Introduction to Accelerator Physics

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Lecture Dates

https://uebungen.physik.uni-heidelberg.de/vorlesung/20222/1611/lecture

Date	Topic
19.10.2022	Introduction and basic definitions
26.10.2022	Accelerating structures
02.11.2022	Accelerator Components
09.11.2022	Optics with magnets (1)
16.11.2022	Optics with magnets (2)
23.11.2022	Equations of motion
30.11.2022	Phase ellipses and magneto-optical system / Transverse beam dynamics
07.12.2022	Transverse beam dynamics, beam stability / Longitudinal beam dynamics
14.12.2023	Phase space and beam cooling (Invitation)
11.01.2023	Space charge and beam-beam dynamics
18.01.2023	Physics at Storage Rings
25.01.2023	Physics at Colliders
01.02.2023	New accelerator technologies
08.02.2023	Student seminar
15.02.2023	reserve
22.02.2023	reserve

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

Wednesdays, 14:15-16:00

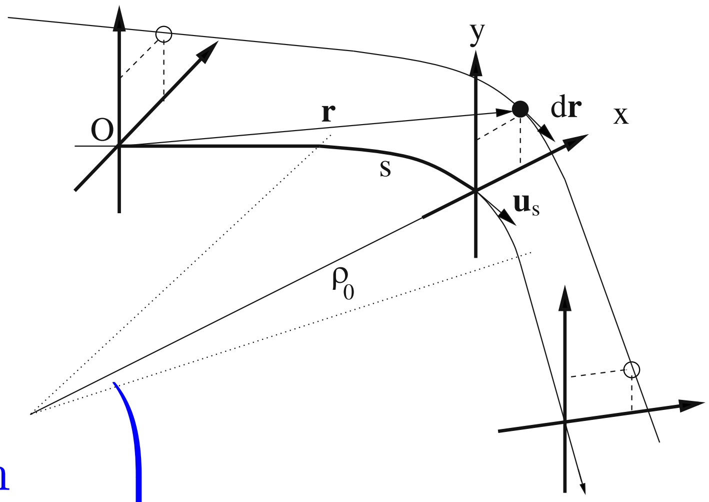
„Leistungskontrolle“

Accelerator Physics Related Applications

- *Particle cancer therapie*
- *Cosmic rays*
- *Accelerator Mass Spectrometry*
- *Accelerator Driven System*
- *Energy recovery accelerator*
- *Superheavy elements*
- *Strongest magnetic field*
- *Tokamak*
- *Photon facility*
- *Isotopes for medicine*
- *Crystalline beams*

Summary of last lecture

Curvilinear coordinate system



Relative coordinates of each particle can be described with a six-dimensional vector

$$\mathbf{x}(s) = \begin{pmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{pmatrix} = \begin{pmatrix} \text{radial orbit deviation} \\ \text{radial direction deviation} \\ \text{axial orbit deviation} \\ \text{axial direction deviation} \\ \text{longitudinal deviation} \\ \text{longitudinal momentum deviation} \end{pmatrix}$$

Since $x,\ x',\ y,\ y',\ l,\ \delta l$ are small $\Longrightarrow$ units are [mm], [mrad], [promil]

$$\boxed{1\ \text{mrad} = 1\ \text{mm}/1\ \text{m}}$$

Linear approximation: x and y planes can be treated independently

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3. Equation of Motion
(linear approximation)

- **Lorentz force**

$$\dot{\vec{p}} = q(\vec{v} \times \vec{B})$$

$$p = mv\gamma$$

$$\ddot{r} = \frac{q}{\gamma m}(\vec{v} \times \vec{B})$$

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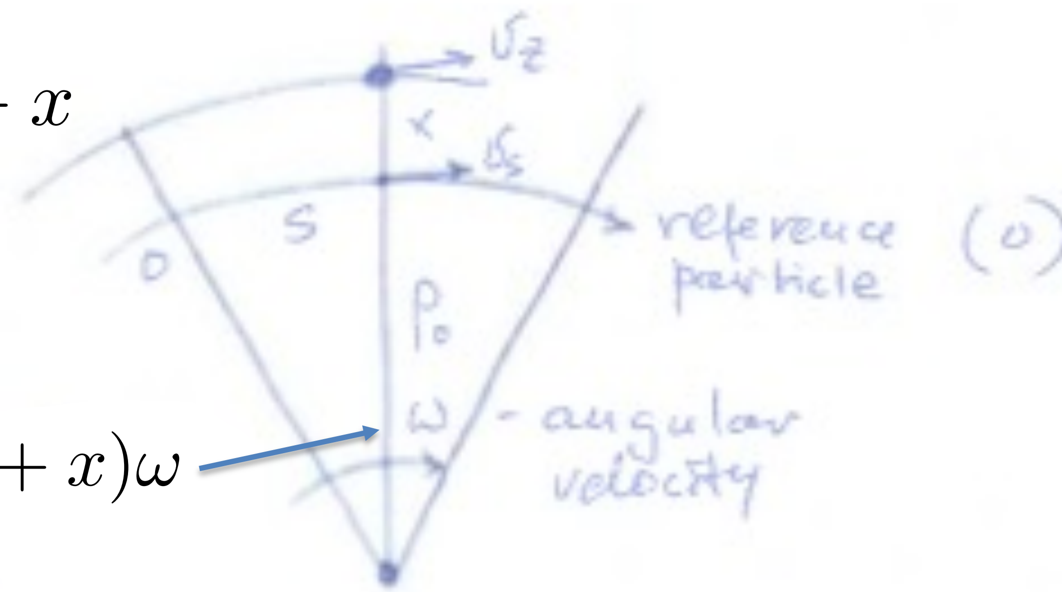
3. Equation of Motion
(linear approximation)

