Standard Model of Particle Physics

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3. Gauge Theory and Standard Model

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Gauge Symmetry in QED

* The Lagrangian density for the free e.m. field is

$$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where $F^{\mu\nu}$ is the field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Thus $\mathcal{L}_{em} = \frac{1}{2}(\underline{E}^2 - \underline{B}^2)$

* In $A^0 = 0$ gauge the momentum density is

$$\underline{\pi} = \partial \underline{A}/\partial t = -\underline{E}$$

hence

$$\mathcal{H}_{em} = \underline{\pi} \cdot \underline{\dot{A}} - \mathcal{L}_{em} = \frac{1}{2}(\underline{E}^2 + \underline{B}^2)$$

* Gauge symmetry: $F^{\mu\nu}$ and hence $\mathcal{L}_{em}$ are invariant under transformation

$$A^\mu \longrightarrow A'^\mu = A^\mu + \partial^\mu \chi$$

N.B. A photon mass term $-\frac{1}{2} m_\gamma^2 A^\mu A_\mu$

is forbidden by gauge symmetry!

* The full Lagrangian for charged Dirac fermions interacting with the e.m. field is

$$\mathcal{L} = \mathcal{L}_{em} + \mathcal{L}_D$$

$$\mathcal{L}_D = i\bar{\psi}\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\bar{\psi}\psi$$

Under a gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \chi$$

$$\psi \to \psi'$$

we have

$$\mathcal{L}_D \to \mathcal{L}_D' = i\bar{\psi}'\gamma^\mu(\partial_\mu + ieA_\mu + ie(\partial_\mu\chi))\psi' - m\bar{\psi}'\psi'$$

* Invariance of $\mathcal{L}_D$ under gauge transformations thus requires $\psi \to \psi'$ such that

(a) $\bar{\psi}'\psi' = \bar{\psi}\psi$

$$\Rightarrow \quad \psi' = e^{i\phi(x)}\psi$$

(b) $\partial_\mu\psi' + ie\psi'\partial_\mu\chi = e^{i\phi(x)}\partial_\mu\psi$

$$\Rightarrow \quad \phi(x) = -e\chi(x)$$

$$\Rightarrow \quad \underline{\psi' = e^{-ie\chi}\psi}$$

Thus we find

$\boxed{\text{Gauge transformation of e.m. field} \Longleftrightarrow \text{Phase transformation of (charged) Dirac field}}$

* Conversely, if we demand symmetry of $\mathcal{L}$ under local phase transformations of ψ, then this requires the existence of a (massless) vector field (to cancel the term involving $\partial_\mu \phi(x)$).

* The gauge symmetry occurs because the derivative ∂_μ only appears in the combination called the covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu .$$

Then $D_\mu \psi$ transforms in the same way as ψ itself

$$D'_\mu \psi' = (\partial_\mu + ieA_\mu + ie(\partial_\mu \chi))\, e^{-ie\chi} \psi$$
$$= e^{-ie\chi} D_\mu \psi$$

* Note that $\mathcal{L}_{em}$ also involves only D_μ since

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = D^\mu A^\nu - D^\nu A^\mu$$

* Recall that e.m. phase (gauge) symmetry $\Rightarrow$ Conservation of (electric) current and charge (Noether)

* Successive gauge transformations commute:

$$e^{-ie\chi_1} e^{-ie\chi_2} = e^{-ie\chi_2} e^{-ie\chi_1} = e^{-ie(\chi_1+\chi_2)}$$

This is called <u>Abelian ($U(1)$) gauge symmetry.</u> ← N.H. Abel

→ <u>Quantum Electrodynamics</u> (QED)

* Recall that photons do not couple to each other (since they are not charged).

* It is believed that all fundamental interactions are described by some form of gauge theory.

Non-Abelian Gauge Symmetry

* Suppose the Lagrangian involves two fermion fields (e.g. ν_e and e^-) and we demand symmetry under transformations that mix them together while preserving normalization and orthogonality:

$$\begin{cases} \psi_1 \longrightarrow \psi_1' = \alpha \psi_1 + \beta \psi_2 \\ \psi_2 \longrightarrow \psi_2' = \gamma \psi_1 + \delta \psi_2 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha\alpha^* + \beta\beta^* = \gamma\gamma^* + \delta\delta^* = 1 \\ \alpha\gamma^* + \beta\delta^* = \gamma\alpha^* + \delta\beta^* = 0 \end{cases}$$

Hence

$$\Psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \longrightarrow \Psi' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

i.e.

$$\mathcal{U}\,\mathcal{U}^+ = \mathbb{1}$$

Hence the matrix of coefficients $\mathcal{U}$ is a <u>unitary</u> matrix.

* $\mathcal{U}$ has 4 complex elements satisfying 4 constraints $\Rightarrow$ 4 real parameters. It can be written as

$$\mathcal{U} = \exp\left[i\alpha_0 + i\alpha_1\tau_1 + i\alpha_2\tau_2 + i\alpha_3\tau_3 \right]$$

where $\tau_{1,2,3}$ are the Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\alpha_{0,1,2,3}$ are the real parameters.

* The exponential can be defined as a power series

$$\exp[A] = \mathbb{1} + A + \frac{1}{2}A^2 + \ldots$$

Note that in general

$$\exp[A_1]\cdot\exp[A_2] \neq \exp[A_2]\cdot\exp[A_1]$$
$$\neq \exp[A_1 + A_2]$$

$\Rightarrow$ the gauge symmetry is non-Abelian [$\mathcal{U}(2)$]. We can write

$\rightsquigarrow$ Yang-Mills theories

$$\mathcal{U} = e^{i\alpha_0}\, V$$

where $e^{i\alpha_0} \in \mathcal{U}(1)$, the Abelian symmetry, and $V = e^{i\underline{\alpha}\cdot\underline{\tau}} \in S\mathcal{U}(2)$, the non-Abelian group of 2×2 unitary matrices with determinant 1.

* Now consider only $SU(2)$ transformations

$$\mathcal{U} = e^{i\underline{\alpha}\cdot\underline{\tau}} \in SU(2)$$

We have

$$\det \mathcal{U} = e^{i\,\mathrm{tr}(\underline{\alpha}\cdot\underline{\tau})} = e^{0} = 1$$

$$\mathcal{U}^{+} = e^{-i\underline{\alpha}^{*}\cdot\underline{\tau}^{+}} = e^{-i\underline{\alpha}\cdot\underline{\tau}} = \mathcal{U}^{-1}$$

$\alpha_{1,2,3}$ real $\qquad$ $\tau_{1,2,3}$ hermitian

* The matrices $\tau_{1,2,3}$ are the generators of the group $SU(2)$. An infinitesimal gauge transformation can be written as

$$\mathcal{U} = e^{i\underline{\alpha}\cdot\underline{\tau}} \simeq 1 + i\underline{\alpha}\cdot\underline{\tau}$$

$\alpha_{1,2,3}$ small

* In fact we can define the exponential for matrices like this:

$$e^{A} = \lim_{N\to\infty}\left(1 + \frac{A}{N}\right)^{N}$$

$$e^{i\underline{\alpha}\cdot\underline{\tau}} = \lim_{\varepsilon\to 0}\left(1 + i\varepsilon\underline{\alpha}\cdot\underline{\tau}\right)^{1/\varepsilon}$$

* For simplicity we shall usually consider infinitesimal gauge transformations. If necessary, we can then build up finite ones as above.

