4. Ion Optics

1. Coordinate system

- various definition exist!

$\vec{s}$ - particle direction
$\vec{y}$ - upwards
$\vec{x}$ - bending plane

} Curvilinear system

$\boxed{\text{Coordinate system is defined on the reference / nominal / sollbahn trajectory}}$!

- metric $\boxed{r(s) = \vec{r}_0(s) + x(s)\vec{u}_x(s) + y(s)\vec{u}_y(s)}$

$$d\vec{r} = \vec{u}_x dx + \vec{u}_y dy + \vec{u}_s (1 + hx) ds$$

$$h = \frac{1}{\rho_0} \qquad \rho_0 \text{ - curvature of nominal trajectory}$$

$$h(s) = \frac{1}{\rho_0(s)} = \frac{q}{p_0} B_y(x=0, y=0, s) = \frac{q}{p_0} B_0(s)$$

q, p_0 - charge and momentum of reference particle

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- deviation of a particle in 3-dimensions

$\Delta x, \Delta y, \Delta z$ - in space
$\Delta p_x, \Delta p_y, \Delta p_z$ - in momentum

} 6 parameters

$\Delta p_x, \Delta p_y, \Delta p_z \ll p_0$

$$x' = \frac{dx}{ds} = \frac{\Delta p_x}{p_0}, \quad y' = \frac{dy}{ds} = \frac{\Delta p_y}{p_0}, \quad \ell = -v_0 (t - t_0)$$

early particle $t < t_0$; $\ell > 0$

$$\delta = \frac{p - p_0}{p_0}$$

$$\cancel{X(s)} \begin{pmatrix} x \\ x' \\ y \\ y' \\ \ell \\ \delta \end{pmatrix} = \begin{pmatrix} \text{radial orbit deviation} \\ \text{radial direction deviation} \\ \text{axial} \quad \text{"} \\ \text{axial} \quad \text{"} \\ \text{logitudinal} \ldots \\ \text{"} \quad \text{momentum deviation} \end{pmatrix}$$

Since $x, x', y, y', \ell, \delta$ are small $\Rightarrow$ mm, mrad, promil

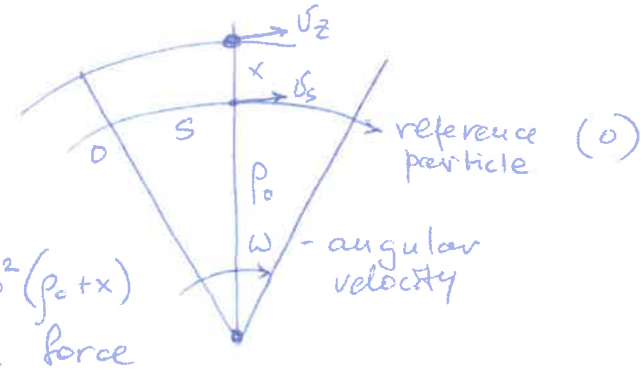
2. Equations of motion

(linear approximation) (dipole, quadrupole magnets..)

- Lorentz force $\dot{\vec{p}} = q(\vec{v}\times\vec{B})$

$$\ddot{r} = q(\vec{v}\times\vec{B})/\gamma m$$

a) dipole magnet



radius $= \rho = \rho_o + x$

velocity $= v_z = (\rho_o + x)\omega$

radial acceleration $a_r = -\omega^2(\rho_o + x)$

- Lorentz force + centrifugal force

$$\ddot{r} = \ddot{x} + a_r = \ddot{x} - \omega^2(\rho_o + x) = \frac{q}{\gamma m}\left(\underline{v_y B_z} - \underline{v_z} B_y\right) \quad (1)$$

$$\ddot{y} = \frac{q}{\gamma m}\left(v_z B_x - \underline{v_x B_z}\right) \quad (2)$$

