4.6 Geometrical optics

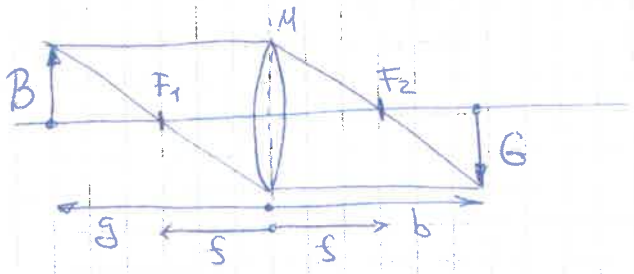
* thin lense approximation

$$\Delta x' = -\frac{1}{f} x_0 \qquad f - \text{focal length}$$

matrix equation

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \qquad \text{similar for } y$$

Example: point-to-point transformation



F - focus, image point

condition $\quad \frac{1}{g} + \frac{1}{b} = \frac{1}{f} \quad$ (optics)

matrix representation

$$R_x = \underbrace{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}}_{\text{drift}} \underbrace{\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}}_{\text{lense}} \underbrace{\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}}_{\text{drift}} = \begin{pmatrix} -b/g & 0 \\ -1/f & -g/b \end{pmatrix} \qquad R_{12} = 0$$

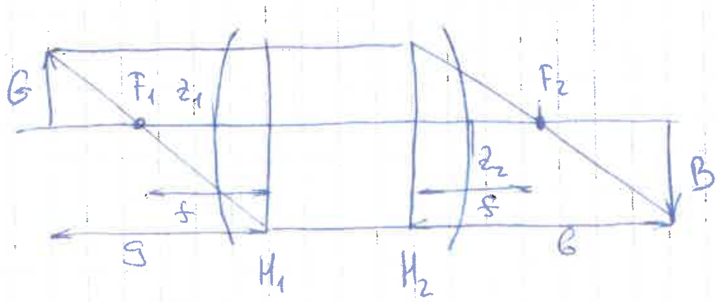
focusing strength $\quad R_{21} = -1/f$

scaling of transformation (amplification)

$$(x/x_0) = M = \frac{B}{G} = -\frac{b}{g} = R_{11}$$

$$(x', x_0') = M^{-1} = -\frac{g}{b} = R_{22}$$

* thick lense - additional drift, principal planes (H), gaps (z)



$$\underbrace{\begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix}}_{\text{real lense}} =$$

$$= \begin{pmatrix} 1 - \frac{z_2}{f} & z_1 + z_2 - \frac{z_1 z_2}{f} \\ -\frac{1}{f} & 1 - \frac{z_1}{f} \end{pmatrix} \Rightarrow$$

$$\frac{1}{f} = -R_{21}$$

$$z_1 = \frac{R_{22} - 1}{R_{21}} \; ; \; z_2 = \frac{R_{11} - 1}{R_{21}}$$

Examples:

- radial focusing quadrupole

$$\frac{1}{f} = \sqrt{K_x}\,\sin\left(\sqrt{K_x}\,L\right)$$

$$z_1 = \frac{\cos\left(\sqrt{K_x}\,L\right) - 1}{-\sqrt{K_x}\,\sin\left(\sqrt{K_x}\,L\right)}$$

$$z_2 = z_1$$

- homogenious dipole

$$z_1 = z_2 = \rho_0 \tan\frac{\alpha}{2} \qquad \left(\alpha = \frac{L}{\rho_0}\right)$$

thick lens = $\overleftarrow{z_1}$ ( ) $\overrightarrow{z_2}$ = drift ( ) drift

thin lens

4.7. Phase ellipse

up to now – single particle, now a bunch

$\vec{x}(s) = R(s)\,\vec{x}(0)$ – the same principle

– (x, x')-plane, $\rho(x, x')$ – particle density can be presented as ellipse in 2D described by matrix

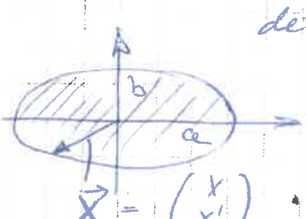
$\sigma_x = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ σ-matrix