* Consider a small $SU(2)$ gauge transformation of the form

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \longrightarrow \Psi' = \left[1 + i\frac{g}{2} \underline{\omega}(x) \cdot \underline{\tau} \right] \Psi$$

i.e. $\alpha_{1,2,3} = \frac{1}{2} g \, \omega_{1,2,3}$ = small.

* By analogy with QED, we expect gauge invariance to require the presence of vector fields W_1^μ, W_2^μ, W_3^μ, coupling to the fermions via the covariant derivative

$$D^\mu = \partial^\mu + i\frac{g}{2} \underline{W}^\mu \cdot \underline{\tau}$$

see later

i.e. $\mathcal{L}_D = i \bar{\Psi} \gamma^\mu D_\mu \Psi \quad \left[- m \bar{\Psi} \Psi \right]$

* What transformation law must $W^\mu_{1,2,3}$ have to make $\mathcal{L}_D$ gauge invariant? Clearly we need $D_\mu \Psi$ to transform just like Ψ itself:

$$D'_\mu \Psi' = \left[1 + i\frac{g}{2} \underline{\omega} \cdot \underline{\tau} \right] D_\mu \Psi$$

$$\bar{\Psi}' = \bar{\Psi} \left[1 - i\frac{g}{2} \underline{\omega} \cdot \underline{\tau} \right]$$

$$\Rightarrow \quad \bar{\Psi}' \gamma^\mu D'_\mu \Psi' = \bar{\Psi} \gamma^\mu D_\mu \Psi + \mathcal{O}(\omega^2)$$

neglect

as wanted

* This implies a more complicated gauge transformation law for $W^\mu_{1,2,3}$. With

$$D^\mu \longrightarrow D'^\mu = \partial^\mu + i\frac{g}{2}\underline{W}'^\mu\cdot\underline{\tau}$$

we have

$$D'^\mu \Psi' = \left(\partial^\mu + i\frac{g}{2}\underline{W}'^\mu\cdot\underline{\tau}\right)\left(1 + i\frac{g}{2}\underline{\omega}\cdot\underline{\tau}\right)\Psi .$$

As we have seen this should equal

$$\left[1 + i\frac{g}{2}\underline{\omega}\cdot\underline{\tau}\right] D^\mu \Psi =$$

$$= \left[1 + i\frac{g}{2}\underline{\omega}\cdot\underline{\tau}\right]\left(\partial^\mu + i\frac{g}{2}\underline{W}^\mu\cdot\underline{\tau}\right)\underbrace{\left(1 - i\frac{g}{2}\underline{\omega}\cdot\underline{\tau}\right)\left(1 + i\frac{g}{2}\underline{\omega}\cdot\underline{\tau}\right)}_{=1+O(\omega^2)}\Psi$$

Thus

$$\partial^\mu + i\frac{g}{2}\underline{W}'^\mu\cdot\underline{\tau} =$$

$$= \left(1 + i\frac{g}{2}\underline{\omega}\cdot\underline{\tau}\right)\left(\partial^\mu + i\frac{g}{2}\underline{W}^\mu\cdot\underline{\tau}\right)\left(1 - i\frac{g}{2}\underline{\omega}\cdot\underline{\tau}\right)$$

$$= \partial^\mu + i\frac{g}{2}\underline{W}^\mu\cdot\underline{\tau} - i\frac{g}{2}\left(\partial^\mu\underline{\omega}\right)\cdot\underline{\tau}$$

$$- \frac{g^2}{4}\left(\omega_j\tau_j W^\mu_k\tau_k - W^\mu_k\tau_k\omega_j\tau_j\right) + O(\omega^2)$$

Now

$$\tau_j\tau_k - \tau_k\tau_j = 2i\,\varepsilon_{jkl}\,\tau_l$$

Hence

$$\boxed{W'^\mu_l = W^\mu_l - \partial^\mu\omega_l - g\,\varepsilon_{jkl}\,\omega_j W^\mu_k}$$

* To preserve gauge invariance we have introduced gauge fields $W^{\mu}_{1,2,3}$ via the covariant derivative. Clearly we also have to add a gauge field part $\mathcal{L}_G$ ($\rightarrow$ propagation) to the Lagrangian.

* We might expect for $\mathcal{L}_G$

$$-\frac{1}{4} F_j^{\mu\nu} F_{j\mu\nu}$$

where $F_j^{\mu\nu} = \partial^{\mu} W_j^{\nu} - \partial^{\nu} W_j^{\mu}$ by analogy with QED.

But this is not gauge invariant!

$$F_j'^{\mu\nu} = F_j^{\mu\nu} - g\,\varepsilon_{jkl}\,\omega_k F_l^{\mu\nu}$$
$$- g\,\varepsilon_{jkl}\left[(\partial^{\mu}\omega_k) W_l^{\nu} - (\partial^{\nu}\omega_k) W_l^{\mu}\right]$$

and

$$-\frac{1}{4} F_j'^{\mu\nu} F'_{j\mu\nu} = -\frac{1}{4} F_j^{\mu\nu} F_{j\mu\nu}$$
$$+ \frac{1}{2} g\,\varepsilon_{jkl}\,F_j^{\mu\nu}\left[(\partial_{\mu}\omega_k) W_{l\nu} - (\partial_{\nu}\omega_k) W_{l\mu}\right]$$
$$+ \mathcal{O}(\omega^2)$$

(using antisymmetry of ε_{jkl})

* Thus to get rid of the extra term we must define

$$\boxed{\mathcal{L}_G = -\frac{1}{4} G_j^{\mu\nu} G_{j\mu\nu}}$$

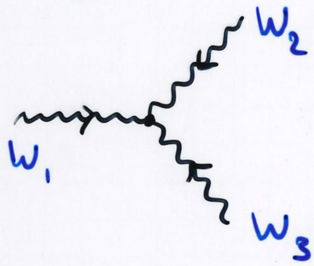
where

$$\boxed{G_j^{\mu\nu} = \partial^\mu W_j^\nu - \partial^\nu W_j^\mu - g\,\varepsilon_{jkl} W_k^\mu W_l^\nu}$$

* Note that $\mathcal{L}_G$ now contains terms representing self-interactions of the gauge fields.

