$$\cos\mu = \frac{1}{2}\,Tr(M) = \cos\mu_0 \underbrace{- \frac{1}{2}\frac{\beta_0}{f}\sin\mu_0}_{\text{additional term}}$$

$$\Delta\mu = 2\pi\Delta Q = \frac{1}{2}\frac{\beta_0}{f}\,, \quad \Delta Q = \frac{1}{4\pi}\frac{\beta_0}{f}$$ ← can be used to measure β_0

Result: Shift of operation point

$$\Delta Q = \frac{1}{4\pi}\oint \beta(\bar{s})\,\delta K(\bar{s})\,d\bar{s}$$

similarly as for the dispersion case (not $\sqrt{\beta}$ but β)

$$\Delta\beta(s) = \frac{\beta(s)}{2\sin 2Q\pi}\int_s^{s+C} \delta K(\bar{s})\,\beta(\bar{s})\cos 2[\psi(\bar{s}) - \psi(s) - Q\pi]\,d\bar{s}$$

important: $2Q\pi \to$ Q - half-integer $\Rightarrow$ resonance

7.~~1~~.2 Floquet transformation

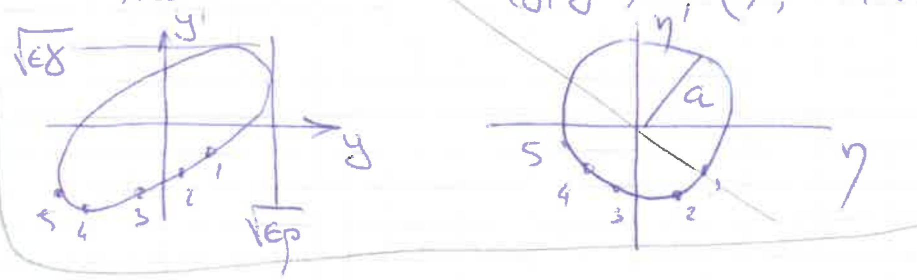
used to discuss resonances and instabilities

Courant - Snyder - variables

$$\boxed{\frac{y^2}{\beta} + \frac{(\alpha y + \beta y')^2}{\beta} = a^2 = \epsilon}$$ C-S - invariant

tilted ellipse → aim: circular diagram

Transformation $(y, y') \to (\eta, d\eta/d\varphi)$



$$y(s) \longleftrightarrow \eta(\psi) \quad \text{with} \quad \eta = \frac{y}{\sqrt{\beta}}$$

$$y'(s) \longleftrightarrow \frac{d\eta}{d\psi} \quad \text{with} \quad \frac{d\eta}{d\psi} = \alpha\frac{y}{\sqrt{\beta}} + \sqrt{\beta}\,y'$$

$$\begin{pmatrix} \eta \\ \frac{d\eta}{d\psi} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

↑ "dimensionless" units $\sqrt{\text{mm mrad}}$

~~$\eta = a\cos(\psi+\psi_0)$~~

~~$\frac{d\eta}{d\psi} = -a\sin(\psi+\psi_0)$~~

$\sqrt{\epsilon}$

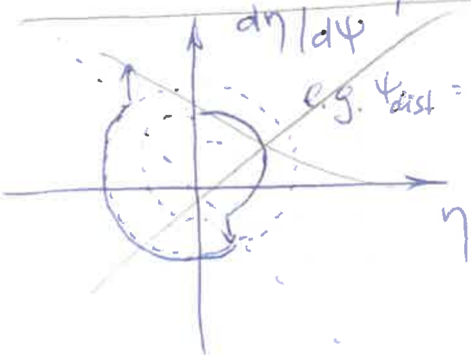
~~diagram -~~
~~- solution of Hill's equations~~

7.6.3 Stop-band (2nd order)

kick: $\Delta y' = -\frac{y}{f} = -\frac{1}{f} a\sqrt{\beta}\cos\psi$

$\Rightarrow \quad \Delta\eta' = -\frac{1}{f} a\beta\cos\psi$

leads to an amplitude shift and phase shift



~~$\Delta a = \Delta\left(\frac{d\eta}{d\psi}\right)\sin\psi = -\frac{a\beta}{f}\cos\psi\sin\psi$~~