- Dipole magnet

- radius of the orbit $\rho = \rho_0 + x$



- velocity $v_z = (\rho_0 + x)\omega$

- radial acceleration $a_r = -\omega^2(\rho_0 + x)$

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3. Equation of Motion
(linear approximation)

- Dipole magnet

Lorentz force + centrifugal force

$$\ddot{r} = \ddot{x} + a_r = \ddot{x} - \omega^2(\rho_0 + x) =$$

$$= \frac{q}{\gamma m}(v_y B_z - v_z B_y) \quad (1)$$

$$\ddot{y} = \frac{q}{\gamma m}(v_z B_x - v_x B_z) \quad (2)$$

$$v_x,\ v_y << v_z;\ B_x,\ B_z << B_y \ \Rightarrow$$

$v_x B_z,\ v_y B_z -$ can be neglected

(we are still in linear approximation)

3. Equation of Motion
(linear approximation)

- Dipole magnet

$$v = v_z\sqrt{1 + x'^2 + y'^2} \approx v_z$$

$$p = \gamma m v \approx \gamma m v_z = \gamma m \rho_0 \omega \left(1 + \frac{x}{\rho_0}\right)$$

$$\frac{x}{\rho_0} << 1 \quad ([\mathrm{mm}] \text{ vs. } [\mathrm{m}])$$

3. Equation of Motion
(linear approximation)

- Dipole magnet

Replacing $\gamma m = p/v$ obtain for (1) and (2)

$$\ddot{x} - \omega^2(\rho_0 + x) = -\frac{q}{p} v_z^2 B_y \qquad (3)$$

$$\ddot{y} = +\frac{q}{p} v_z^2 B_x \qquad (4)$$

3. Equation of Motion
(linear approximation)

- Dipole magnet

Now, get rid of time $$\frac{ds}{dt} = \rho_0 \omega$$

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = \rho_0 \omega \frac{d}{ds}$$

$$\frac{d^2}{dt^2} = (\rho_0 \omega)^2 \frac{d^2}{ds^2}$$

3. Equation of Motion
(linear approximation)

- Dipole magnet

Divide (3) and (4) by $(\rho_0 \omega)^2$

$$x'' - \frac{1}{\rho_0}\left(1 + \frac{x}{\rho_0}\right) = -\frac{q}{p} B_y \left(1 + \frac{x}{\rho_0}\right)^2 \qquad (5)$$

$$y'' = +\frac{q}{p} B_x \left(1 + \frac{x}{\rho_0}\right)^2 \qquad (6)$$

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3. Equation of Motion
(linear approximation)

- Dipole magnet

Using the definition of field index

$$n \equiv \frac{-dB/B}{dr/r} = -\frac{r}{B}\frac{dB}{dr}$$

radius

$$n = -\frac{\partial B_y}{\partial dx}\frac{\rho_0}{B_0}$$

$$B_y = \int_{\rho_0}^{\rho_0+x} \frac{-nB_0}{\rho_0} dx = -\frac{nB_0}{\rho_0}\Bigg|_0^x + B_0$$

$$B_y = B_0\left(1 - \frac{nx}{\rho_0}\right)$$

$$B_x = B_0\left(0 - \frac{ny}{\rho_0}\right)$$

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3. Equation of Motion
(linear approximation)

- Dipole magnet

Now we can rewrite (5) and (6)

$$x'' = -B_0 \frac{q}{p}\left(1 - \frac{nx}{\rho_0}\right)\left(1 + \frac{1}{\rho_0}\right)^2 + \frac{1}{\rho_0}\left(1 + \frac{x}{\rho_0}\right) \qquad (7)$$

$$y'' = -B_0 \frac{q}{p}\frac{ny}{\rho_0}\left(1 + \frac{1}{\rho_0}\right)^2 \qquad (8)$$

3. Equation of Motion
(linear approximation)

- Dipole magnet

Relative momentum deviation $$\delta = \frac{p - p_0}{p_0}$$

$$\frac{1}{p} = \frac{1-\delta}{p_0}$$ where $$p_0 = qB_0\rho_0$$

$$x'' + \frac{1-n}{\rho_0^2}x = \frac{1}{\rho_0}\delta \qquad (9)$$

$$y'' + \frac{n}{\rho_0^2}y = 0 \qquad (10)$$

3. Equation of Motion
(linear approximation)

- Dipole magnet

Define

$$k_x = \frac{1-n}{\rho_0^2}$$

$$k_y = \frac{n}{\rho_0^2}$$

curvature

$$h = \frac{1}{\rho_0}$$

3. Equation of Motion

(linear approximation)

- Dipole magnet

$$x'' + k_x x = h\delta$$
$$y'' + k_y y = 0$$

Second-order linear ordinary differential equation

Hill equation, k_y has to be periodic

3. Equation of Motion
(linear approximation)

- Quadrupole magnet

3. Equation of Motion

(linear approximation)

- Quadrupole magnet

Easier, nominal trajectory is straight

Components of magnetic field

$$B_x = gy$$

$$B_y = gx$$

gradient

$$g = \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}$$

3. Equation of Motion

(linear approximation)

