7 Transverse beam dynamics with Dispersion

7.1 (*) $\boxed{x'' + K_x(s)\,x = h(s)\,\delta}$; $h(s) = \frac{1}{\rho_0(s)}$, $\delta = \frac{\Delta p}{p_0}$

$K_x(s)$ - periodic coefficients : $K_x(s+C) = K_x(s)$, $h(s+C) = h(s)$

leads to a modified equillibrium trajectory

$$x_D(s) = \delta D(s)$$

$D(s)$ - dispersion function, periodic function of the machine

7.2 - Solution of (*)

(**) $x_\delta(s) = x(s) + \delta D(s)$

$x(s)$: homogenious solution $(\delta = 0)$

Setting (**) into (*)

(***) $D'' + K_x(s) D = h_x(s)$

periodic conditions

$D(s+C) = D(s)$

$D'(s+C) = D'(s)$ set a starting point s - any start

- general solution

$$D(s) = D_0 C(s) + D_0' S(s) + d(s)$$

$d(s)$: special solution (4). Lecture

D_0, D_0': coefficients

$C(s)$, $S(s)$: cos/sin - like solutions basis-solutions of homogenious equations of motions

after lengthy calculations (Hinterberger)

$$\boxed{D(s) = \frac{\sqrt{\beta}}{2\sin\frac{\mu}{2}} \int_s^{s+C} h(\bar{s})\sqrt{\beta(\bar{s})}\,\cos\left[\Psi(\bar{s}) - \Psi(s) - \frac{\mu}{2}\right] d\bar{s}}$$ (6.67)

- Dispersion is an integral effect of ALL bending magnets!
- Perturbations in $h(s)$ contribute with $\sqrt{\beta(s)}$
- Contribution of an individual bending magnet is $\propto h = \frac{1}{\rho_0}$
- if $\frac{\mu}{2} = 0 \Rightarrow$ dispersion $\to \infty$ $\equiv$ resonance catastrophy

$\mu = 2\pi \pmod{2\pi}$ means that $Q \in \mathbb{N}$ (number of betatron oscillations)

7.3 Calculation of $D(s)$

- either numerically
- from Twiss matrix $\quad M(s) = R(s, s+C)$

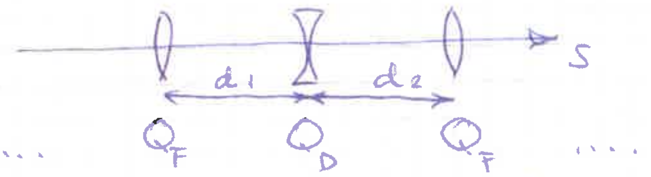
$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{16} \\ M_{21} & M_{22} & M_{26} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}$$

in each accelerator due to symmetry conditions points at which

$$D'(s) = 0$$

then $\quad D = \dfrac{M_{16}}{1 - M_{11}}$

7.4 Example FODO $= \frac{1}{2}F \; O \; D \; O \; \frac{1}{2}F$



$d_1 = d_2 = d$

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_1} & 1 \end{pmatrix}$$

$$M = \cos\mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\mu \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

- SPS / CERN (Super proton synchrotron)
  - 108-cells $Q_F + 4D(\rho_0 = 741{,}2\,m)\ (\alpha = 8{,}445\ mrad) + Q_D$
  - 6 superperiods of 14-cells + 4 insertions with missing dipoles
  - edge focusing $\sim 1 \cdot 10^{-5}\ m^{-1}$ – neglect

$\mu_c = 91{,}8620°$ (per cell)

$\mu = 91{,}8620 \cdot 108 = 9921{,}10°$ (per revolution) $\Rightarrow Q = 27{,}559$

Due to slightly different field gradients in Q_F and Q_D (3‰)

$Q_x = 27{,}574$ and $Q_y = 27{,}554$

↳ „separate function" machine

! however the running point is now at 27,4 why?