~~$\Delta\psi = -\frac{1}{a}\Delta\left(\frac{d\eta}{d\psi}\right)\cos\psi = \frac{\beta}{f}\cos^2\psi$~~

$$\Delta Q = \frac{1}{2\pi}\frac{\beta}{f}\cos^2\psi = \frac{1}{4\pi}\frac{\beta}{f}(1-2\cos 2\psi)$$

average shift of the working point by $\overline{\Delta Q} = \frac{1}{4\pi}\frac{\beta}{f}$

with superimposed modulation $\delta Q = \frac{1}{4\pi}\frac{\beta}{f}\cos 2\psi$

~~modulation & shift are small if β small and f large~~

7.6.4. Sextupoles (similar to quadrupole)

resonance if 3Q - integer

band: $\delta Q = \frac{\beta^{3/2}}{16\pi}\left(\frac{\partial^2 B_y}{\partial x^2}\right)\frac{\Delta s}{B\rho} a \cos 3\psi$

Stop-band 3rd-order

amplitude of betatron oscillation

- non-linear effect
- dynamic aperture

7.7. Stability condition

Stay away from resonances, resonance diagram (slide)

1^{st} order dipoles
2^{nd} order Q-poles
3^{rd} order S-poles

resonance conditions

$$p = n Q_x$$
$$p = n Q_y$$
$$p = \ell Q_x + m Q_y \quad \text{(coupled)} \qquad n = |\ell| + |m|$$

summa or difference resonance

these are bands

depending on β (QP)
emittance (SP)

7.8. α_p and stability (phase focusing)

Synchrotron principle ω - depends on p

$$\frac{\Delta\omega}{\omega_0} = \eta \frac{\Delta p}{p_0}, \qquad \omega = 2\pi \frac{v}{C}$$

$$\frac{\Delta\omega}{\omega_0} = \frac{\Delta v}{v_0} - \frac{\Delta C}{C_0}$$

$$\frac{\Delta v}{v_0} = \frac{1}{\gamma^2} \frac{\Delta p}{p_0} \quad \text{and} \quad \frac{\Delta C}{C_0} = \alpha_p \frac{\Delta p}{p_0}$$

relativistic relation between v and p

relation between relative change of C corresp to Δp

$$\eta = \frac{1}{\gamma^2} - \alpha_p = \frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2}$$

transition energy

momentum compaction factor

$\alpha_p \approx \frac{1}{Q_x^2}$ $\qquad Q_x$ - large, α_p - small = compact trajectories

acceleration

γ - small	$\eta > 0$	stable
$\gamma = \gamma_{tr}$	$\eta = 0$	unstable
γ - large	$\eta < 0$	stable

large energy range $\Leftrightarrow$ small α_p
need for pre-accelerators / dynamic range

7.9. Chromaticity

Problem: particles with $\delta = \frac{\Delta p}{p_0}$ see differen bending and focusing in the machine

QP:

$$\Delta Q_x = \xi_x \frac{\Delta p}{p_0} \quad , \quad \Delta Q_y = \xi_y \frac{\Delta p}{p_0}$$

ξ - (natural) chromaticity (ξ^n) slide

- the ~~coefficients~~ focusing strength in the Hill's equations

$$\Delta K_x = -K_x \frac{\Delta p}{p_0}$$

tune shift

$$\Delta Q_x = \underbrace{\left[-\frac{1}{4\pi} \oint \beta(\bar{s}) K_x(\bar{s}) d\bar{s} \right]}_{\xi_x^n} \frac{\Delta p}{p} \leftarrow \text{slow part, large difference}$$

ξ_x^n - always negative

additional chromaticity comes from sextupole fields

$$\xi_x = \underset{\uparrow \atop Q}{\xi_x^n} + \underset{\uparrow \atop S}{\xi_x^S} \quad ; \quad \xi_y = \underset{\uparrow \atop Q}{\xi_y^n} + \underset{\uparrow \atop S}{\xi_y^S}$$