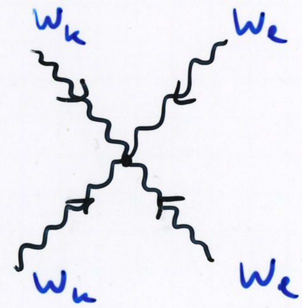
$$-\frac{1}{2} g\, \varepsilon_{jkl} \left(\partial^\mu W_j^\nu - \partial^\nu W_j^\mu\right) W_{k\mu} W_{l\nu}$$

$\Rightarrow$ $W_1 W_2 W_3$ vertex



$$-\frac{1}{4} g^2 \varepsilon_{jkl} \varepsilon_{jmn} W_k^\mu W_l^\nu W_{m\mu} W_{n\nu}$$

$\Rightarrow$ $W_k W_l W_m W_n$ vertices



Weak Interactions

* We can use a two-component notation to represent the two charge states of a given species of lepton or quark

$$\Psi_e = \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix} \qquad \Psi_q = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$$

* Then the weak interaction is described by an SU(2) gauge theory: the gauge invariance w.r.t. $\Psi \to \Psi' = U\,\Psi$ is weak isospin symmetry.

* The interaction term is

$$\frac{g}{2}\,\overline{\Psi}\,\gamma^\mu\,\underline{W}_\mu \cdot \underline{\tau}\,\Psi$$

where

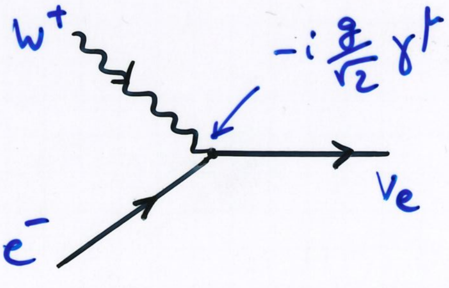
$$\Psi = \begin{pmatrix} \psi_{\nu_e} \\ \psi_e \end{pmatrix} \qquad \overline{\Psi} = \left(\overline{\psi}_{\nu_e}, \overline{\psi}_e\right)$$

$$\underline{W}^\mu \cdot \underline{\tau} = \begin{pmatrix} W_3^\mu & W_1^\mu - i W_2^\mu \\ W_1^\mu + i W_2^\mu & -W_3^\mu \end{pmatrix}$$

Define $\quad W^\pm = \frac{1}{\sqrt{2}}\left(W_1 \mp i W_2\right)$

then we get a term

$$\frac{g}{\sqrt{2}}\,\overline{\psi}_{\nu_e}\,\gamma^\mu\,W^+_\mu\,\psi_e \quad \Longrightarrow$$



* We know from experiment that the $W^\pm$ in fact only interact with the left-handed fermion states

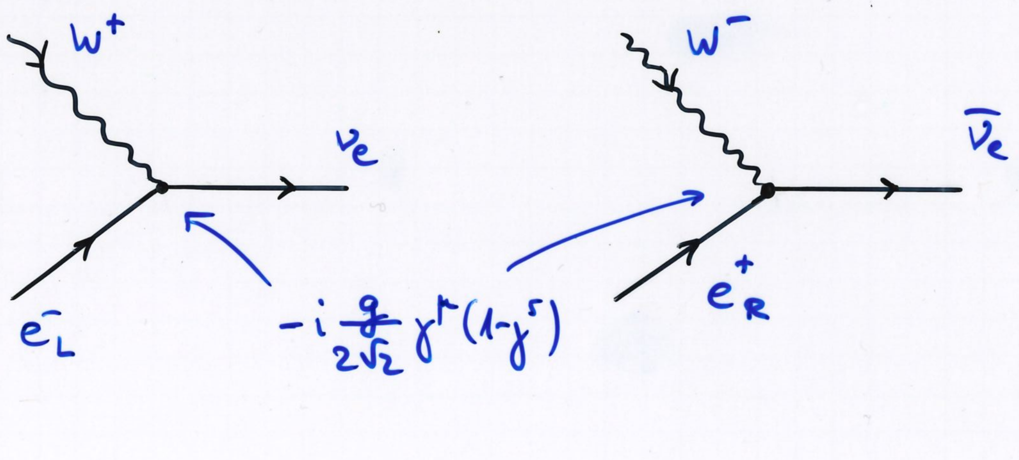
$$f^L \equiv \frac{1}{2}(1-\gamma^5) f$$
$$f^R \equiv \frac{1}{2}(1+\gamma^5) f$$

and correspondingly with right-handed antifermions

$$(\bar{f})^R = \overline{f^L} = \frac{1}{2}\bar{f}(1+\gamma^5)$$

* Thus by Ψ we really mean Ψ^L:

$$\overline{\Psi^L}\gamma^\mu \underline{W}_\mu \cdot \underline{\tau}\, \Psi^L = \frac{1}{4}\bar{\Psi}(1+\gamma^5)\gamma^\mu \underline{W}_\mu \cdot \underline{\tau}\,(1-\gamma^5)\Psi$$
$$= \frac{1}{4}\bar{\Psi}\gamma^\mu \underline{W}_\mu \cdot \underline{\tau}\, \underbrace{(1-\gamma^5)(1-\gamma^5)}_{= 1 + (\gamma^5)^2 - 2\gamma^5 = 2(1-\gamma^5)} \Psi$$
$$= \frac{1}{2}\bar{\Psi}\gamma^\mu(1-\gamma^5)\, \underline{W}_\mu \cdot \underline{\tau}\, \Psi$$



* Thus we say that the left-handed fermions have weak isospin $I_W = 1/2$ ($\rightarrow$ $2I_W + 1 = 2$ states ψ_1^L, ψ_2^L, e.g. ν_e, e_L^-), transforming under weak isospin gauge transformations as

$$\Psi^L = \begin{pmatrix} \psi_1^L \\ \psi_2^L \end{pmatrix} \longrightarrow e^{ig\underbrace{\tfrac{1}{2}}_{I_W = \frac{1}{2}}\underline{\omega}\cdot\underline{\tau}}\, \Psi^L$$

* The right-handed fermions have $I_W = 0$, i.e. 1 state (e_R^-), invariant under weak isospin transformations

$$\psi^R \longrightarrow e^0 \psi^R = \psi^R$$

* This is fine except that it implies that fermions can not have mass!

The mass term in the Dirac Lagrangian is

$$m\bar{\psi}\psi = \frac{1}{4}m\bar{\psi}(1-\gamma^5)(1-\gamma^5)\psi + \frac{1}{4}m\bar{\psi}(1+\gamma^5)(1+\gamma^5)\psi$$

$$= m\overline{\psi^R}\psi^L + m\overline{\psi^L}\psi^R$$

* Since ψ^R and $\overline{\psi^R}$ do not transform while ψ^L and $\overline{\psi^L}$ do, this is clearly not gauge invariant (unless $m=0$).