σ_x - symmetric $\sigma_x = \sigma_x^T$ $(\sigma_{12} = \sigma_{21})$

$\det(\sigma_x) > 0$

① $\vec{x}^T \sigma_x^{-1} \vec{x} = 1$ – equation of ellipse

$\hookrightarrow \frac{1}{\det(\sigma_x)} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}$



$\frac{x^2}{a^2} + \frac{x'^2}{b^2} = 1$

$\vec{x} = \begin{pmatrix} x \\ x' \end{pmatrix}$

vector from origin to ellipse boundary

① $\sigma_{22} x^2 - 2\sigma_{12} x x' + \sigma_{11} x'^2 = \det(\sigma_x) = \epsilon_x^2$

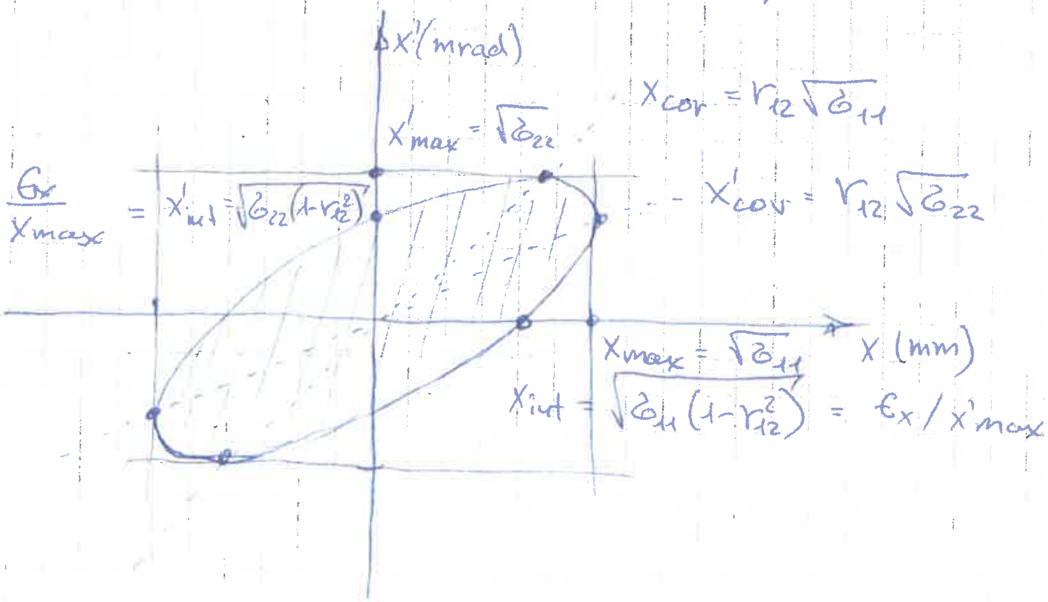
emittance $E_x = \pi \epsilon_x = \pi \sqrt{\det(\sigma_x)}$ (often ϵ_x is emittance) surface of the ellipse

maximal values $x_{max} = \sqrt{\sigma_{11}}$

$x'_{max} = \sqrt{\sigma_{22}}$

σ_{12} – correlation between x and x'

dimensionless correlation parameter $r_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}} \in [-1, 1]$



4.8. Density distribution in phase ellipse

Gaussian in 2D

$$\rho(\vec{x}) = \frac{1}{2\pi\epsilon_x} \exp\Big(-\frac{1}{2}\underbrace{\vec{x}^T \sigma_x^{-1}\vec{x}}_{\text{standard deviation}\,\wedge 2\, = \,\text{STD}\wedge 2}\Big)$$

in 6D: $(2\pi)^3 \epsilon_x \epsilon_y \epsilon_z$

STD^2 = 1 encloses 39,3% of particles $\epsilon_x^{1\sigma} = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}$

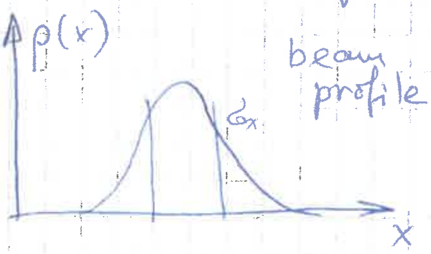
STD^2 = 4 — " — 86,5% — " — $\epsilon_x^{2\sigma} = 4\,\epsilon_x^{1\sigma}$