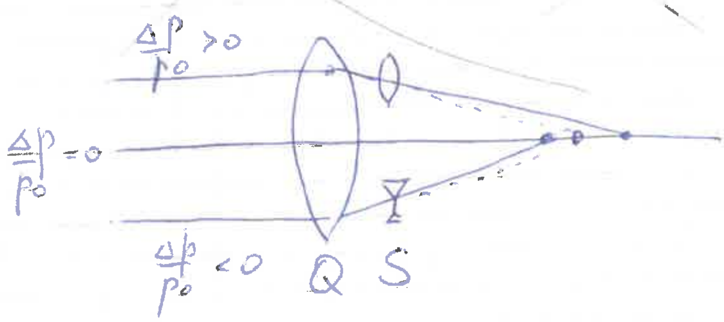
- Chromaticity correction

employing additional sextupoles in dispersion regions

$$\left\{ g_s = \frac{\partial^2 B_y}{\partial x^2} \right.$$

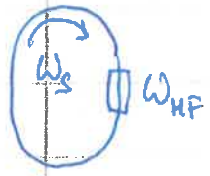
$$\oplus \quad \xi_x^S = \frac{1}{4\pi} \oint g_s(\bar{s}) \frac{\beta_x(\bar{s}) D(\bar{s})}{B\rho} d\bar{s}$$

$$\ominus \quad \xi_y^S = \ominus \quad \frac{\beta_y(\bar{s})}{}$$



8. Longitudinal beam dynamics

8.1 phase focusing and synchrotron oscillations



$\omega_{HF} = h\,\omega_s$ — h - integer, harmonic number

the condition for acceleration ← synchronous particle

Change of energy per revolution $[\Delta E_s]_u$ of synchronous particle

$$[\Delta E_s]_u = C_s \frac{dp_s}{dt}$$

C_s — circumference of synchronous particle; $\frac{dp_s}{dt}$ ← change of momentum per time unit

- the energy a particle gains per revolution is a function of the phase φ with which the particle passes the HF-unit

$$\Delta E^{HF} = q U_0 \sin\varphi$$

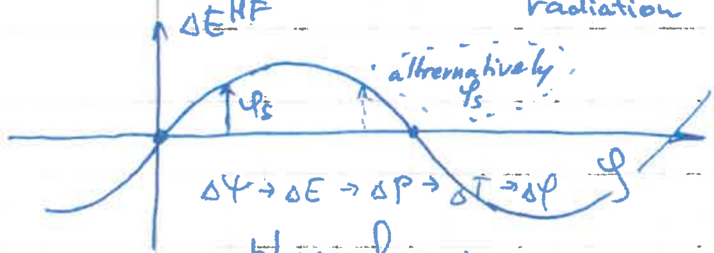
← relative to the middle of the acceleration unit

U_0 — HF-amplitude; q — charge of the particle

- the „zero" is defined as $\varphi = 0$, $\Delta E^{HF} = 0$ → $\varphi > 0$ - later; $\varphi < 0$ - earlier } part.

$$\Delta E^{loss} = -\Delta E_{rad}(E) - \Delta E_{loss}(E) \ldots$$

radiation, energy loss, etc.



$$[\Delta E]_u = q U_0 \sin\varphi + \Delta E^{loss}$$
$$[\Delta E_s]_u = q U_0 \sin\varphi_s + \Delta E^{loss}_s$$

- phase focusing

only possible if momentum change leads to ω-change

$$\frac{\Delta\omega}{\omega} = -\eta \frac{\Delta p}{p} = \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2}\right)\frac{\Delta p}{p}$$

η — dispersion of particle frequencies

$$\alpha_p = \frac{1}{\gamma_{tr}^2} \text{ - momentum compaction factor}$$

8.2 Synchrotron oscillations with small amplitude

$$\Delta\varphi = \varphi - \varphi_s$$
$$\Delta p = p - p_s$$
$$\Delta E = E - E_s$$
$$\Delta\omega = \omega - \omega_s$$

change of $\Delta\varphi$ and ΔE per revolution:

$$\delta(\Delta\varphi) = -\eta_s \frac{\Delta p}{p_s} h 2\pi$$

← phase change of synchr. particle per revolution

$$\delta(\Delta E) = q U_0 (\sin\varphi - \sin\varphi_s)$$

$T_s = \frac{2\pi}{\omega_s}$ - revolution period

- time derivative:

(1) $\Delta\dot{\varphi} = \frac{-1}{T_s}\eta_s \frac{\Delta p}{p_s} h 2\pi = -\frac{h\eta_s\omega_s}{p_s v_s}\Delta E$

(2) $\Delta\dot{E} = \frac{1}{T_s} q U_0 (\sin\varphi - \sin\varphi_s) = \frac{\omega_s}{2\pi} q U_0 (\underbrace{\sin\varphi}_{\varphi_s + \Delta\varphi} - \sin\varphi_s)$

φ_s - is the phase where losses and gain (acceleration) equilize

- adiabatic approximation $\Rightarrow$

$\eta_c, \omega_s, p_s, v_s, U_0$ and φ_s change little

↳ set them constant

- Set (2) and (1) together + differentiation

(3) $\Delta\ddot{\varphi} = -\frac{h\eta_s\omega_s}{p_s v_s}\Delta\dot{E} =$

$= -\frac{h\eta_s\omega_s^2}{2\pi p_s v_s} q U_0 (\sin\varphi - \sin\varphi_s) = \delta[\sin\varphi - \sin\varphi_s]$

in linear approximation $\sin\varphi - \sin\varphi_s \cong \cos\varphi_s \Delta\varphi$

$$\boxed{\frac{d^2}{dt^2}\Delta\varphi + \omega_{syn}^2 \Delta\varphi = 0}$$ - oscillations

analytically not solvable

$$\omega_{syn}^2 = \frac{h\eta_s\omega_s^2}{2\pi p_s v_s} q U_0 \cos\varphi_s$$

$$\omega_{syn} = \omega_s \underbrace{\sqrt{\frac{h\eta_s}{2\pi p_s v_s} q U_0 \cos\varphi_s}}_{Q_{syn}}$$

Q_{syn} - number of synchrotron oscillations per turn

if we assume $\Delta\varphi = \Delta\varphi_0 \cos\omega_{syn} t$ and set in (1)

$\Delta E = \underbrace{\frac{\omega_{syn}}{\omega_s}\frac{p_s v_s}{h\eta_s}\Delta\varphi_0}_{E_0} \sin\omega_{syn} t$ ellipse in $(\Delta\varphi, \Delta E)$ plane

$\Delta E_0 = Q_{syn}\frac{p_s v_s}{h\eta_s}\Delta\varphi_0$ - connection between two amplitudes

$$\left(\frac{\Delta\varphi}{\Delta\varphi_0}\right)^2 + \left(\frac{\Delta E}{E_0}\right)^2 = 1$$

8.3 Stability conditions

$\eta_s \cos\varphi_s > 0$

$\cos\varphi_s > 0, \ \eta_s > 0, \ \gamma_s < \gamma_{tr}$

$\cos\varphi_s < 0, \ \eta_s < 0, \ \gamma_s > \gamma_{tr}$

$\left.\begin{array}{l}\varphi_s > 0 \text{ for acceleration} \\ \varphi_s < 0 \text{ for slowing down}\end{array}\right\}$ 4 regions

1) $\eta_s > 0, \ \gamma_s < \gamma_{tr}, \ \cos\varphi_s > 0, \ \sin\varphi_s > 0, \quad 0 < \varphi_s < \pi/2$

2) $\eta_s < 0, \ \gamma_s > \gamma_{tr}, \ \cos\varphi_s < 0, \ \sin\varphi_s > 0 \quad \pi/2 < \varphi_s < \pi$

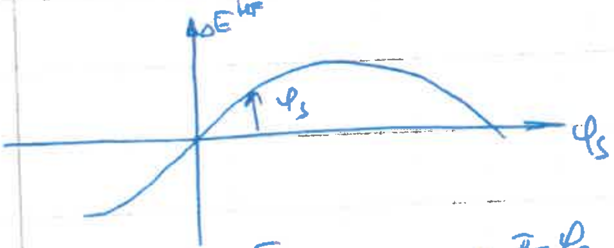
3) $\eta_s > 0, \ \gamma_s < \gamma_{tr}, \ \cos\varphi_s > 0, \ \sin\varphi_s < 0 \quad -\pi/2 < \varphi_s < 0$