$v_x, v_y \ll v_z \rightarrow$ can neglect $v_x B_z$, $v_y B_z$ (we are still in linear appoximation)

$$v = v_z\sqrt{1 + x'^2 + y'^2} \approx v_z$$

$$p = \gamma m v \approx \gamma m v_z = \gamma m \rho_o \omega\,(1 + x/\rho_o)\,;\quad x/\rho_o \ll 1 \ (\text{mm vs m})$$

- replacing $\gamma m = p/v$ obtain for (1) and (2)

$$\ddot{x} - \omega^2(\rho_o + x) = -\frac{q}{p} v_z^2 B_y \quad (3)$$

$$\ddot{y} = +\frac{q}{p} v_z^2 B_x \quad (4)$$

now: get rid of time $\frac{ds}{dt} = \rho_o\omega$

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = \rho_o\omega\frac{d}{ds}\,,\quad \frac{d^2}{dt^2} = (\rho_o\omega)^2\frac{d^2}{ds^2}$$

- divide (3) and (4) by $(\rho_o\omega)^2$:

$$x'' - \frac{1}{\rho_o}\left(1 + \frac{x}{\rho_o}\right) = -\frac{q}{p} B_y\left(1 + \frac{x}{\rho_o}\right)^2 \quad (5)$$

$$y'' = +\frac{q}{p} B_x\left(1 + \frac{x}{\rho_o}\right)^2 \quad (6)$$

- using the definition of field index:

$$n = \frac{-dB/B}{dr/r} = -\frac{r}{B}\cdot\frac{dB}{dr} \quad \text{where } r\text{ - radius}$$

$$n = -\frac{\partial B_y}{\partial x}\frac{\rho_o}{B_o} \qquad B_y = B_o + \int_{\rho_o}^{x} \frac{-nB_o}{\rho_o}\,dx = -\frac{nBx}{\rho_o}\Big|_{\rho_o}^{x} + B_o$$

$$B_x = B_o\left(0 - \frac{ny}{\rho_o}\right)\,;\qquad B_y = B_o\left(1 - \frac{nx}{\rho_o}\right)$$

- we can now rewrite 5 and 6

$$x'' = -B_0 \frac{q}{p}\left(1 - \frac{nx}{\rho_0}\right)\left(1+\frac{x}{\rho_0}\right)^2 + \frac{1}{\rho_0}\left(1+\frac{x}{\rho_0}\right) \quad (7)$$

$$y'' = -B_0 \frac{q}{p}\frac{ny}{\rho_0}\left(1+\frac{x}{\rho_0}\right)^2 \quad (8)$$

- relative momentum deviation

$$\frac{1}{p} = \frac{1-\delta}{P_0}, \text{ where } P_0 = qB_0\rho_0 \qquad \left(\delta = 1 - \frac{P_0}{p} \simeq \frac{p-P_0}{P_0}\right)$$

$$\left.\begin{aligned} x'' + \frac{1-n}{\rho_0^2}x &= \frac{1}{\rho_0}\delta \\ y'' + \frac{n}{\rho_0^2}y &= 0 \end{aligned}\right\} \quad (9)$$

$$\boxed{\left.\begin{aligned} K_x &= \frac{1-n}{\rho_0^2} \\ K_y &= n/\rho_0^2 \\ h &= 1/\rho_0 \end{aligned}\right\} \text{define}}$$

← curvature

$$\boxed{\begin{aligned} x'' + K_x x &= h\delta \\ y'' + K_y y &= 0 \end{aligned}} \quad (10)$$

second-order linear ordinary differential equation = Hill
K_y - has to be periodic

b) Quadrupole magnet (easier, nominal-orbit → straight)

- Components of magnetic field

$B_x = gy$ (g - gradient of the field $g = \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}$)
$B_y = gx$

$$\left.\begin{aligned} \ddot{x} &= -\frac{q}{\gamma m} v_z g x \\ \ddot{y} &= +\frac{q}{\gamma m} v_z g y \end{aligned}\right\} \quad (11)$$