STD^2 = 9 — " — 98,5% — " — $\epsilon_x^{3\sigma} = 9\,\epsilon_x^{1\sigma}$

real distribution deviates from Gaussian: beam tube instead of ∞ scattered particles - halo

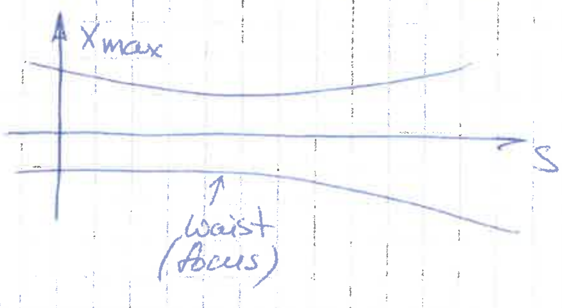
slide

4.9. Beam envelope



beam envelope

$$x_{max}(s) = \sqrt{\sigma_{11}(s)}$$



4.10 Transformation of phase ellipse

$$\sigma_x(s) = R_x(s)\,\sigma_x(0)\,R_x^T(s) \quad \text{proof see textbook}$$

examples

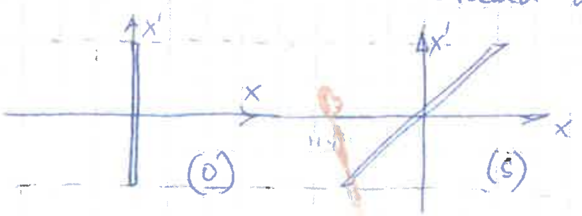
* drift with length L

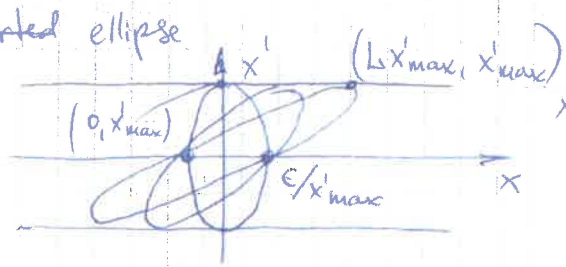
upright ellipse

$$\sigma_x'(L) = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{11}(0) & 0 \\ 0 & \sigma_{22}(0) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ L & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \sigma_{11}(0) + L^2\sigma_{22}(0) & L\sigma_{22}(0) \\ L\sigma_{22}(0) & \sigma_{22}(0) \end{pmatrix}$$

rotated ~~distorted~~ ellipse





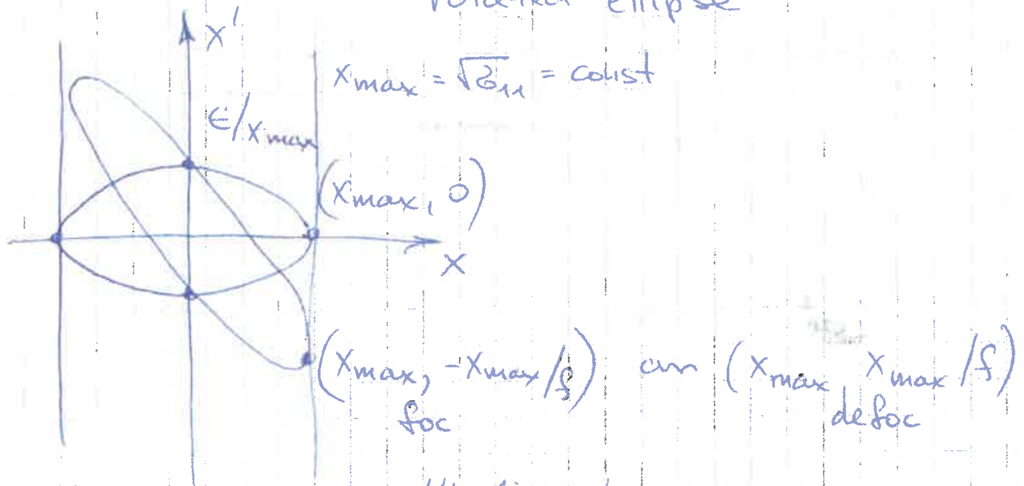
$x_{max}' = \sqrt{\sigma_{22}} = \text{const}$