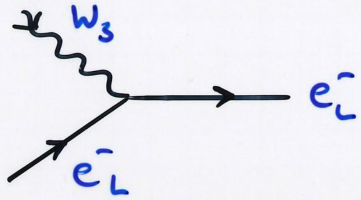
We postpone this problem for the moment...

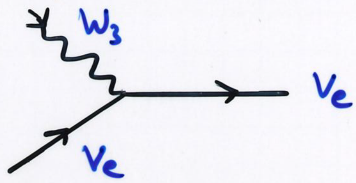
Electroweak Interactions

→ S. Glashow, S. Weinberg, A. Salam

* We interpreted $(W_1 \mp i W_2)/\sqrt{2}$ as $W^{\pm}$ - boson fields, but what about W_3? Could W_3 be the Z^0 boson?

* It has the right kind of vertices





* However, the Z^0 also interacts with right-handed electrons. Also it has a different mass from $W^{\pm}$, so they cannot belong in an exact symmetry multiplet (weak isospin triplet, $I_W = 1$).

* But there is another neutral gauge boson, the photon. Therefore we suppose that W_3 may be a mixture:

$$W_3^{\mu} = \cos\theta_W \, Z^{\mu} + \sin\theta_W \, A^{\mu}$$

Here θ_W is the Weinberg angle ('weak mixing angle')

$$\sin^2\theta_W = 0.23121(4)$$

* The combination orthogonal to W_3 is

$$B^\mu \equiv -\sin\theta_W \, Z^\mu + \cos\theta_W \, A^\mu \,.$$

In the Standard Model, B^μ is the gauge boson field for an additional Abelian gauge symmetry. Thus the overall electroweak symmetry is

$$\text{gauge bosons} \longrightarrow \boxed{\underbrace{SU(2)}_{W^+,\,W^-,\,W_3} \times \underbrace{U(1)}_{B}}$$

* The coupling constant for the $U(1)$ interactions is $g' \neq g$. The coupling of any fermion to B^μ is proportional to the weak hypercharge, Y

$$Y = Q - I_3$$

where I_3 is the third component of the weak isospin, i.e.

$$I_3 = \begin{cases} \pm 1/2 & \text{for upper/lower component of a weak isospin doublet } (I_W = \tfrac{1}{2}) \\ 0 & \text{for singlet } (I_W = 0) \end{cases}$$

Q = charge in units of $|e|$

Electroweak Quantum Numbers

Particle	Q	I_3	Y
ν_e, ν_μ, ν_τ	0	$\frac{1}{2}$	$-\frac{1}{2}$
e_L, μ_L, τ_L	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
e_R, μ_R, τ_R	-1	0	-1
u_L, c_L, t_L	$+\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
d_L, s_L, b_L	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{6}$
u_R, c_R, t_R	$+\frac{2}{3}$	0	$\frac{2}{3}$
d_R, s_R, b_R	$-\frac{1}{3}$	0	$-\frac{1}{3}$

* N.B: Hyperchange is the average charge of the weak isospin multiplet.

* Weak isospin doublets are

$$\begin{pmatrix} \nu_e \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau_L \end{pmatrix}$$

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix}$$

* The electroweak Lagrangian becomes

$$\mathcal{L}_{EW} = -\frac{1}{4} G_j^{\mu\nu} G_{j\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu}$$
$$+ i \bar{\Psi} \gamma^\mu \left(\partial_\mu + i g \underline{W}_\mu \cdot \underline{I} + i g' B_\mu Y \right) \Psi$$

where $B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$,

$\underline{I} = \frac{1}{2} \underline{\tau}$ for doublets, 0 for singlets,

Y as given in the table

* General gauge transformation is

$$\Psi \longrightarrow \Psi' = \exp\left(i g \underline{\omega} \cdot \underline{I} + i g' \omega_0 Y \right) \Psi$$

where $\omega_0, \omega_1, \omega_2, \omega_3$ are arbitrary functions of space-time coordinates

* The terms in $\mathcal{L}_{EW}$ involving the neutral gauge bosons are

$$- \bar{\Psi} \gamma^\mu \left[g \left(\cos\theta_W Z_\mu + \sin\theta_W A_\mu \right) I_3 \right.$$
$$\left. + g' \left(-\sin\theta_W Z_\mu + \cos\theta_W A_\mu \right) Y \right] \Psi$$

Hence the coupling to the photon is

$$g \sin\theta_W \cdot I_3 + g' \cos\theta_W \cdot Y$$

which must be equal to

$$eQ = e \cdot I_3 + e \cdot Y$$

$$\Rightarrow \quad \boxed{g \sin\theta_W = g' \cos\theta_W = e}$$

* Note that since Q is the same for left- and right-handed states, the photon couples to the currents

$$\bar{\psi}_L \gamma^\mu \psi_L + \bar{\psi}_R \gamma^\mu \psi_R =$$

$$= \frac{1}{2} \bar{\psi} \gamma^\mu (1-\gamma^5) \psi + \frac{1}{2} \bar{\psi} \gamma^\mu (1+\gamma^5) \psi$$

$$= \bar{\psi} \gamma^\mu \psi$$

→ photon interactions are parity-conserving (no γ^5)

* The coupling of the Z^0 on the other hand:

$$g \cos\theta_W \cdot I_3 - g' \sin\theta_W \cdot Y =$$

$$= \frac{e}{\sin\theta_W \cos\theta_W} \left(\cos^2\theta_W \cdot I_3 - \sin^2\theta_W \cdot Y \right)$$

$$= \frac{2e}{\sin 2\theta_W} \left(I_3 - \sin^2\theta_W \cdot Q \right)$$

involves the current

$$\bar{\psi}_L \gamma^\mu (I_3 - \sin^2\theta_W \cdot Q) \psi_L + \bar{\psi}_R \gamma^\mu (-\sin^2\theta_W \cdot Q) \psi_R$$

$$= \frac{1}{2} \bar{\psi} \gamma^\mu \left[(I_{3L} - \sin^2\theta_W \cdot Q)(1-\gamma^5) - \sin^2\theta_W \cdot Q (1+\gamma^5) \right] \psi$$

$$= \frac{1}{2} \bar{\psi} \gamma^\mu \left(I_{3L} - 2Q \sin^2\theta_W - I_{3L} \gamma^5 \right) \psi$$

γ^5-term ⇒ parity violation

The Higgs Mechanism

The electroweak theory we have discussed so far is perfectly self-consistent but it cannot be correct because all the fermions and gauge bosons have to be massless to preserve gauge-invariance of the Lagrangian.

* Recall that a fermion mass term converts left-handed particles into right-handed ones and vice-versa

$$m \bar{f} f = m \overline{f^R} f^L + m \overline{f^L} f^R$$

* Since the left-handed fermions are weak isospin doublets and the right-handed ones are singlets, we need to replace m by a doublet ($I_W = 1/2$) type of quantity to restore gauge invariance, e.g. a Yukawa interaction

$$g_f \, \Phi^+ \overline{f^R} \, \overline{\Psi}^L + g_f \, \overline{\Psi}^L f^R \Phi$$

where $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ is a weak isospin doublet of scalar fields.