- approximations $v \simeq v_z$, $\gamma m v = p = P_0(1+\delta) \approx P_0$

$$\left.\begin{aligned} x'' + \frac{gq}{P_0}x &= 0 \\ y'' - \frac{gq}{P_0}y &= 0 \end{aligned}\right\} \quad (12)$$

$$\boxed{\left.\begin{aligned} K_x &= +\frac{gq}{P_0} \\ K_y &= -\frac{gq}{P_0} \end{aligned}\right\} \text{define}}$$

$$\boxed{\begin{aligned} x'' + K_x x &= 0 \\ y'' + K_y y &= 0 \end{aligned}} \quad (13)$$

3. Solutions of equations of motion

- for a reference particle, or monoenergetic particles

$$\boxed{\begin{aligned} x'' + K_x(s)\,x &= 0 \\ y'' + K_y(s)\,y &= 0 \end{aligned}}$$ (14)

- General solution

$$x(s) = x_0\,C_x(s) + x_0'\,S_x(s)$$ x_0, x_0', y_0, y_0' -

$$y(s) = y_0\,C_y(s) + y_0'\,S_y(s)$$ - start values

C_x, C_y (S_x, S_y) are cosine/sinus) - similar solutions

$C_x(0) = 1$ $S_x(0) = 0$

$C_x'(0) = 0$ $S_x'(0) = 1$ and the same for y, y'

- If particles are not monoenergetic ($\delta \neq 0$) → inhomogenious Eq

$$x'' + K_x(s)\,x = h(s)\,\delta \qquad \text{(dipole)}$$

Solution from homogenious differential Eq. plus solution of inhomogenious eq.

$$x(s) = x_0\,C_x(s) + x_0'\,S_x(s) + \delta\, d_x(s)$$

↑
- particular solution
- dispersion function

Particles are sorted dependent on the field index = Correlation of δ and x = dispersion

$$d_x(s) = \int_0^s h(\bar{s})\,G_x(s,\bar{s})\,d\bar{s}$$ - with green function

$$G_x(s,\bar{s}) = S_x(s)\,C_x(\bar{s}) - C_x(s)\,S_x(\bar{s})$$

with initial conditions $d_x(0) = 0$; $d_x'(0) = 0$

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$x_0 = 1\,\text{mm}$, $x_0' = 1\,\text{mrad}$, $\delta = 1$ promil

$y_0 = 1\,\text{mm}$, $y_0' = 1\,\text{mrad}$

- Characteristic solutions for transverse motion
  drift, quadrupole, dipole
  „Sharp cut-off" – no fringing fields, effective lengths
  $K_x, K_y = const$ within L_i $\Rightarrow$ (14) – harmonic oscillator

	$K_x(s)>0$ & $K_y(s)>0$	$K_x(s)=K_y(s)=0$	$K_x(s)<0$, $K_y(s)<0$
$C_x(s)$	$\cos(\sqrt{K_x}\, s)$	1	$\cosh(\sqrt{\lvert K_x\rvert}\, s)$
$S_x(s)$	$\frac{1}{\sqrt{K_x}}\sin(\sqrt{K_x}\, s)$	s	$\frac{1}{\sqrt{\lvert K_x\rvert}}\sinh(\sqrt{\lvert K_x\rvert}\, s)$
$d_x(s)$	$\frac{h}{K_x}\left[1-\cos(\sqrt{K_x}\, s)\right]$	0	$\frac{h}{\sqrt{\lvert K_x\rvert}}\left[-1+\cosh(\sqrt{\lvert K_x\rvert}\, s)\right]$
$C_y(s)$	$\cos(\sqrt{K_y}\, s)$	1	$\cosh(\sqrt{\lvert K_y\rvert}\, s)$
$S_y(s)$	$\frac{1}{\sqrt{K_y}}\sin(\sqrt{K_y}\, s)$	s	$\frac{1}{\sqrt{\lvert K_y\rvert}}\sinh(\sqrt{\lvert K_y\rvert}\, s)$
	oscillatory behavior		~~divergent~~