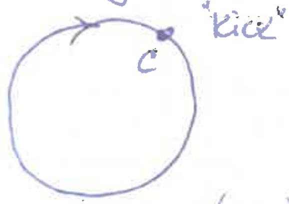
$5 \cdot 27{,}6 = 138$ – 5^{th} order resonance

with 6 SP ⇒ 4.6 beta-oscillations per period

other examples Model Synchrotron / ELSA slide

7.5. Distortions and resonances

- Starting from equilibrium trajectory



$$\Delta x' = \frac{-\delta B}{B\rho}\Delta s = F(s_0)\Delta s$$

„Closed orbit distortion"

$$\begin{pmatrix} x_c \\ x_c' \end{pmatrix}_{s_0} = M(s_0)\begin{pmatrix} x_c \\ x_c' + \Delta x' \end{pmatrix}_{s_0}$$

↑ describe the distorted reference orbit

Solution is analogous to the case of dispersion.

distorted orbit:

$$\cancel{x_c(s) = \frac{\sqrt{\beta}}{2\sin Q\pi}\Delta x'\sqrt{\beta(s_0)}\cos\left[\psi(s)-\psi(s_0)-Q\pi\right]}$$

$$x_c(s) = \frac{\sqrt{\beta(s)}}{2\sin Q\pi}\int_s^{s+C} F(\bar{s})\sqrt{\beta(\bar{s})}\cos\left[\psi(\bar{s})-\psi(s)-Q\pi\right]d\bar{s}$$

$F(s)$ - distortion function $\qquad x_c'' + K_x(s)\,x_c = F(s)$

Consequences:

$$x_c(s) \propto \Delta x'$$

$$\left.\begin{array}{l} x_c(s) \propto \sqrt{\beta(s_0)} \\ x_c(s) \propto \sqrt{\beta(s)} \end{array}\right\}$$ essential for correction magnets (trial & error)

$$x_c(s) \propto 1/\sin Q\pi$$ ⇐ Q cannot be integer

7.6. Quadrupole and sextupole fields

7.6.1. - Q-magnet is a lense with focus f

⇒ small distortions → additional focusing

$$\frac{1}{f} = \delta K \Delta s$$

Twiss matrix $\qquad M_0 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$

$$M' = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} M_0$$ an additional thin lense.

$$\mu' = \mu_0 + \Delta\mu \quad (\Delta\mu \ll 1)$$ $\Delta\mu =$

$$\cos\mu = \frac{1}{2}\mathrm{Tr}(M) = \cos\mu_0 \underbrace{- \frac{1}{2}\frac{\beta_0}{f}}_{\text{additional term}} \sin\mu_0$$

$$\Delta\mu = 2\pi\Delta Q = \frac{1}{2}\frac{\beta_0}{f}, \quad \Delta Q = \frac{1}{4\pi}\frac{\beta_0}{f}$$ ← can be used to measure β_0

Result: Shift of operation point

$$\Delta Q = \frac{1}{4\pi}\oint \beta(\bar{s})\,\delta K(\bar{s})\,d\bar{s}$$

similarly as for the dispersion case (not $\sqrt{\beta}$ but β)

$$\Delta\beta(s) = \frac{\beta(s)}{2\sin^2 2Q\pi}\int_s^{s+C} \delta K(\bar{s})\,\beta(\bar{s})\cos 2[\psi(\bar{s})-\psi(s)-Q\pi]\,d\bar{s}$$

important: $2Q\pi$ → Q - half-integer ⇒ resonance

7.6.2 Floquet transformation

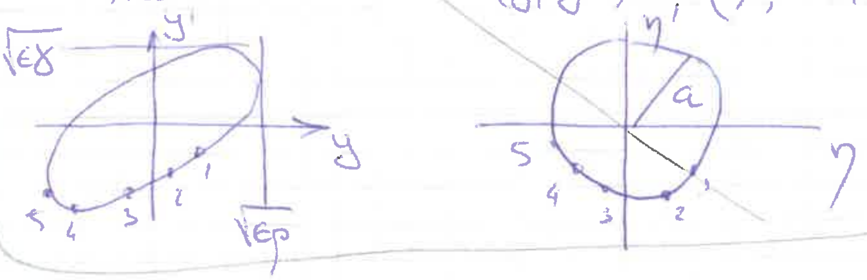
used to discuss resonances and instabilities