- **Quadrupole magnet**

$$\ddot{x} = -\frac{q}{\gamma m} v_z g x$$

$$\ddot{y} = +\frac{q}{\gamma m} v_z g y$$

3. Equation of Motion
(linear approximation)

- Quadrupole magnet

Approximation: $v_z \approx v, \;\; \gamma m = p = p_0(1+\delta) \approx p_0$

$$x'' + \frac{gq}{p_0} x = 0$$

$$y'' - \frac{gq}{p_0} y = 0$$

2. Equation of Motion

(linear optics)

- Quadrupole magnet

Define

$$k_x = +\frac{gq}{p_0}$$

$$k_y = -\frac{gp}{p_0}$$

3. Equation of Motion
(linear approximation)

- Quadrupole magnet

$$x'' + k_x x = 0$$
$$y'' + k_y y = 0$$

4. Solution of the Equation of Motion

(linear approximation)

- For a reference particle (or monoenergetic particles)

$$x'' + k_x(s)x = 0$$

$$y'' + k_y(y)y = 0$$

- General solution:

$$x(s) = x_0 C_x(s) + x'_0 S_x(s)$$

$$y(s) = y_0 C_y(s) + y'_0 S_y(s)$$

Where $x_0, \ x'_0, \ y_0, \ y'_0$ are start values

$C_x \ (S_x), \ C_y \ (S_y)$ - cosine (sine) – like solutions

$$C_x(0) = 1, \ S_x(0) = 0$$

$$C'_x(0) = 0, \ S'_x(0) = 1$$ And similar for *y* and *y'*

4. Solution of the Equation of Motion
(linear approximation)

- If particles are NOT monoenergetic $(\delta \neq 0) \Longrightarrow$ Inhomogenious equation

$$x'' + k_x(s)x = h(s)\delta$$ (dipole magnet)

Solution: solution of homogenious differential equation plus a particular solution of inhomogenious equation

$$x(s) = x_0 C_x(s) + x'_0 S_x(s) + \delta d_x(s)$$

Dispersion function

Particles are sorted dependent on field index
Correlation of δ and x - dispersion

4. Solution of the Equation of Motion
(linear approximation)

$$d_x(s) = \int_0^s h(\bar{s}) G_x(s, \bar{s}) d\bar{s}$$

With Green function

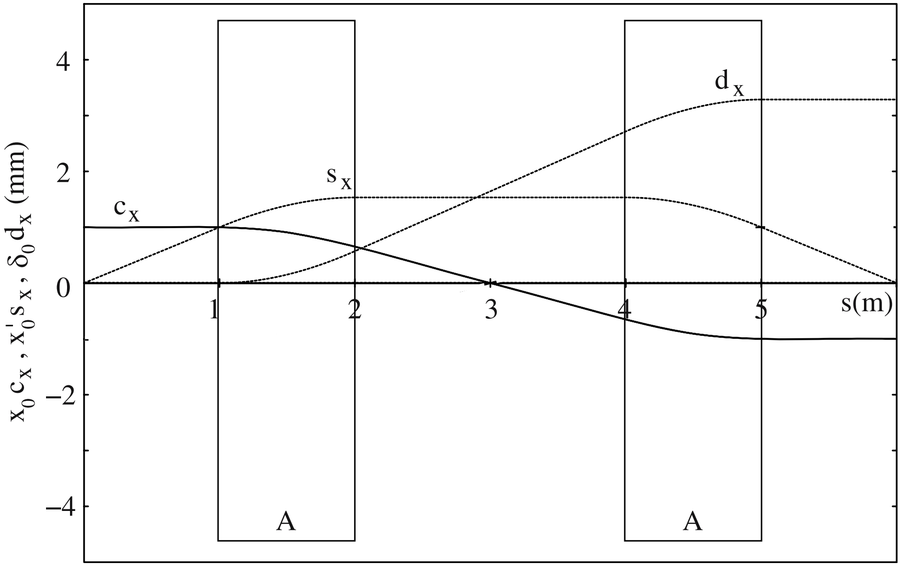
$$G_x(s, \bar{s}) = S_x(s) C_x(\bar{s}) - C_x(s) S_x(\bar{s})$$

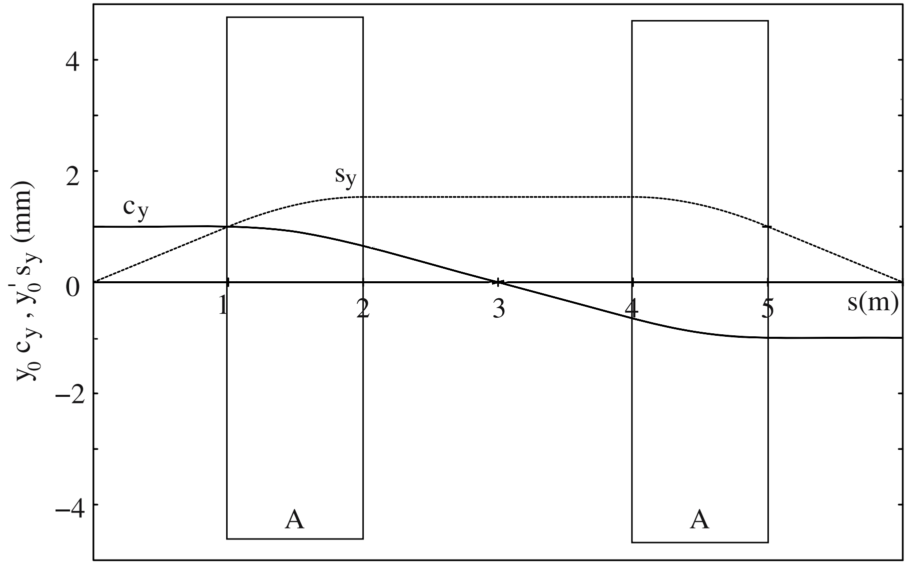
And initial conditions:

$$d_x(0) = 0, \;\; d'_x(0) = 0$$

4. Solution of the Equation of Motion
(linear approximation)

$$x(s) = x_0 C_x(s) + x'_0 S_x(s) \qquad y(s) = y_0 C_y(s) + y'_0 S_y(s)$$





drift dipole drift dipole drift

drift dipole drift dipole drift

$$x_0 = 1 \text{ mm},\ x'_0 = 1 \text{ mrad},\ \delta = 1 \text{ promil}$$

$$y_0 = 1 \text{ mm},\ y'_0 = 1 \text{ mrad}$$

$$n = 0.5 \text{ field index}$$

$$L_{\text{drift}} = 1 \text{ m}$$

$$L_{\text{dipole}} = 1 \text{ m}$$

$$\rho_0 = 0.8219 \text{ m}$$

$$\alpha = 69.71^\circ$$

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4. Solution of the Equation of Motion
(linear approximation)

Characteristic solutions for transverse motion (drift, quadrupole, dipole)

„Sharp cut-off“ (no fringing fields, effective lengths)
k_x, k_y = *const* within *L*

$$x'' + k_x x = 0$$
$$y'' + k_y y = 0$$

$\Rightarrow$ **Harmonic oscillator**

4. Solution of the Equation of Motion
(linear approximation)

Characteristic solutions for transverse motion (drift, quadrupole, dipole)

$k_x(s) > 0 \;\&\; k_y(s) > 0$	$k_x(s) = k_y(s) = 0$	$k_x(s) < 0 \;\&\; k_y(s) < 0$
$C_x(s) = \cos(\sqrt{k_x} s)$	$C_x(s) = 0$	$C_x(s) = \cosh(\sqrt{\lvert k_x\rvert} s)$
$S_x(s) = \frac{\sin(\sqrt{k_x} s)}{\sqrt{k_x}}$	$S_x(s) = s$	$S_x(s) = \frac{\sinh(\sqrt{\lvert k_x\rvert} s)}{\sqrt{\lvert k_x\rvert}}$
$d_x(s) = \frac{h}{k_x}[1 - \cos(\sqrt{k_x} s)]$	$d_x(s) = 0$	$d_x(s) = \frac{h}{\lvert k_x\rvert}[\cosh(\sqrt{\lvert k_x\rvert} s) - 1]$
$C_y(s) = \cos(\sqrt{k_y} s)$	$C_y(s) = 1$	$C_y(s) = \cosh(\sqrt{\lvert k_y\rvert} s)$
$S_y(s) = \frac{\sin(\sqrt{k_y} s)}{\sqrt{k_y}}$	$S_y(s) = s$	$S_y(s) = \frac{\sinh(\sqrt{\lvert k_y\rvert} s)}{\sqrt{\lvert k_y\rvert}}$

periodic

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

4. Solution of the Equation of Motion
(linear approximation)

Characteristic solutions for longitudinal motion

Orbital deviation at start: $l(0) = l_0$ At the end: $l(s) = l$

$$l(s) - l(0) = v_0(t_0 - t) \Rightarrow$$

$$\Rightarrow \quad l(s) = -x_0 \int_0^s h(\bar{s}) C_x(\bar{s}) d\bar{s} -$$

$$-x_0' \int_0^s h(\bar{s}) Sx(\bar{s}) d\bar{s} +$$

$$+l_0 - \delta \left(\int_0^s h(\bar{s} d_x(\bar{s}) d\bar{s} - \frac{s}{\gamma^2} \right)$$

2. Transfer matrix

Transfer matrix / transport matrix / R-matrix

$$\vec{x}(s) = \mathbf{R}(s)\vec{x}(0)$$

$$\mathbf{x}(s) = \begin{pmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{pmatrix} = \begin{pmatrix} \text{radial orbit deviation} \\ \text{radial direction deviation} \\ \text{axial orbit deviation} \\ \text{axial direction deviation} \\ \text{longitudinal deviation} \\ \text{longitudinal momentum deviation} \end{pmatrix}$$

$$det(\mathbf{R}) = 1$$ (Liouville‘s theorem)

2. Transfer matrix

(linear approximation)

In linear approximation we have decoupled planes *x* and *y*

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Transfer matrix

(linear approximation)

Units

$$\mathbf{R} = \begin{bmatrix} 1 & \mathrm{m} & 0 & 0 & 0 & \mathrm{m} \\ \mathrm{m}^{-1} & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \mathrm{m} & 0 & 0 \\ 0 & 0 & \mathrm{m}^{-1} & 1 & 0 & 0 \\ 1 & \mathrm{m} & 0 & 0 & 1 & \mathrm{m} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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2. Transfer matrix

(linear approximation)

In linear approximation we have decoupled planes *x* and *y*

Position dispersion

Angle dispersion

$$\mathbf{R} = \begin{bmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & R_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Independent description

2. Transfer matrix

(linear approximation)

In linear approximation we have decoupled planes *x* and *y*

Independent description with dispersion - (3x3) matrices:

$$\mathbf{R_x} = \begin{bmatrix} R_{11} & R_{12} & R_{16} \\ R_{21} & R_{22} & R_{26} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_x & S_x & d_x \\ C'_x & S'_x & d'_x \\ 0 & 0 & 1 \end{bmatrix}$$

