L7 (2018)

→ Electrostatic elements (lenses, deflectors, drifts, acc. sections) ⇒

⇒ very similar descriptions as for magnetic elements (book)

6. Transverse beam dynamics

6.1. Coordinate system

equilibrium orbit, reference particle moves each turn

closed orbit

The goal: describe beam position as a function of $\vec{s}$ in the accelerator / lattice) structure and after several revolutions

6,2 Hill's equations (linear approximation)

$$\left[\begin{array}{l} \dfrac{d^2x}{ds^2} + k_x(s)\,x = \dfrac{1}{\rho_0(s)} \dfrac{\Delta p}{p_0} \\ \dfrac{d^2y}{dy^2} + k_y(s)\,y = 0 \end{array}\right] \quad (*)$$

assume first $\frac{\Delta p}{p_0} = 0$, monochomatic/monoenergetic beam

⇒ result: oscillation along the reference orbit $\vec{s}$ with variable amplitude $a\sqrt{\beta(s)}$ and variable wave number $1/\beta(s)$

6,3 Twiss matrix

- If formal solution of (*) - motion of a particle obviously depends on the start values $[y(s), y'(s)]$ at a given s.
- special matrix $M = R(s+C)$ C-circumference
- general form

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \Rightarrow \det(M) = 1$$

$$\Rightarrow \cos\mu \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} + \sin\mu \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{J}$$

α, β, γ - Twiss parameters

$\det(M) = 1 \;:\Rightarrow\; \det(J) = \beta\gamma - \alpha^2 = 1 \;;\; J^2 = J \cdot J = -I$

- Stability criterion

for many turns N

$$M^N = (I\cos\mu + J\sin\mu)^N = \underline{I\cos N\mu + J\sin N\mu}$$

similar to von Moirre formula: $(\cos\mu + i\sin\mu)^N = \cos\mu N + i\sin N\mu$

$$\cos\mu = \frac{1}{2}\operatorname{Tr} M = \frac{1}{2}(M_{11} + M_{22})$$

$$\sin\mu = \operatorname{sign}(M_{12})\sqrt{1-\cos^2\mu}$$

$$\left.\begin{aligned} \beta &= \frac{M_{12}}{\sin\mu} \\ \alpha &= (M_{11} - M_{22})/2\sin\mu \\ \gamma &= -M_{21}/\sin\mu \end{aligned}\right\} \text{real}$$

$\alpha(s), \beta(s), \gamma(s)$ - optical, betatron, amplitude, lattice functions

Goal: describe machine

μ - independent of s, machine parameter defined by M

to 2π = phase advance of $\beta(s)$ per revolution

6.3. Solution of Hill's equations

$$y'' + K_y(s)y = 0$$

(6.24) in Hinterberger $\boxed{y(s) = a\sqrt{\beta(s)}\cos[\psi(s) + \psi_0]}$ (**)

a, ψ_0 are defined for each particle, which are the amplitude and the phase of oscillations, resp.

$a\sqrt{\beta(s)}$ - variable amplitude along $\vec{s}$

$\frac{d\psi}{ds} = \frac{1}{\beta(s)}$ - variable wave number / wavelength

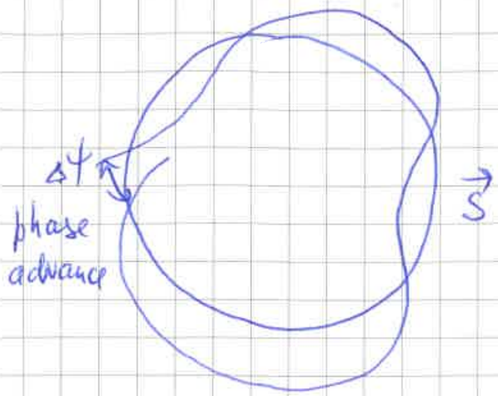
$\lambda(s) = 2\pi\beta(s)$

phase shift / phase advance

$$\mu = \int_s^{s+C} \frac{d\bar{s}}{\beta(\bar{s})} = \oint \frac{d\bar{s}}{\beta(\bar{s})}$$

number of betatron oscillations per revolution, betatron tune

$$Q = \frac{\mu}{2\pi} = \frac{1}{2\pi}\oint \frac{d\bar{s}}{\beta(\bar{s})}$$



if $\Delta\psi = 0$
the particle ~~fits~~ moves always on the same orbit ! resonance !
disturbances will be multiplied $\Rightarrow$ instability