a special solution $\cos\varphi_s = 0$ & $\eta_s = 0$ is excluded

$\gamma = \gamma_{tr}$ – no solution

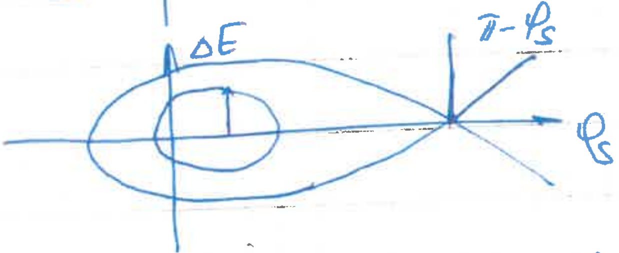
8.4 Separatrix

Shows for a given φ_s stable region



$\gamma < \gamma_{tr}$

$\Delta E = E - E_s$



8.5. Large amplitudes $\Delta\varphi \rightarrow$ formation of buckets

8.6. Optics in longitudinal plane

$$\begin{pmatrix} \ell \\ \delta \end{pmatrix} = M \begin{pmatrix} \ell_0 \\ \delta_0 \end{pmatrix}$$

ℓ — longitudinal position deviation

δ — relative momentum deviation

- Acceleration unit, HF-field acts as a thin lense with $\frac{1}{f}$ focusing strength

$$\frac{1}{f} = \frac{dE}{dt}\frac{1}{p\gamma^2} = \frac{qU_0\omega_{HF}}{p\gamma^2}$$

if $\frac{dE}{dt} > 0$ focusing lense „buncher"

— " — < 0 defocusing „debuncher"

$$\begin{pmatrix} \ell \\ \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} \ell_0 \\ \delta_0 \end{pmatrix}$$

- drift

$$M = \begin{pmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{pmatrix}$$

- buncher

$$\begin{pmatrix} \ell \\ \delta \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{pmatrix}}_{\text{drift}} \underbrace{\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}}_{\text{HF}} \underbrace{\begin{pmatrix} 1 & L/\gamma^2 \\ 0 & 1 \end{pmatrix}}_{\text{drift}}$$

exercise!

If one describes the motion of the whole ensemble we can define the longitudinal ~~phse~~ phase ellipse

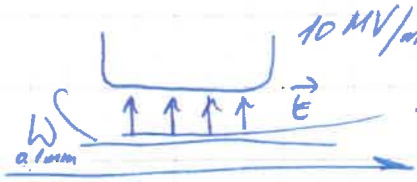
$$\sigma_{\text{longitudinal}} = \sigma_\ell = \begin{pmatrix} \sigma_{55} & \sigma_{56} \\ \sigma_{65} & \sigma_{66} \end{pmatrix}$$

the transformation

$$\sigma_\ell(s) = R_\ell(s)\,\sigma_\ell(0)\,R_\ell^T(s)$$

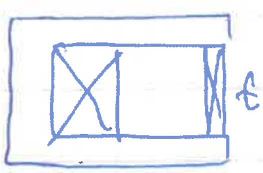
8.7. Injection

- electrostatic septum



$$\Theta = \frac{|E| L}{Bp\,v}$$

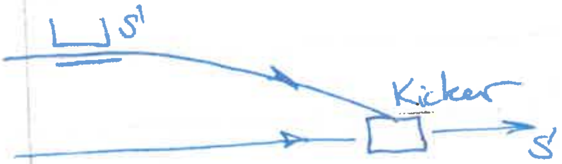
- magnetic septum



$$\Theta = \frac{BL}{Bp}$$

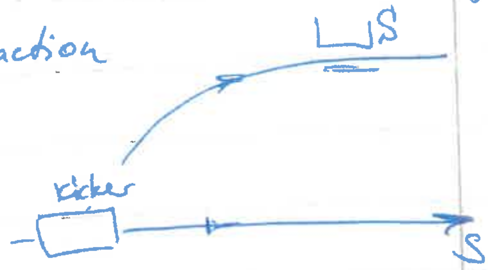
pulsed mode up to 2T

8.8. Injection



- single-turn injection
- multi-turn injection
  stacking
- charge-exchange injection

Extraction



- single turn extraction
- slow resonant extraction
  $3Q_x$ - resonanse.
- super slow extraction