For y-coordinate in an analogous way

2. Transfer matrix

(linear approximation)

Often the representation is used indicating the dependence of elements on each other

$$\mathbf{R_x} = \begin{bmatrix} (x|x) & (x|x') & 0 & 0 & 0 & (x|\delta) \\ (x'|x) & (x'|x') & 0 & 0 & 0 & (x'|\delta) \\ 0 & 0 & (y|y) & (y|y') & 0 & 0 \\ 0 & 0 & (y'|y) & (y'|y') & 0 & 0 \\ (l|x) & (l|x') & 0 & 0 & (l|l) & (l|\delta) \\ 0 & 0 & 0 & 0 & 0 & (\delta|\delta) \end{bmatrix}$$

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

4. Solution of the Equation of Motion
(linear approximation)

Characteristic solutions for longitudinal motion

The corresponding Transfer matrix elements:

$$R_{51}(s) = (l|x_0) = -\int_0^s h(\bar{s}C_x(\bar{s})d\bar{s}$$

$$R_{52}(s) = (l|x_0') = -\int_0^s h(\bar{s}S_x(\bar{s})d\bar{s}$$

$$R_{55}(s) = (l|l_0) = 1$$

$$R_{56}(s) = (l|\delta) = -\int_0^s h(\bar{s}d_x(\bar{s})d\bar{s} + s/\gamma^2$$

For drifts, quadrupoles, (sextupoles, octupoles) *h=0*

$$R_{51}(s) = (l|x_0) = 0$$

$$R_{52}(s) = (l|x_0') = 0$$

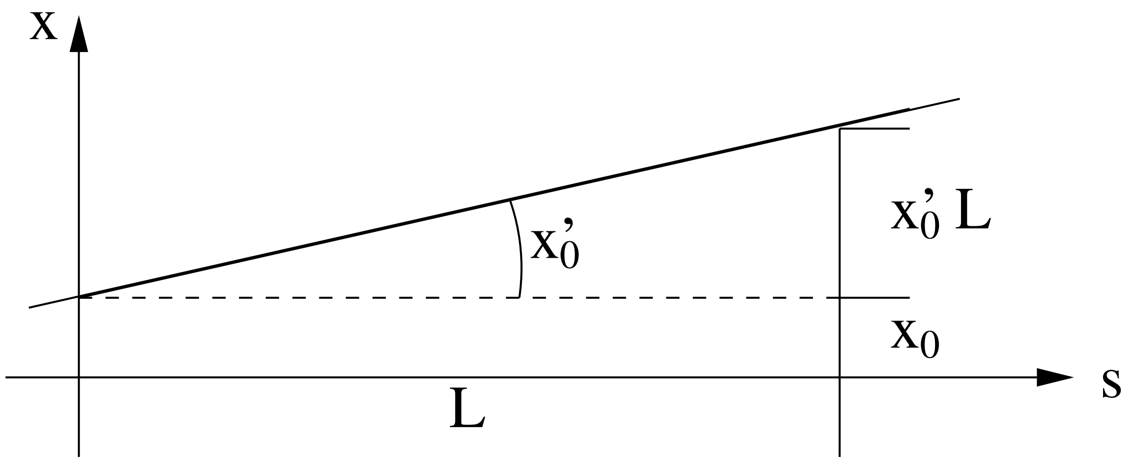
$$R_{55}(s) = (l|l_0) = 1$$

$$R_{56}(s) = (l|\delta) = +s/\gamma^2$$

4. Transfer matrix

- Drift

$$x = x_0 + Lx'_0$$
$$x' = x'_0$$
$$y = y_0 + Ly'_0$$
$$y' = y'_0$$
$$l = l_0 + \delta_0 \frac{L}{\gamma^2}$$
$$\delta = \delta_0$$



$$\mathbf{R}_{\text{drift}} = \begin{bmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Transfer matrices

- Quadrupole

$$k_x = +\frac{gq}{p_0}$$
$$k_y = -\frac{gp}{p_0}$$

Magnetic rigidity $B\rho = \frac{p}{q}$

Momentum (p)

Charge (q)

$$k = \frac{|g|}{(B\rho)_0} = \frac{B_0}{a}\frac{1}{(B\rho)_0}$$

Field at the pole (B_0)

Aperture (a)

Radially focusing and axially defocusing quadrupole: $k_x = k \text{ and } k_y = -k$

4. Transfer matrix

Radially focusing and axially de-focusing quadrupole: $k_x = k \text{ and } k_y = -k$

$$\mathbf{R} = \begin{bmatrix} \cos(\sqrt{k}L) & \frac{\sin(\sqrt{k}L)}{\sqrt{k}} & 0 & 0 & 0 & 0 \\ -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(\sqrt{k}L) & \frac{\sinh(\sqrt{k}L)}{\sqrt{k}} & 0 & 0 \\ 0 & 0 & \sqrt{k}\sinh(\sqrt{k}L) & \cosh(\sqrt{k}L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Radially de-focusing and axially focusing quadrupole: $k_x = -k \text{ and } k_y = k$

$$\mathbf{R} = \begin{bmatrix} \cosh(\sqrt{k}L) & \frac{\sinh(\sqrt{k}L)}{\sqrt{k}} & 0 & 0 & 0 & 0 \\ \sqrt{k}\sinh(\sqrt{k}L) & \cosh(\sqrt{k}L) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\sqrt{k}L) & \frac{\sin(\sqrt{k}L)}{\sqrt{k}} & 0 & 0 \\ 0 & 0 & -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Transfer matrix