* For first-generation leptons, for example, we find

$$g_e \phi_1^+ \bar{\psi}_{e_R} \psi_{\nu_e} + g_e \phi_2^+ \bar{\psi}_{e_R} \psi_{e_L} + \text{c.c.}$$

and thus for an electron mass we need

$$\Phi_0 = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} , \qquad g_e \frac{v}{\sqrt{2}} = m_e .$$

* This is the vacuum expectation value of the Higgs field, Φ_0. Note that ϕ_2 must be neutral:

$$Q = 0 , \quad I_3 = -\frac{1}{2} \implies Y_{\text{Higgs}} = \frac{1}{2} .$$

Then ϕ_1 must have $Q = +1$, i.e. in general

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

* Of course, the particular choice of $\Phi_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$ breaks gauge invariance, so we do not seem to have achieved much. But we shall see that this type of symmetry breaking is 'natural', and does not spoil the good properties of gauge theories.

* The interactions between the Higgs field and the gauge fields are generated by the usual 'kinetic' term in the Higgs (Klein-Gordon) part of the Lagrangian

see later

$$\mathcal{L}_H = (D^\mu \Phi^+)(D_\mu \Phi) + \dots$$

where D^μ is the electroweak covariant derivative

$$D^\mu = \partial^\mu + ig\,\underline{W}^\mu \cdot \underline{I} + ig' B^\mu Y$$

* For the vacuum Higgs field we have explicitly (v = const.)

$$D^\mu \Phi_0 = \frac{iv}{2\sqrt{2}} \begin{pmatrix} \sqrt{2}\, g\, W^{+\mu} \\ g' B^\mu - g W_3^\mu \end{pmatrix}$$

$$g' B^\mu - g W_3^\mu = \frac{g}{\cos\theta_W} \left(\sin\theta_W \cdot B^\mu - \cos\theta_W \cdot W_3^\mu \right)$$

$$= - \frac{g}{\cos\theta_W} \cdot Z^\mu$$

Hence

$$\mathcal{L}_H = \frac{v^2 g^2}{8} \left(2 W^{-\mu} W^+_\mu + \frac{1}{\cos^2\theta_W} Z^\mu Z_\mu \right) + \dots$$

which corresponds to W and Z mass terms

$$\boxed{m_W = \frac{1}{2} v g = m_Z \cdot \cos\theta_W}$$

Note that $\sin^2\theta_W = 1 - \frac{m_W^2}{m_Z^2}$

Parameters of the Standard Model

* The Standard (Glashow - Weinberg - Salam) Model describes all electroweak interactions at present energies in terms of three basic parameters

$$g \,,\quad g' \;(\text{or } \theta_W)\,,\quad v$$

* These are most accurately measured from
  * the fine structure constant

$$\alpha = \frac{e^2}{4\pi} = \frac{g^2 \sin^2\theta_W}{4\pi} = 1/137.035999084(21)$$

  * the Fermi weak coupling constant

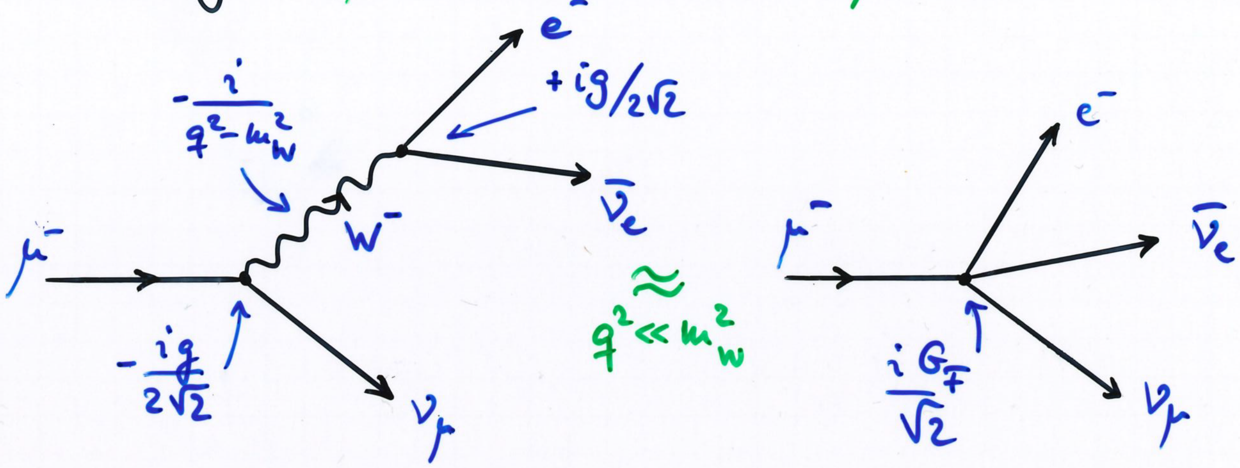
$$G_F = \frac{g^2\sqrt{2}}{8\, m_W^2} = \frac{1}{\sqrt{2}\, v^2} = 1.1663788(6)\times 10^{-5}\ \text{GeV}^{-2}$$

  * the Z^0 boson mass

$$M_Z = \frac{g v}{2\cos\theta_W} = 91.1876(21)\ \text{GeV}$$

* These relations are subject to higher-order (electroweak + strong) corrections
* In addition there are further parameters of the Higgs sector, including Yukawa couplings

* The Fermi constant is the effective 4-fermion coupling for the decay $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$



$G_F = 1.166 \times 10^{-5}\ \mathrm{GeV}^{-2} \implies \underline{v = 246\ \mathrm{GeV}}$

* The Yukawa couplings can then be deduced from the fermion masses

$$g_f = m_f \sqrt{2} / v$$

* The numerical values of the Yukawa couplings (and thus the fermion masses) are presently not understood. ($\to$ 1 parameter per fermion)

<u>NB</u>: $m_t = 173\ \mathrm{GeV} \implies g_t = 1.0$

Is this accidental ??

Spontaneous Symmetry Breaking

The advantage of the Higgs mechanism for mass generation is that the gauge-symmetry-breaking vacuum Higgs field Φ_0 can arise 'spontaneously' even though the Lagrangian is exactly gauge-invariant.
We consider first for simplicity the spontaneous breaking of a global (i.e. space-time independent) Abelian symmetry.
Consider a classical complex scalar field ϕ with Lagrangian density

$$\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 .$$

Clearly $\mathcal{L}$ is invariant w.r.t.