Courant-Snyder-variables

$$\boxed{\frac{y^2}{\beta} + \frac{(\alpha y + \beta y')^2}{\beta} = a^2 = \epsilon}$$ C-S - invariant

tilted ellipse → aim: circular diagram

~~Get~~ Transformation $(y, y') \to (\eta, d\eta/d\varphi)$



$$y(s) \leftrightarrow \eta(\psi) \quad \text{with} \quad \eta = \frac{y}{\sqrt{\beta}}$$

$$y'(s) \leftrightarrow \frac{d\eta}{d\psi} \quad \text{with} \quad \frac{d\eta}{d\psi} = \alpha\frac{y}{\sqrt{\beta}} + \sqrt{\beta}\,y'$$

$$\begin{pmatrix} \eta \\ \frac{d\eta}{d\psi} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

↑ "dimensionless" units $\sqrt{\text{mm mrad}}$

$$\eta = a\cos(\psi+\psi_0)$$
$$\frac{d\eta}{d\psi} = -\underset{\uparrow\atop \sqrt{\epsilon}}{a}\sin(\psi+\psi_0)$$

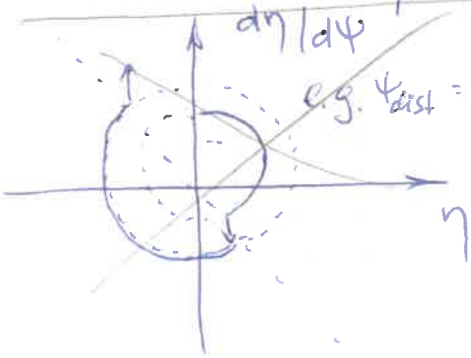
~~diagram - - solution of Hill's equations~~

7.6.3 Stop-band (2nd order)

kick: $\Delta y' = -\frac{y}{f} = -\frac{1}{f}a\sqrt{\beta}\cos\psi$

$\Rightarrow \quad \Delta\eta' = -\frac{1}{f}a\beta\cos\psi$

leads to an amplitude shift and phase shift



$$\Delta a = \Delta\left(\frac{d\eta}{d\psi}\right)\sin\psi = -\frac{a\beta}{f}\cos\psi\sin\psi$$
$$\Delta\psi = -\frac{1}{a}\Delta\left(\frac{d\eta}{d\psi}\right)\cos\psi = \frac{\beta}{f}\cos^2\psi$$

$$\Delta Q = \frac{1}{2\pi}\frac{\beta}{f}\cos^2\psi = \frac{1}{4\pi}\frac{\beta}{f}(1-2\cos2\psi)$$

average shift of the working point by $\overline{\Delta Q} = \frac{1}{4\pi}\frac{\beta}{f}$

with superimposed modulation $\delta Q = \frac{1}{4\pi}\frac{\beta}{f}\cos2\psi$

~~modulation & shift are small if β small and f large~~

7.6.4. Sextupoles (similar to quadrupole)

resonance if $3Q$ - integer

band: $\delta Q = \frac{\beta^{3/2}}{16\pi}\left(\frac{\partial^2 B_y}{\partial x^2}\right)\frac{\Delta s}{B\rho}\,a\cos3\psi$

Stop-band 3rd-order

amplitude of betatron oscillation

- non-linear effect
- dynamic aperture

7.7. Stability condition

Stay away from resonances, resonance diagram (slide)