Radially focusing and axially de-focusing quadrupole: $k_x = k \text{ and } k_y = -k$

$$\mathbf{R} = \begin{bmatrix} \cos(\sqrt{k}L) & \frac{\sin(\sqrt{k}L)}{\sqrt{k}} & 0 & 0 & 0 & 0 \\ -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(\sqrt{k}L) & \frac{\sinh(\sqrt{k}L)}{\sqrt{k}} & 0 & 0 \\ 0 & 0 & \sqrt{k}\sinh(\sqrt{k}L) & \cosh(\sqrt{k}L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

divergent

Radially de-focusing and axially focusing quadrupole: $k_x = -k \text{ and } k_y = k$

$$\mathbf{R} = \begin{bmatrix} \cosh(\sqrt{k}L) & \frac{\sinh(\sqrt{k}L)}{\sqrt{k}} & 0 & 0 & 0 & 0 \\ \sqrt{k}\sinh(\sqrt{k}L) & \cosh(\sqrt{k}L) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\sqrt{k}L) & \frac{\sin(\sqrt{k}L)}{\sqrt{k}} & 0 & 0 \\ 0 & 0 & -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

divergent

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4. Transfer matrix

- Homogeneous dipole (n=0), sektor magnet

$$k_x = \frac{1-n}{\rho_0^2}$$
$$k_y = \frac{n}{\rho_0^2}$$

Radius of reference orbit	ρ_0	Major parameters
Bending angle	α	
Effective length	L	

$$k_x = \frac{1}{\rho_0^2}, \; k_y = 0, \; h = \frac{1}{\rho_0}, \; \alpha = \frac{L}{\rho_0}$$

Axial direction: drift length $L = \rho_0 \alpha$

Transverse direction: particles with $\delta = \Delta p / p_0$ travel on radius ρ deviating from ρ_0

$\Longrightarrow$ Position (***R*₁₆**) and angle dispersion (***R*₂₆**)

4. Transfer matrix

- Homogeneous dipole (n=0), sector magnet

dispersion

drift

$$\mathbf{R} = \begin{bmatrix} \cos(\alpha) & \rho_0 \sin(\alpha) & 0 & 0 & 0 & \rho_0(1-\cos(\alpha)) \\ \frac{\sin(\alpha)}{\rho_0} & \cos(\alpha) & 0 & 0 & 0 & \sin(\alpha) \\ 0 & 0 & 1 & \rho_0\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin(\alpha) & -\rho_0(1-\cos(\alpha)) & 0 & 0 & 1 & \rho_0 \frac{\alpha}{\gamma^2} - \rho_0(\alpha - \sin(\alpha)) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Orbit (particle traveling on the inner or outter trajectory relative to the reference orbit) and velocity (velocity spread) effects

4. Transfer matrix

- Weak-focusing magnet (0<n<1)

Radius of reference orbit ρ_0
Bending angle α
Effective length L

} Major parameters

$$k_x = \frac{1-n}{\rho_0^2}$$
$$k_y = \frac{n}{\rho_0^2}$$

$$k_x = \frac{1-n}{\rho_0^2},\ k_y = \frac{n}{\rho_0^2},\ h = \frac{1}{\rho_0},\ \alpha = \frac{L}{\rho_0}$$

$$\mathbf{R_x} = \begin{bmatrix} \cos(\sqrt{1-n}\alpha) & \frac{\rho_0 \sin(\sqrt{1-n}\alpha)}{\sqrt{1-n}} & \frac{\rho_0(1-\cos(\sqrt{1-n}\alpha))}{1-n} \\ -\frac{\sqrt{1-n}\sin(\sqrt{1-n}\alpha)}{\rho_0} & \cos(\sqrt{1-n}\alpha) & \frac{\sin(\sqrt{1-n}\alpha)}{\sqrt{1-n}} \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{R_y} = \begin{bmatrix} \cos(\sqrt{n}\alpha) & \frac{\rho_0 \sin(\sqrt{n}\alpha)}{\sqrt{n}} \\ -\frac{\sqrt{n}\sin(\sqrt{n}\alpha)}{\rho_0} & \cos(\sqrt{n}\alpha) \end{bmatrix}$$

For $\boldsymbol{R_{51}}$, $\boldsymbol{R_{52}}$ and $\boldsymbol{R_{56}}$ characteristic solutions for the longitudinal motion shall be used

4. Transfer matrix

- Strong-focusing magnet (|n|>>1)

$$k_x = \frac{1-n}{\rho_0^2}$$
$$k_y = \frac{n}{\rho_0^2}$$

Radius of reference orbit	ρ_0	Major parameters
Bending angle	α	
Effective length	L	

$$k_x = \frac{1-n}{\rho_0^2}, \; k_y = \frac{n}{\rho_0^2}, \; h = \frac{1}{\rho_0}, \; \alpha = \frac{L}{\rho_0}$$

For $\boldsymbol{R_{51}}$, $\boldsymbol{R_{52}}$ and $\boldsymbol{R_{56}}$ characteristic solutions for the longitudinal motion shall be used

