^2 \, k\cdot k'\right)$$

see example sheet

* Removing the arbitrary vectors k_μ and k'_ν,

$$L_a^{\mu\nu} = 2\left[p_a^\mu p_a'^\nu + p_a'^\mu p_a^\nu - (p_a \cdot p_a' - m_a^2)\, g^{\mu\nu} \right]$$

and similarly

$$L^b_{\mu\nu} = 2\left[p_{b\mu} p'_{b\nu} + p'_{b\mu} p_{b\nu} - (p_b \cdot p_b' - m_b^2)\, g_{\mu\nu} \right]$$

and so

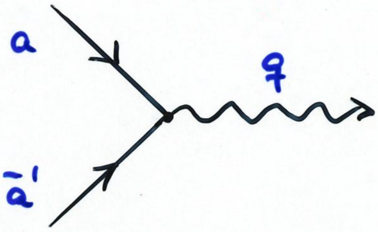
$$L_a^{\mu\nu} L^b_{\mu\nu} = 8\Big[(p_a \cdot p_b)(p_a' \cdot p_b') + (p_a \cdot p_b')(p_a' \cdot p_b) - m_a^2 (p_b \cdot p_b') - m_b^2 (p_a \cdot p_a') + 2 m_a^2 m_b^2 \Big]$$

* Expressing this in terms of the Mandelstam variables s, t, u we find an invariant differential cross section

$$\frac{d\sigma}{dt} = \frac{e_a^2 e_b^2}{32\pi s t^2 (p_a^*)^2} \Big[s^2 + u^2 - 4(m_a^2 + m_b^2)(s+u) + 6(m_a^2 + m_b^2)^2 \Big]$$

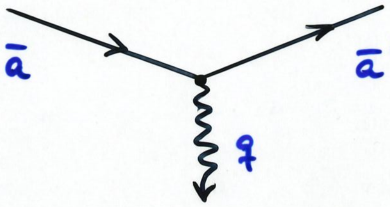
where p_a^* is the momentum of a in the c.m. frame.

* For processes involving Dirac antiparticles, we should use the v-spinors in place of u's:



"u_i" $= u_a$, "$\bar{u}_f$" $= \bar{v}_a'$

$\Rightarrow$ vertex factor $-ie_a \bar{v}_a' \gamma^\mu u_a$



"u_i" $= v_a'$, "$\bar{u}_f$" $= \bar{v}_a$

$\Rightarrow$ vertex factor $-ie_a \bar{v}_a \gamma^\mu v_a'$

* We also need

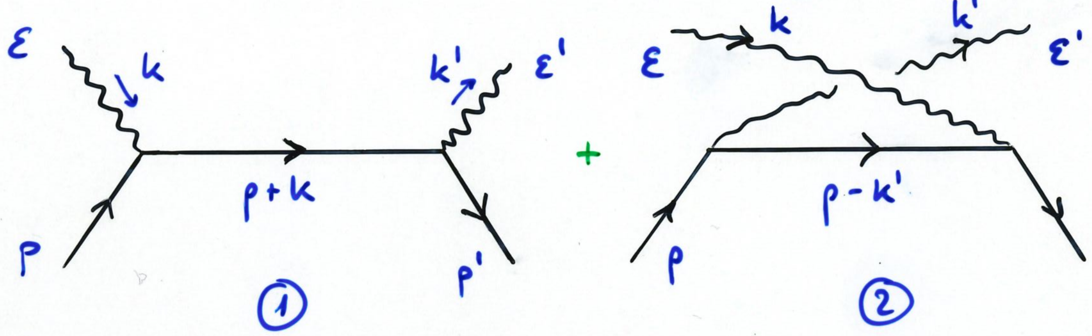
$$\sum_{\text{spins}} v\bar{v} \equiv v^\uparrow \bar{v}^\uparrow + v^\downarrow \bar{v}^\downarrow = \not{p} - m$$

different sign!