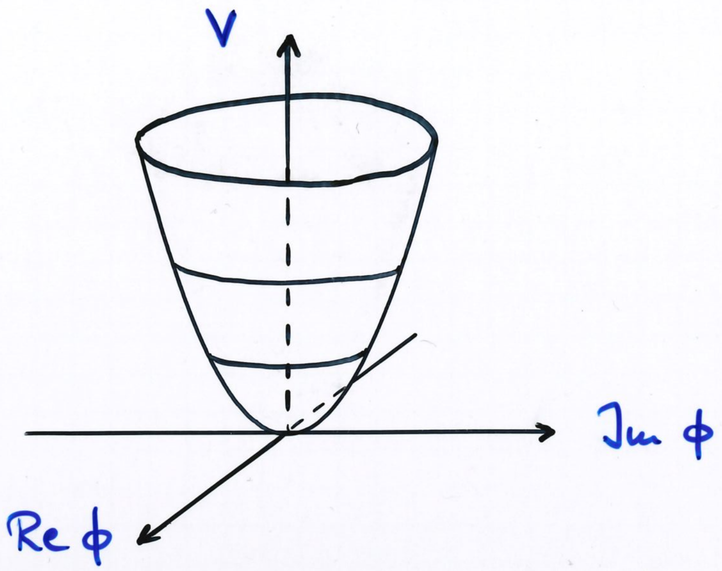
$$\phi \to e^{i\alpha}\phi \quad , \quad \phi^* \to e^{-i\alpha}\phi^*$$

for any real constant α.
The Hamiltonian density is

$$\mathcal{H} = \left|\frac{\partial \phi}{\partial t}\right|^2 + \underline{\nabla}\phi^* \cdot \underline{\nabla}\phi + V(\phi)$$

where the 'potential energy' is

$$V = \mu^2 |\phi|^2 + \lambda |\phi|^4 .$$



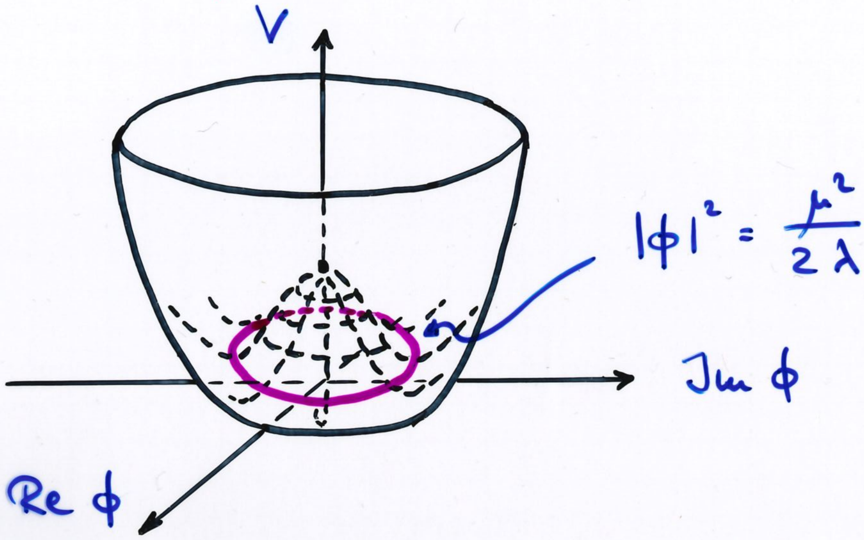
* This potential has its minimum at $\phi = 0$ and hence the minimum-energy (vacuum) state has ϕ = const. = 0 (classically).
* After second quantization there will be zero-point fluctuations but the vacuum expectation value $\langle \hat{\phi} \rangle$ will still be zero.
* The curvature of $V(\phi)$ at the minimum tells us the mass of the scalar bosons created and annihilated by the field operator (c.f. Klein-Gordon equation)

$$m^2 = \frac{1}{2} \left. \frac{d^2 V}{d\phi^2} \right|_{\phi = \phi_{min}} = \mu^2$$

* The term $\lambda |\phi|^4$ represents a 4-boson interaction that can be treated as a perturbation (coupling constant $\propto \lambda$)

* Now suppose we change the sign of the first term in $V(\phi)$:

$$V = -\mu^2 |\phi|^2 + \lambda |\phi|^4$$



The potential has a 'hump': the minimum is now anywhere on the circle

$$\phi = \sqrt{\frac{\mu^2}{2\lambda}} \cdot e^{i\theta} \qquad (0 \leq \theta \leq 2\pi)$$

* The physical vacuum can be any one of these (infinitely many) degenerate vacua. But choosing a particular one (a value of θ) breaks the $U(1)$ symmetry, since $e^{i\alpha}\phi$ will be a different vacuum. (Compare falling rod, ferromagnet.)
Since a particular vacuum is realized, there is spontaneous symmetry breaking.

* The two principal curvatures of $V(\phi)$ at the vacuum solution are now different: one is zero (around the circle), indicating a massless boson (Goldstone boson):

Goldstone's theorem (1961)

spontaneous symmetry breaking
$\Rightarrow$ massless boson

[We shall see this is not true in gauge theories.]

* The other curvature (radial) is non-zero, indicating a massive boson is also present.

* Without loss of generality (because of gauge invariance) we can define the vacuum value to be real:

$$\phi_0 = \frac{v}{\sqrt{2}} \quad , \quad v \equiv \sqrt{\frac{\mu^2}{\lambda}} .$$

Then setting

$$\phi = \frac{1}{\sqrt{2}}\left[v + \sigma(x) + i\eta(x)\right]$$

we expect σ and η to represent massive and massless boson fields, respectively.

Substituting in the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left[(\partial^\mu\sigma)(\partial_\mu\sigma) + (\partial^\mu\eta)(\partial_\mu\eta)\right]$$
$$+ \frac{1}{2}\mu^2\left[(v+\sigma)^2 + \eta^2\right]$$
$$- \frac{1}{4}\lambda\left[(v+\sigma)^2 + \eta^2\right]^2$$
$$= \frac{1}{2}\left[(\partial^\mu\sigma)(\partial_\mu\sigma) - m_\sigma^2\sigma^2\right] + \frac{1}{2}(\partial^\mu\eta)(\partial_\mu\eta)$$
$$+ \text{const.} + \text{interaction terms}$$

where $m_\sigma = \sqrt{2}\,\mu$

* Now suppose our model scalar field theory is a $U(1)$ gauge theory (Higgs model, 1964), i.e. the Lagrangian is invariant w.r.t.

$$\phi \to e^{i\alpha(x)}\phi \quad , \quad \phi^* \to e^{-i\alpha(x)}\phi^*$$

for any real function $\alpha(x)$. We know this means there must be a gauge field $B^\mu(x)$ and the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}K^{\mu\nu}K_{\mu\nu} + D^\mu\phi^* D_\mu\phi + \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2$$

where

$$K^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$$

$$D^\mu = \partial^\mu + ig' \underset{\substack{\uparrow \\ Y = \text{hypercharge}}}{Y} B^\mu$$

* The spontaneous symmetry breaking in the Higgs field ϕ works as before, giving a vacuum with $B^\mu = 0$, $\phi = \phi_0 \neq 0$. Choosing $\phi_0 = v/\sqrt{2}$ (real), this generates a mass term for the gauge boson:

$$\frac{1}{2}(D^\mu v)(D_\mu v) = \frac{1}{2}(g' Y v)^2 B^\mu B_\mu$$

corresponding to a mass $m_B = g' Y v$.