4. Transfer matrix

- Strong-focusing magnet (n>1)

$$\mathbf{R_x} = \begin{bmatrix} \cosh(\sqrt{|1-n|}\alpha) & \frac{\rho_0 \sinh(\sqrt{|1-n|}\alpha)}{\sqrt{|1-n|}} & \frac{\rho_0 (1-\cosh(\sqrt{|1-n|}\alpha)}{1-n} \\ -\frac{\sqrt{|1-n|}\sinh(\sqrt{|1-n|}\alpha)}{\rho_0} & \cosh(\sqrt{|1-n|}\alpha) & \frac{\sinh(\sqrt{|1-n|}\alpha)}{\sqrt{|1-n|}} \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{R_y} = \begin{bmatrix} \cos(\sqrt{n}\alpha) & \frac{\rho_0 \sin(\sqrt{n}\alpha)}{\sqrt{n}} \\ -\frac{\sqrt{n}\sin(\sqrt{n}\alpha)}{\rho_0} & \cos(\sqrt{n}\alpha) \end{bmatrix}$$

de-focusing   focusing

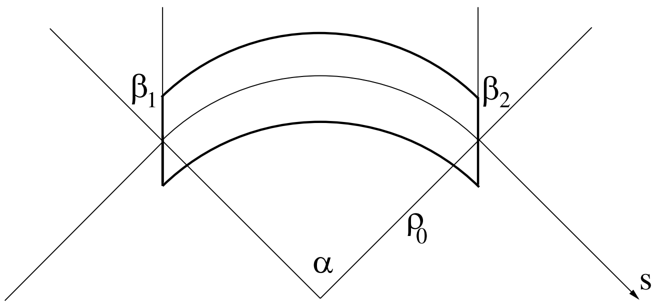
- Strong-focusing magnet (n<1)

$$\mathbf{R_x} = \begin{bmatrix} \cos(\sqrt{1-n}\alpha) & \frac{\rho_0 \sin(\sqrt{1-n}\alpha)}{\sqrt{1-n}} & \frac{\rho_0 (1-\cos(\sqrt{1-n}\alpha)}{1-n} \\ -\frac{\sqrt{1-n}\sin(\sqrt{1-n}\alpha)}{\rho_0} & \cos(\sqrt{1-n}\alpha) & \frac{\sin(\sqrt{1-n}\alpha)}{\sqrt{1-n}} \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{R_y} = \begin{bmatrix} \cosh(\sqrt{|n|}\alpha) & \frac{\rho_0 \sinh(\sqrt{|n|}\alpha)}{\sqrt{|n|}} \\ -\frac{\sqrt{|n|}\sinh(\sqrt{|n|}\alpha)}{\rho_0} & \cosh(\sqrt{|n|}\alpha) \end{bmatrix}$$

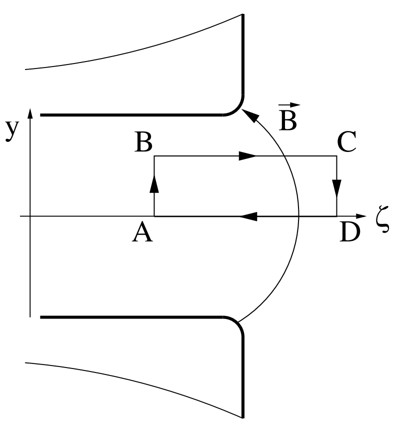
focusing   de-focusing

5. Edge focusing

Transversal direction



Axial direction



A particle on the orbit inside (outside) of the reference orbit spends more (less) time in magnetic field.
It receives negative (positive) angle $\Delta x'$ relative to the reference particle.

Pathlength difference

$$\Delta x' = \frac{\Delta z}{\rho_0} = \frac{\tan(\beta)}{\rho_0} x_0$$

$$\Delta y' = -\frac{\tan(\beta_{\text{eff}})}{\rho_0} y_0$$

$$\beta_{\text{eff}} \approx \beta$$

β_{eff} takes into account the effect of the extended fringing-field leading to a slightly smaller angle change Δy'

5. Edge focusing

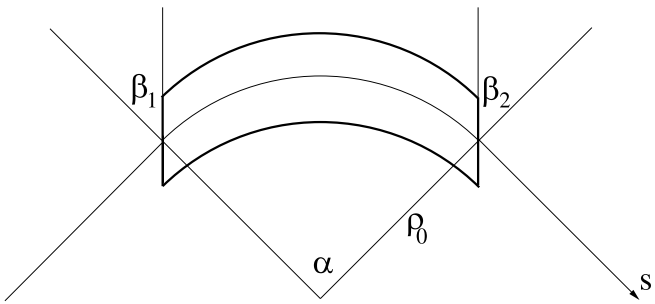
de-focusing

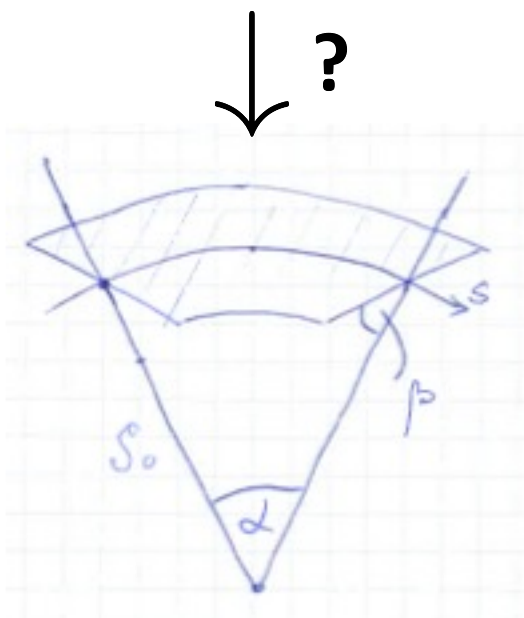
focusing

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\tan(\beta)}{\rho_0} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\tan(\beta_{\text{eff}})}{\rho_0} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5. Edge focusing