1^{st} order dipoles
2^{nd} order Qpoles
3^{rd} order S-poles

resonance conditions

$$p = nQ_x$$
$$p = nQ_y$$
$$p = \ell Q_x + m Q_y \quad \text{(coupled)} \qquad n = |\ell| + |m|$$

summe or difference resonan

these are bands

depending on β (QP)
emittance (SP)

7.8. α_p and Stability (phase focusing)

Synchrotron principle ω - depends on p

$$\frac{\Delta\omega}{\omega_0} = \eta \frac{\Delta p}{p_0}, \qquad \omega = 2\pi \frac{v}{C}$$

$$\frac{\Delta\omega}{\omega_0} = \frac{\Delta v}{v_0} - \frac{\Delta C}{C_0}$$

$$\frac{\Delta v}{v_0} = \frac{1}{\gamma^2} \frac{\Delta p}{p_0} \quad \text{and} \quad \frac{\Delta C}{C_0} = \alpha_p \frac{\Delta p}{p_0}$$

relativistic relation between v and p

relation between relative change of C corresp to Δp

$$\eta = \frac{1}{\gamma^2} - \alpha_p = \frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2}$$

transition energy

momentum compaction factor

$\alpha_p \approx \frac{1}{Q_x^2}$ $\quad$ Q_x - large, α_p - small = compact trajectories

Acceleration

γ - small	$\eta > 0$	stable
$\gamma = \gamma_{tr}$	$\eta = 0$	unstable
γ - large	$\eta < 0$	stable

large energy range $\Leftrightarrow$ small α_p
need for pre-accelerators / dynamic range

7.9. Chromaticity

Problem: particles with $\delta = \frac{\Delta p}{p_0}$ see differen bending and focusing in the machine

QP:

$$\Delta Q_x = \xi_x \frac{\Delta p}{p_0} \quad , \quad \Delta Q_y = \xi_y \frac{\Delta p}{p_0}$$

ξ - (natural) chromaticity (ξ^n) (slide)

- the ~~coefficients~~ focusing strength in the Hill's equations

$$\Delta K_x = -K_x \frac{\Delta p}{p_0}$$

tune shift

$$\Delta Q_x = \underbrace{\left[-\frac{1}{4\pi} \oint \beta(\bar{s}) K_x(\bar{s}) d\bar{s} \right]}_{\xi_x^n} \frac{\Delta p}{p} \leftarrow \text{slow part. large difference}$$

ξ_x^n - always negative

additional chromaticity comes from sextupole fields

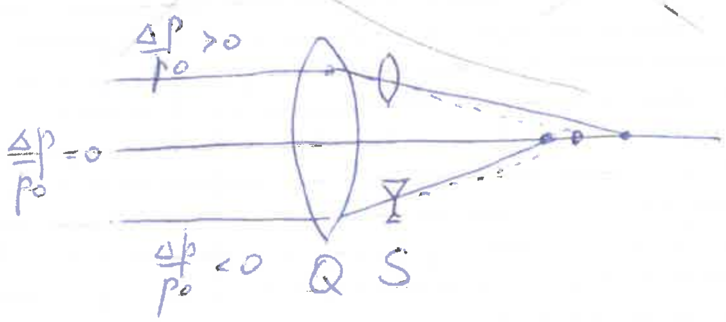
$$\xi_x = \underset{Q}{\xi_x^n} + \underset{S}{\xi_x^s} \; , \; \xi_y = \underset{Q}{\xi_y^n} + \underset{S}{\xi_y^s}$$

- Chromaticity correction

employing additional sextupoles in dispersion regions $\left(g_s = \frac{\partial^2 B_y}{\partial x^2} \right)$

$$\oplus \; \xi_x^s = \frac{1}{4\pi} \oint g_s(\bar{s}) \frac{\beta_x(\bar{s}) D(\bar{s})}{B\rho} d\bar{s}$$

$$\ominus \; \xi_y^s = \ominus \quad \beta_y(\bar{s})$$



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