6.4. Courant - Snyder Invariant

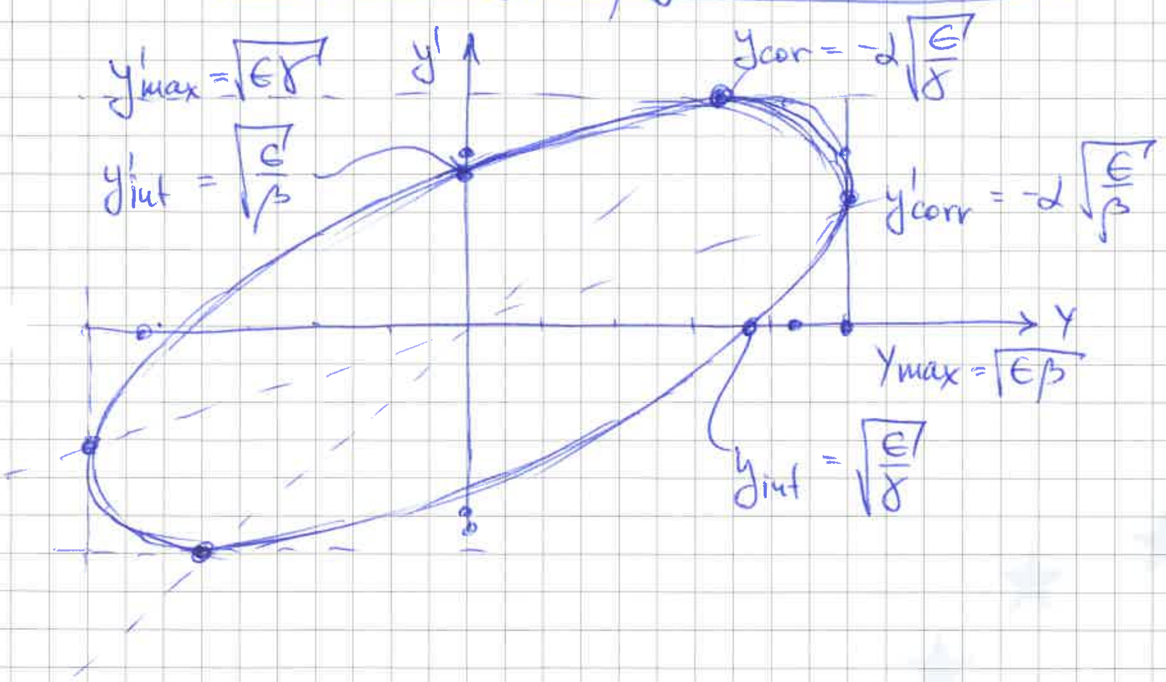
$$y(s) = a\sqrt{\beta(s)}\cos[\psi(s) + \psi_0] \quad (**)$$

<u>what is a ?</u>

one can rewrite (**) after a lengthy math to:

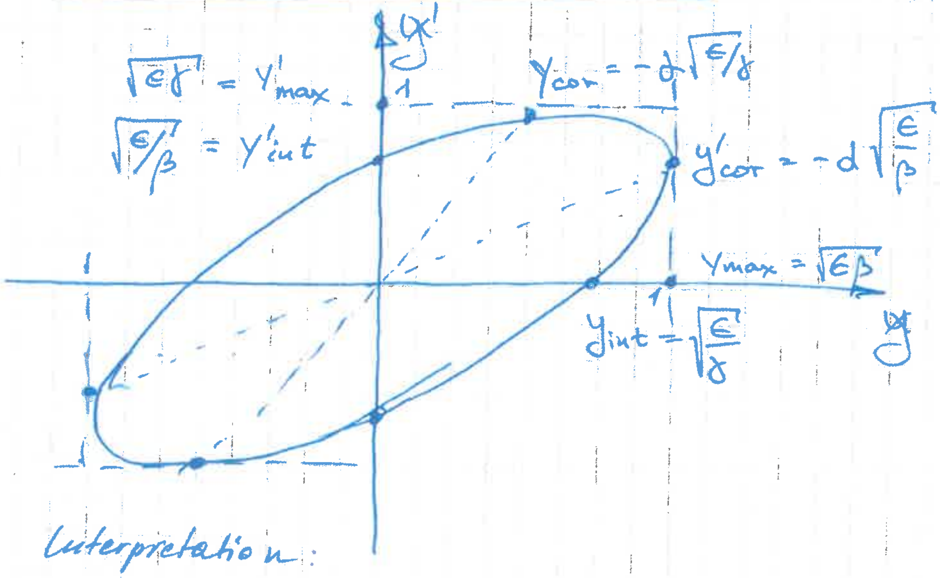
$$\boxed{\frac{y^2}{\beta} + \frac{(\alpha y + \beta y')^2}{\beta} = a^2 = \epsilon} \quad \text{— C-S-Invariant}$$

$$\boxed{\gamma y^2 + 2\alpha y y' + \beta y'^2 = a^2 = \epsilon}$$



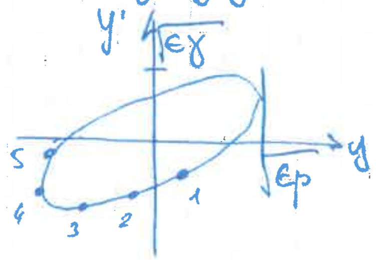
$$\boxed{\gamma y^2 + 2\alpha y y' + \beta y'^2 = a^2 = \epsilon}$$

$$\gamma = \frac{1+\alpha^2}{\beta} = \frac{1+\left(\frac{\beta'}{2}\right)^2}{\beta}$$



Interpretation:

1. a particle with coordinates (y, y') propagates along a changing ellipse
2. the area of the ellipse is constant and defined by a
3. the shape of the ellipse is defined by the machine via $\alpha(s), \beta(s), \gamma(s)$ - functions → machine ellipse
4. plotting (y, y') after each revolution gives an ellipse



5. all particles with smaller a - are enclosed in the ellipse

6.5. Beam emittance & machine acceptance

- beam ~~envelope~~ ~~emittance~~ ellipse, if machine & beam are matched

$$\sigma_x = \sqrt{\epsilon_x \beta_x}$$

1σ - emittance

- RMS-emittance / 1σ-emittance

$$\epsilon_x^{1\sigma} = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle x x'\rangle^2}$$

for N-particles

$$\epsilon_x^{1\sigma} = \frac{1}{N}\sum_i^N \epsilon_{x,i} = \frac{1}{N}\sum_i \gamma_x x_i^2 + 2\alpha_x x_i x_i' + \beta_x x_i^2$$

machine parameters

- Beam envelope

$$y_{max}(s) = \sqrt{\epsilon}\sqrt{\beta(s)}$$

$$y'_{max}(s) = \sqrt{\epsilon}\sqrt{\gamma(s)}$$

Beam   machine property

- acceptance, if a limiting aperture, e.g. through slits

$$\epsilon_{max} = \frac{y_{lim}^2}{\beta_{lim}}, \quad A = \pi\epsilon_{max}$$

acceptance of the machine

6.6. Machine ellipse

Ellipses defined by α, β, γ functions is nothing else but Eigenellipses σ_e of matrix M at each s.

$$\sigma_e = M \sigma_e M^T$$

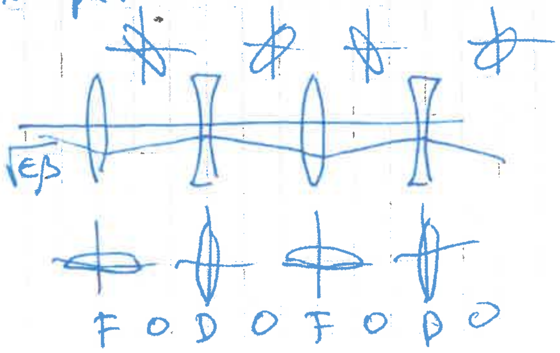
$$\sigma_e(s + C) = \sigma_e$$

Since $M(s) = \cos\mu\, I + \sin\mu\, J \Rightarrow \sigma_e(s) = \epsilon \begin{pmatrix} \beta(s) & -\alpha(s) \\ -\alpha(s) & \gamma(s) \end{pmatrix}$

σ_e - beam profile

σ_e - determined through $\alpha(s), \beta(s), \gamma(s)$

example.