* We seem to have created a degree of freedom: a massive vector field ($S=1$) has 3 polarization states in contrast to the 2 of a massless field: longitudinal polarization (helicity 0) is also possible, as well as transverse (L, R).

* This is because one degree of freedom of the Higgs field has disappeared:
In a gauge theory we can always set the massless field $\eta(x)$ to zero by a gauge transformation: choose $\tan\alpha(x) = -\eta(x)/(v+\sigma(x))$

$$e^{i\alpha}\phi = (\cos\alpha + i\sin\alpha)\frac{1}{\sqrt{2}}(v+\sigma+i\eta)$$

$$= \frac{1}{\sqrt{2}}\Big[\underbrace{(v+\sigma)^2 + \eta^2}_{\text{real for all } x}\Big]^{1/2} = \frac{1}{\sqrt{2}}\Big(v + \underbrace{\sigma + \frac{\sigma^2+\eta^2}{2v} + \ldots}_{\sigma'}\Big)$$

→ 'unitary gauge'

* Thus we can say that the gauge field has eaten the Goldstone boson in order to create its extra polarization state.

* The situation in the electroweak case is a little more complicated.
The original $SU(2) \times U(1)$ symmetry (4 generators) is spontaneously broken to $U(1)_{e.m.}$ [not the original $U(1)$!] (with 1 generator). There should be 3 Goldstone bosons, but these are eaten by the W^+, W^-, Z^0 to produce their longitudinal polarization states.
This leaves one massless gauge boson (the photon) and one massive, neutral scalar boson – the Higgs boson.

Properties of the Higgs Boson

* Note that the **mass** of the Higgs boson σ,

$$m_H = m_\sigma = \sqrt{2}\,\mu$$

is not determined, since only the combination of Higgs parameters

$$v = \mu / \sqrt{\lambda} = 246 \text{ GeV}$$

is fixed by current data.

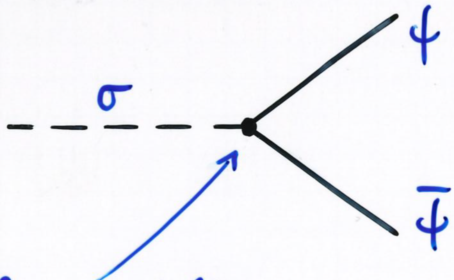
* Higher-order corrections depend (weakly) on μ.

* The Higgs boson has been found at the LHC with the mass

$$m_H = 125.25\,(17) \text{ GeV}$$

* The **coupling** of the Higgs boson to **fermions** is given by the Yukawa coupling, which also gives the fermion mass:

$$g_f \frac{1}{\sqrt{2}} (v + \sigma)\, \bar{f} f$$



$\Rightarrow \quad m_f = g_f \dfrac{v}{\sqrt{2}}$

$\Rightarrow \quad$ vertex factor $\; -i \dfrac{g_f}{\sqrt{2}} = -i \dfrac{m_f}{v}$

* Thus the Higgs boson likes to decay into (and be produced by) the heaviest available fermion.

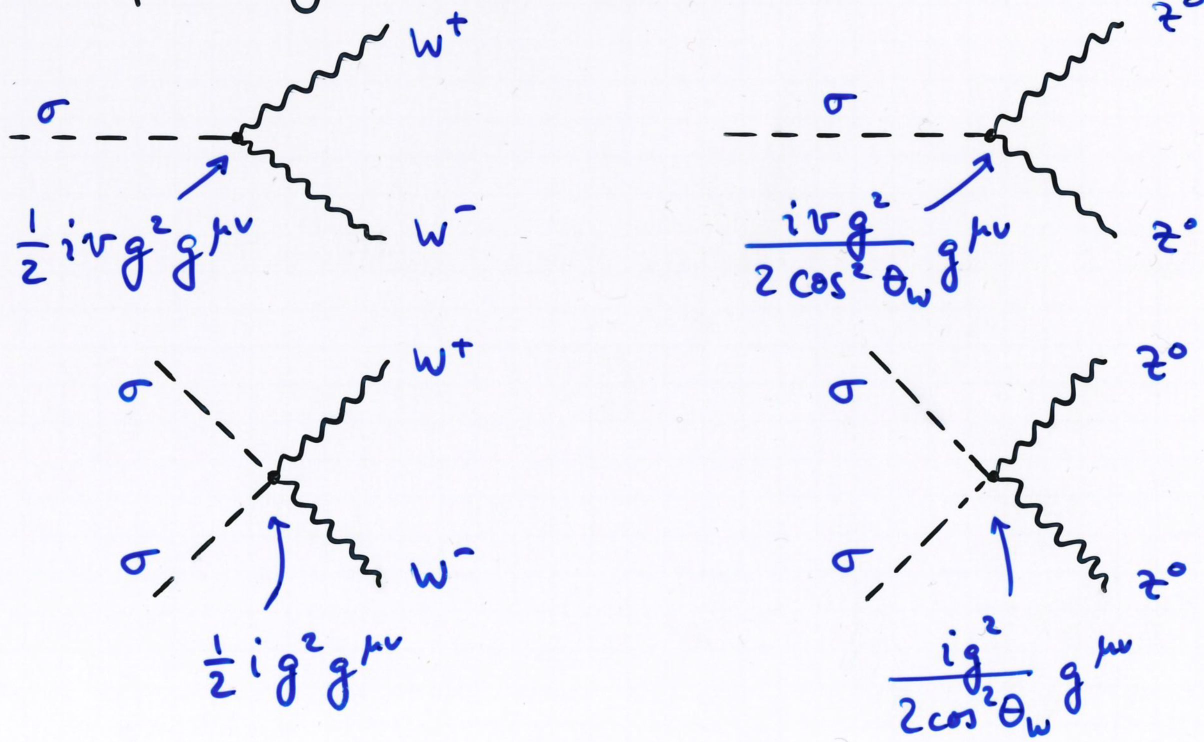
* The coupling to gauge bosons is obtained by replacing v by $(v+\sigma)$ in the term that produces the gauge boson masses:

$$\mathcal{L}_H = \frac{(v+\sigma)^2}{8} g^2 \left(W^{-\mu} W^+_\mu + \frac{1}{\cos^2\theta_W} Z^\mu Z_\mu \right)$$