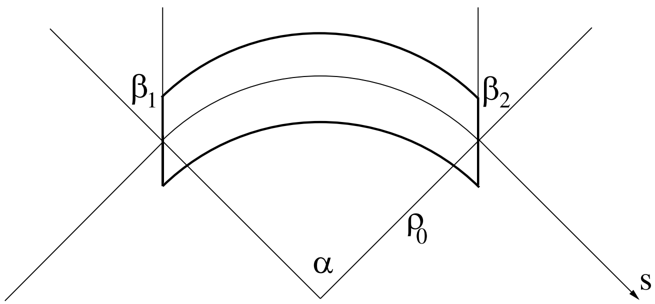
Transversal direction

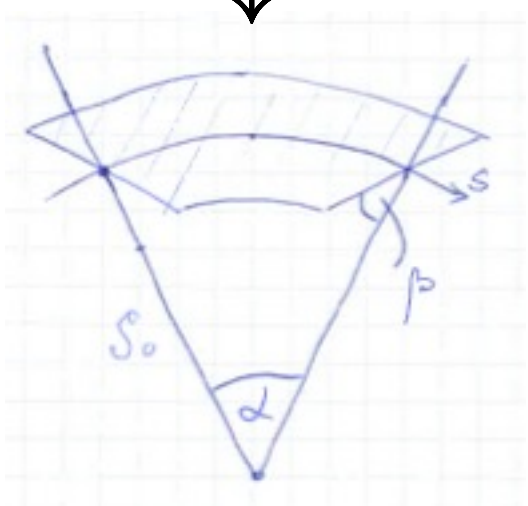


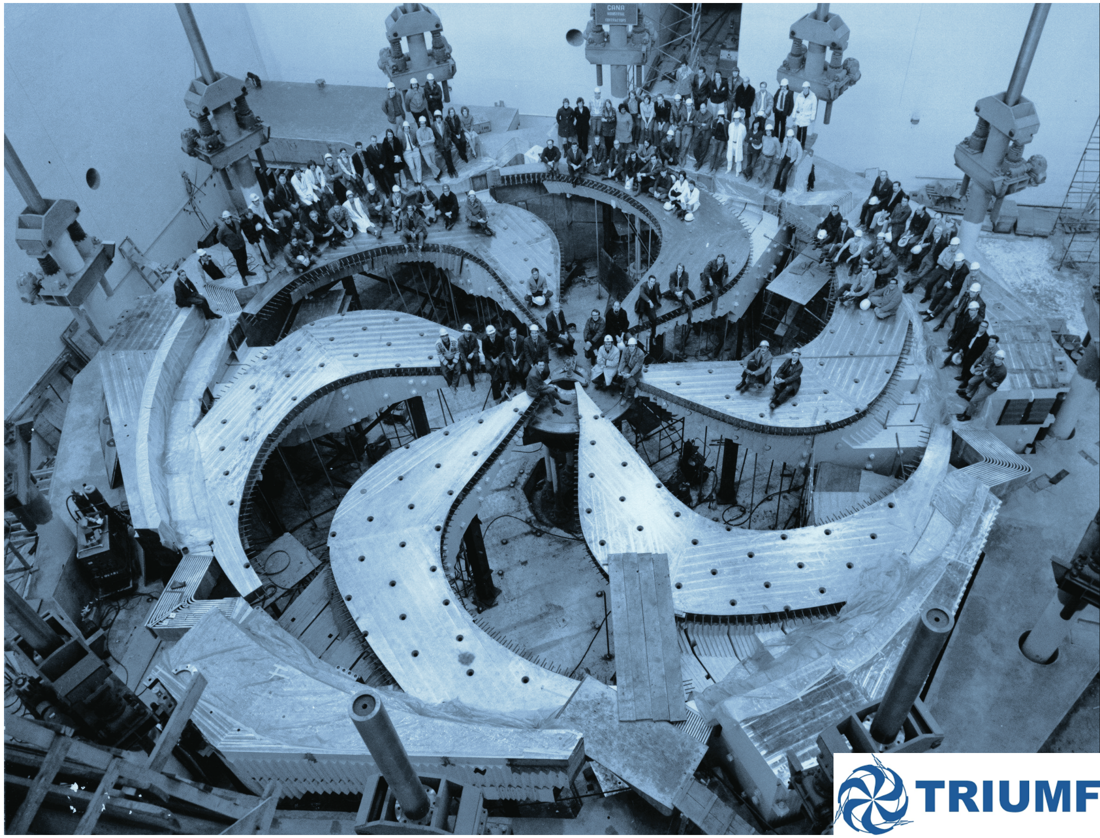


5. Edge focusing

Transversal direction







Summary of the lecture

Curvilinear coordinate system

Transfer / transport / R-matrix

Equation of motion with/without dispersion (Hill equations)

$$x'' + k_x x = h\delta$$
$$y'' + k_y y = 0$$

$$x'' + k_x x = 0$$
$$y'' + k_y y = 0$$

Solution of equation of motion with/without dispersion

Transfer Matrix for

- Drift
- Qudrupole magnet
- Dipole magnet (sector, weak and strong focusing)

Edge focusing