* thin lense with $1/f_x$

$$\sigma_x\left(\frac{1}{f_x}\right) = \begin{pmatrix} 1 & 0 \\ -1/f_x & 1 \end{pmatrix}\begin{pmatrix} \sigma_{11}(0) & 0 \\ 0 & \sigma_{22}(0) \end{pmatrix}\begin{pmatrix} 1 & -1/f_x \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \sigma_{11}(0) & -\sigma_{11}(0)/f_x \\ -\sigma_{11}(0)/f_x & \sigma_{22}(0) + \sigma_{11}(0)/f_x^2 \end{pmatrix}$$

rotated ellipse



4.11. Phase space in multi-dimensions

analogously! example: dispersive broadening of phase ellipse (Slide)

4.12. Ion-Optical systems

* Quadrupole singlett

real magnets $L \gg 0$

$$\left.\begin{aligned} f_x &= (\sqrt{k}\,\sin\sqrt{k}L)^{-1} \approx \frac{1}{kL} \\ f_y &= (-\sqrt{k}\,\sinh\sqrt{k}L)^{-1} \approx -\frac{1}{kL} \end{aligned}\right\}$$

principal planes

$$z_{1x} = z_{2x} = 1/k \tan\frac{kL}{2} \approx L/2$$

$$z_{1y} = z_{2y} = \frac{1}{k}\tanh\frac{kL}{2} \approx L/2$$

in real computer simulations ($\approx$) are calculated exactly

a single quadrupole makes little sense

* quadrupole duplet at a distance d

$$R = \begin{pmatrix}1 & 0\\ -1/f_2 & 1\end{pmatrix}\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1/f_1 & 1\end{pmatrix} = \begin{pmatrix}1-d/f_1 & d\\ -\frac{1}{f_1}-\frac{1}{f_2}+\frac{d}{f_1 f_2} & 1-\frac{d}{f_2}\end{pmatrix}$$

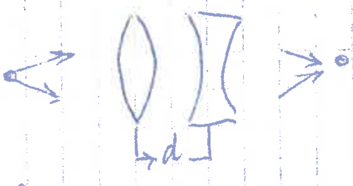
if $f = f_1 = f_2(-1) = -f_2$

(*) $R = \begin{pmatrix}1-d/f & d\\ -d/f^2 & 1+d/f\end{pmatrix} \Rightarrow$ thick lense with $f_D = f^2/d$ slide

! no sign = focussing in both planes x, y !

- Principal planes $z_1 = -f$, $z_2 = +f$
  typically $d < |f| \Rightarrow$ on one side
- point-to-point transformation (important to match apertures)

$R_{12} = 0$ $\quad x = R_{11} x_0 + R_{12} x_0'$



R from (*) plus 2x drifts (before & after)

$$R = \begin{pmatrix}1 & l\\ 0 & 1\end{pmatrix}\begin{pmatrix}1-d/f & d\\ -d/f^2 & 1+d/f\end{pmatrix}\begin{pmatrix}1 & l\\ 0 & 1\end{pmatrix} =$$

$$= \begin{pmatrix}1-d/f-ld/f^2 & 2l+d-d(l/f)^2\\ -d/f^2 & 1+d/f-ld/f^2\end{pmatrix}$$

$R_{12} = 2l+d-d(l/f)^2 \overset{!}{=} 0 \Rightarrow f = \pm l\sqrt{\frac{d}{2l+d}}$

first lense foc & the other way around for "-"

lengths:

$g = l + z_1 = l - f$ and $b = l + z_2 = l + f$

$R = \begin{pmatrix}-b/g & 0\\ -d/f^2 & -g/b\end{pmatrix} \Rightarrow$ dependent on the sign of f either scaling up or down

* quadrupole triplet

example: $\frac{1}{f} \to -\frac{2}{f} \to \frac{1}{f}$

$$R_x = \begin{pmatrix} -1 & 0 \\ -1/f_x & -1 \end{pmatrix} \qquad R_y = \begin{pmatrix} -1 & 0 \\ -1/f_y & -1 \end{pmatrix}$$

no scaling, point-to-point (slide)

* telescopical system

2x: point-to-point & parallel-to-parallel (Slide)

* accelerators - repetitive structures of telescopes! (Slide)