6.7. Transformation of Twiss parameters

$$R = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

from machine ellipses

$$\sigma = R\,\sigma_0\,R^T$$

ellipse

$$\begin{pmatrix} +\beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = R \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} R^T$$

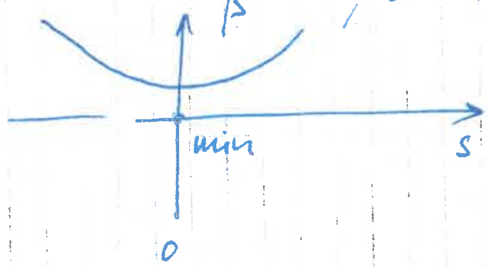
$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

example <u>drift</u>

start at beam waist $\alpha_0 = 0$; $\gamma_0 = \frac{1}{\beta_0}$ (around minimum)

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

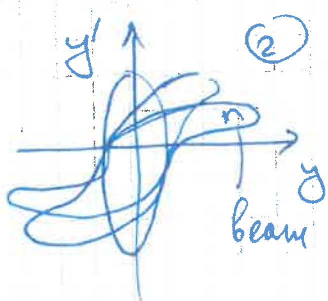
$C \approx 1,\ S \approx s$

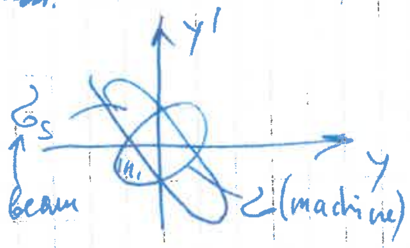


6.8. Matching of a beam and a machine

errors of 2^{nd} and higher order $\Rightarrow$

particles with $a > a_{mach.}$ have $\Delta\varphi$ different from "accepted" ~~machine~~ el.





6.9 Summary

- Twiss matrix
$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} = \cos\mu \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} + \sin\mu \underbrace{\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}}_{J}$$

- periodicity $\quad \beta(s+C) = \beta(s)$
- $\alpha(s)$ & $\gamma(s)$ $\quad \alpha(s) = -\frac{1}{2}\beta'(s)$
$$\gamma(s) = \frac{1-\alpha^2(s)}{\beta(s)} = \frac{1+\left(\beta'(s)/2\right)^2}{\beta(s)}$$
- Eigenellipse of machine
$$\sigma_e = \epsilon \begin{pmatrix} \beta(s) & -\alpha(s) \\ -\alpha(s) & \gamma(s) \end{pmatrix}$$
- beam envelope $\quad y_{max}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$
- orbit of an individual particle
$$y(s) = a\sqrt{\beta(s)}\cos(\psi(s)+\psi_0) = a\sqrt{\beta(s)}\cos\left[\int_{s_0}^{s}\frac{d\bar{s}}{\beta(\bar{s})} + \psi_0\right]$$
- local wave number $\quad k(s) = \frac{2\pi}{\lambda(s)} = \frac{d\psi}{ds} = \frac{1}{\beta(s)}$
- local wave length of betatron oscillation
$$\lambda(s) = 2\pi\beta(s)$$
- phase advance $\quad \Delta\psi = \int_{s_0}^{s}\frac{d\bar{s}}{\beta(\bar{s})}$
- phase advance per revolution
$$\mu = \oint \frac{d\bar{s}}{\beta(\bar{s})}$$
- betatron tune, number of betatron oscillations
$$Q = \frac{1}{2\pi}\mu = \frac{1}{2\pi}\oint\frac{d\bar{s}}{\beta(\bar{s})}$$