examples

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

quadrupole $\quad K_x = -K_y \mapsto$ foc/defoc

solenoid $\quad K_x = K_y \mapsto$ foc/foc or defoc/defoc

dipole $\quad K_x = \frac{1-n}{\rho_0^2} \qquad K_y = \frac{n}{\rho_0}$

$\qquad 0 < n < 1 \qquad$ foc/foc

- Characteristic solutions in the longitudinal direction
  at start: orbital deviation $\ell(0)=\ell_0$ at the end $\ell(s)=\ell$

$$\ell(s)-\ell(0) = v_0(t_0 - t) \Rightarrow$$

$$\mapsto -x_0\int_0^s h(\bar{s})C_x(\bar{s})\,d\bar{s} - x_0'\int_0^s h(\bar{s})S_x(\bar{s})\,d\bar{s} +$$

$$+\ell_0 - \delta\left(\int_0^s h(\bar{s})d_x(\bar{s})\,d\bar{s} - \frac{s}{\gamma^2}\right)$$

4. Transfermatrix

$\vec{x}(s) = R(s)\,\vec{x}(0)$ $\quad$ det $R = 1$ (Liouville's Theorem)

in linear approximation – we have decoupled planes

$$R = \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{pmatrix} x \\ x' \\ y \\ y' \\ \ell \\ \delta \end{pmatrix}$$

R_x, R_y

position dispersion

angle dispersion

independent description of dispersion then 3×3

$$R_x = \begin{pmatrix} R_{11} & R_{12} & R_{16} \\ R_{21} & R_{22} & R_{26} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} C_x & S_x & d_x \\ C_x' & S_x' & d_x' \\ 0 & 0 & 1 \end{pmatrix}$$

- accelerator structure $R = \prod R_i$

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a) drift

$$\begin{cases} x = x_0 + L x_0' \\ x' = x_0' \\ y = y_0 + L y_0' \\ y_1 = y_0' \\ \ell = \ell_0 + \left(\frac{L}{\gamma^2}\right)\delta_0 \\ \delta = \delta_0 \end{cases}$$

$$R = \begin{bmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

b) ~~di~~ quadrupole $\quad$ Slide

$$k = \frac{|g|}{(B\rho)_0} = \frac{|B_0|}{a}\frac{1}{(B\rho)_0}$$

field strength at the pole

radius aperture

$k_x = k$, $k_y = -k$ – foc

c) dipole (homogenious)

$$k_x = \frac{1}{\rho_0^2}, \quad k_y = 0, \quad h = \frac{1}{\rho_0}, \quad \alpha = \frac{L}{\rho_0}$$

bending angle

radial → foc.

axial → drift

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d) weak focusing dipole (cyclotron)

$0 \leq n < 1$ $\qquad$ $n = -\frac{\partial B_y}{\partial x} \frac{\rho_0}{B_0}$

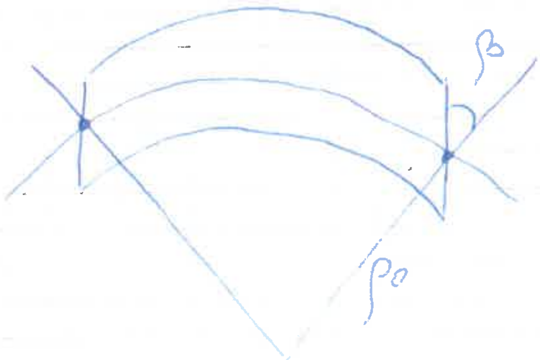
$K_x = \frac{1-n}{\rho_0^2}, \quad K_y = \frac{n}{\rho_0^2}, \quad h = \frac{1}{\rho_0}, \quad \alpha = \frac{L}{\rho_0}$ $\qquad$ both planes focusing

e) strongly focusing dipole (synchrotron)

$n > 0$ or $n > 1$ $\qquad$ $|n| \sim 20$

focus in one plane → defocus in the other

5. Edge focusing



$x = x_0, \quad x' = \frac{\tan\beta}{\rho_0} x_0 + x_0'$

$y = y_0, \quad y' = -\frac{\tan\beta}{\rho_0} y_0 + y_0'$

focusing in one plane
defocusing in the other