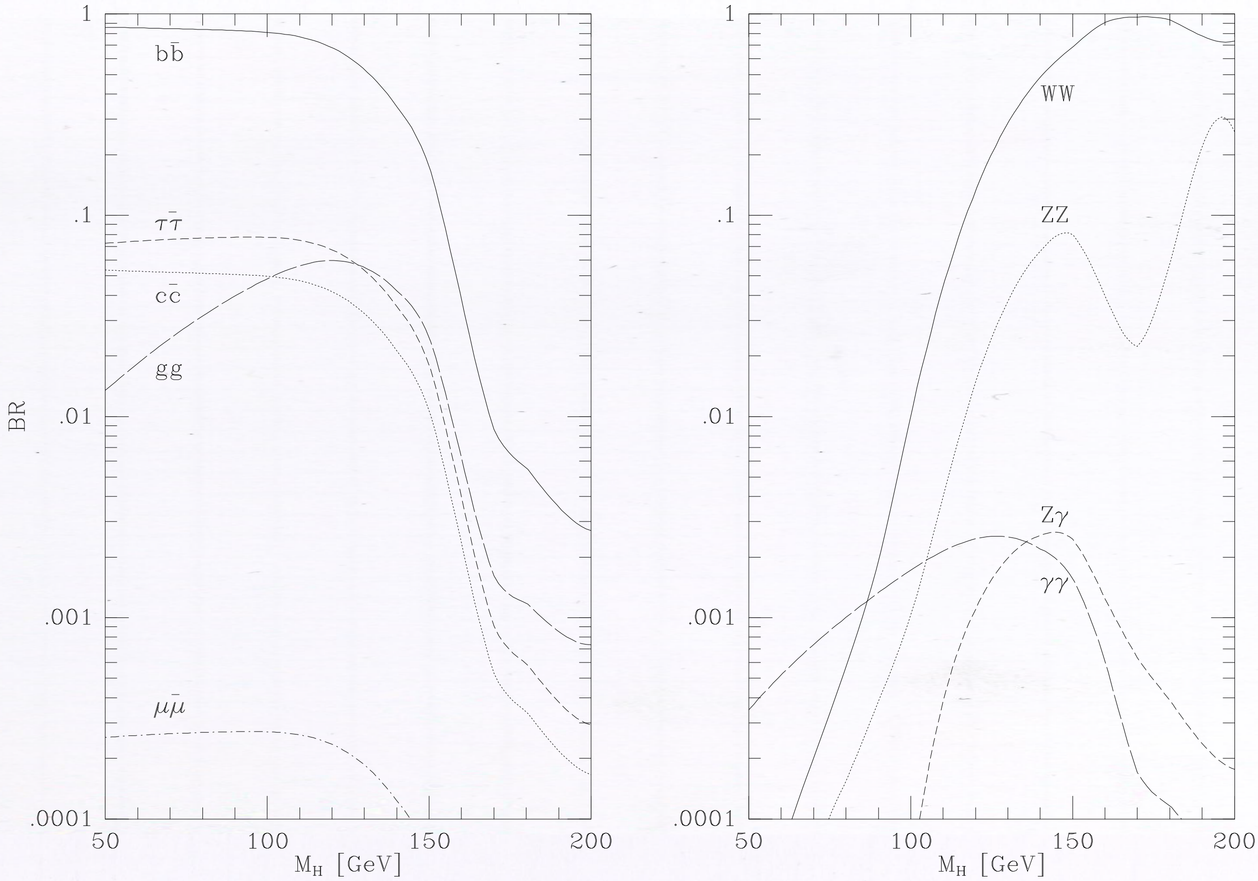
$$= \left(m_W^2 + \frac{1}{2} v g^2 \sigma + \frac{1}{4} g^2 \sigma^2 \right) W^{-\mu} W^+_\mu$$

$$+ \frac{1}{2} \left(m_Z^2 + \frac{v g^2}{2\cos^2\theta_W} \sigma + \frac{g^2}{4\cos^2\theta_W} \sigma^2 \right) Z^\mu Z_\mu$$

Corresponding to vertices



* Branching ratios of Higgs boson decays to various final states:



from:

R.K. Ellis, W.J. Stirling, B.R. Webber

'QCD and Collider Physics'

Cambridge University Press

Quark Mixing

* In general the quark fields entering the doublets of the electroweak interaction need not be mass eigenstates. We should thus write the doublets as

$$\begin{pmatrix} u \\ d' \end{pmatrix}, \quad \begin{pmatrix} c \\ s' \end{pmatrix}, \quad \begin{pmatrix} t \\ b' \end{pmatrix}$$

with the primed down-type quarks q' differing from the mass eigenstates q by a unitary transformation, V:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = V \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

The matrix V is known as the Cabibbo - Kobayashi - Maskawa (CKM) matrix.

* The CKM matrix implies a mixing of mass eigenstates in the electroweak interactions.

* Note that without quark mixing s and b quarks would be stable — contrary to observation.

* It is only a convention to shift the mixing to the down-type quarks.

* Writing the CKM matrix V as

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

the constraint of unitarity ($V^\dagger V = \mathbb{1}$) implies 9 conditions

$$\sum_{j=1}^{3} |V_{ij}|^2 = \sum_{i=1}^{3} |V_{ij}|^2 = 1$$

$$\sum_{k=1}^{3} V_{ki}^* V_{kj} = 0 \qquad (i \neq j)$$

* A unitary 3×3 matrix hence has 9 independent real parameters. The phases of 5 quark fields can be redefined.

* This leaves 4 real parameters:

3 independent mixing angles θ_{ij}

+ 1 phase δ

* The standard parametrization of the CKM matrix is

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

with

$$s_{ij} = \sin\theta_{ij} \,, \qquad c_{ij} = \cos\theta_{ij}$$

* In the Standard Model the phase δ is responsible for CP violation arising from the mixing matrix.

* Note that if we had only two generations there would be one mixing angle (the Cabibbo angle θ_{12}) and no phases. Hence there would be no possibility for CP violation from the mixing matrix.

* The experimental limits (taking into account unitarity)

$$V = \begin{pmatrix} 0.9741 - 0.9756 & 0.219 - 0.226 & 0.0025 - 0.0048 \\ 0.219 - 0.226 & 0.9732 - 0.9748 & 0.038 - 0.044 \\ 0.004 - 0.014 & 0.037 - 0.044 & 0.9990 - 0.9993 \end{pmatrix}$$

imply that the mixing angles are small, and satisfy a hierarchy:

$$1 \gg \theta_{12} \gg \theta_{23} \gg \theta_{13}$$

* A simplified (small-angle) parametrization was introduced by Wolfenstein:

Setting

$$\lambda = \sin\theta_{12} = s_{12}$$

one obtains with $V_{cb} \simeq s_{23} = A\lambda^2$ and $V_{ub} = s_{13}\, e^{-i\delta} = A\lambda^3(\rho - i\eta)$, $A \simeq 1$, and $|\rho - i\eta| < 1$ finally

$$V = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

* Mixing is smallest between the first and third generation.