* Note, however, that the tensor $L_a^{\mu\nu}$ only involves m_a^2. Replacing a by $\bar{a}$ reverses sign of m_a, which does not affect the (unpolarized) scattering cross section.

Hence ab, $\bar{a}b$, $a\bar{b}$ and $\bar{a}\bar{b}$ scattering (by single photon exchange) are all the 'same'.

Compton Scattering



In the Compton scattering process, $\gamma + e \longrightarrow \gamma + e$, we need the propagator factor for a virtual Dirac particle.

* This is $$\frac{i}{q^2 - m^2} \sum_{\text{spins}} u \bar{u} = \frac{i(\not{q} + m)}{q^2 - m^2}$$

* Compare $$\frac{i}{q^2} \sum_{\text{spins}} \varepsilon_\mu \varepsilon_\nu^* \text{ "=" } \frac{i(-g_{\mu\nu})}{q^2}$$ for photon

See later

Thus the 2 graphs give $M_{fi} = M_1 + M_2$ where

$$M_1 = \varepsilon'_\nu \bar{u}' (-ie\gamma^\nu) \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2} (-ie\gamma^\mu) u \varepsilon_\mu$$

$$M_2 = \varepsilon_\mu \bar{u}' (-ie\gamma^\mu) \frac{i(\not{p} - \not{k}' + m)}{(p-k')^2 - m^2} (-ie\gamma^\nu) u \varepsilon'_\nu$$

* The relative phase is $+1$ because the graphs differ by exchange of identical bosons.

* For the unpolarized process, we want to average over initial spin states and sum over final ones. We know how to do this for the electrons. For the photons, consider the incoming polarization ε_μ. We can write schematically

$$\sum_{\text{spins}} (M_1 + M_2)^2 = \sum_{\varepsilon = \varepsilon_x, \varepsilon_y} \varepsilon_\mu \varepsilon_\lambda^* M^{\mu\lambda},$$

where the tensor $M^{\mu\lambda}$ is to be determined. However, we know that it must have the properties

$$k_\mu M^{\mu\lambda} = k_\lambda M^{\mu\lambda} = 0 \qquad (*)$$

to ensure gauge invariance, which allows us to replace $\varepsilon_\mu \longrightarrow \varepsilon_\mu + a k_\mu$ for any a.

Choose z axis along $\underline{k}$:

$$k_\mu = |\underline{k}| \, (1, 0, 0, -1).$$

Then $(*)$ implies $M^{00} = M^{03} = M^{30} = M^{33}$, while

$$\begin{aligned} \sum_{\varepsilon = \varepsilon_x, \varepsilon_y} \varepsilon_\mu \varepsilon_\lambda^* M^{\mu\lambda} &= M^{11} + M^{22} \\ &= M^{11} + M^{22} + M^{33} - M^{00} \\ &= -g_{\mu\lambda} M^{\mu\lambda} = -M^\mu{}_\mu \end{aligned}$$

* Thus, due to gauge invariance, we can replace photon polarization sum by $-g_{\mu\lambda}$.

* Applying the same trick to the outgoing photon polarization (ε'_ν) sum, we find a contribution from diagram 1

$$\frac{1}{4}\sum_{\text{spins}} |M_1|^2 = \frac{e^4}{4(s-m^2)^2} \times$$

$$\times \, \mathrm{Tr}\left\{ (\not{p}'+m)\gamma^\nu(\not{p}+\not{k}+m)\gamma^\mu(\not{p}+m)\gamma_\mu(\not{p}+\not{k}+m)\gamma_\nu \right\}$$

* In the extreme relativistic limit ($s, |t|, |u| \gg m^2$), this becomes (using results on the examples sheet)

$$\frac{e^4}{s^2}\,\mathrm{Tr}\left\{ \not{p}'(\not{p}+\not{k})\not{p}(\not{p}+\not{k}) \right\}$$

$$= 8\,\frac{e^4}{s^2}\,(p\cdot k)(p'\cdot k)$$

$$= -2e^4\,\frac{u}{s}$$

* The other diagram and interference terms are left as an exercise.