Introduction to Accelerator Physics

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Physikalisches Institut der Universität Heidelberg

Lecture Dates

https://uebungen.physik.uni-heidelberg.de/vorlesung/20222/1611/lecture

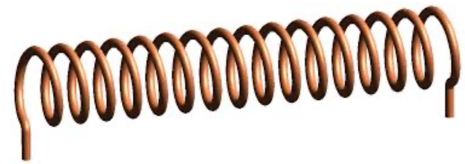
Date	Topic
19.10.2022	Introduction and basic definitions
26.10.2022	Accelerating structures
02.11.2022	Accelerator Components
09.11.2022	Optics with magnets (1)
16.11.2022	Optics with magnets (2)
23.11.2022	Equations of motion
30.11.2022	Phase ellipses and magneto-optical system / Transverse beam dynamics
07.12.2022	Transverse beam dynamics, beam stability / Longitudinal beam dynamics
14.12.2023	Longitudinal beam dynamics and a summary
11.01.2023	Phase space and beam cooling (Invitation)
18.01.2023	Space charge and beam-beam dynamics
25.01.2023	Physics at Storage Rings and Colliders
01.02.2023	New accelerator technologies and final summary
08.02.2023	Student seminar
15.02.2023	Visit GSI
22.02.2023	reserve

Wednesdays, 14:15-16:00

Summary of the lecture

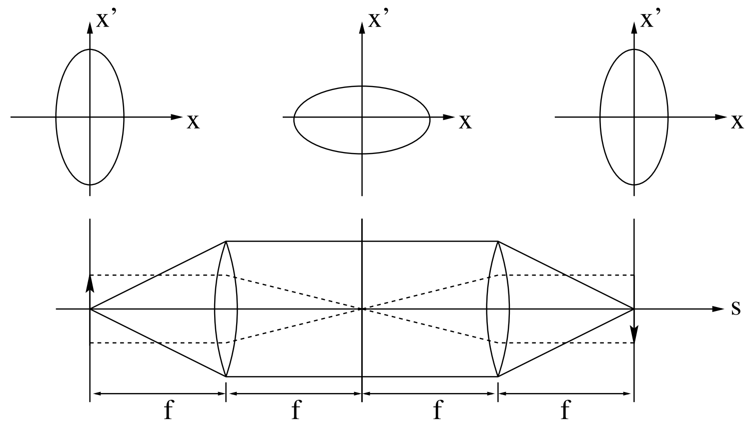
Beam rotations

tilted magnets / solenoid



Ion-optical systems

- Single Quadrupole
- Quadrupole doublet
- Quadrupole triplet
- Telescopic images
- Monochromator
- Achromatic systems
- -- Velocity filter --



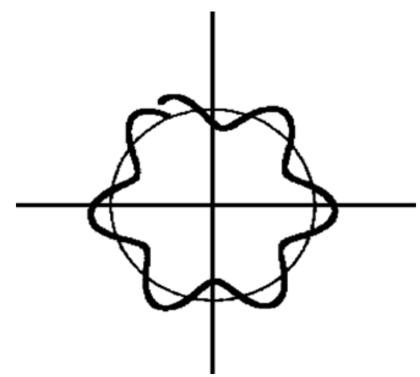
Transverse beam dynamics

- Hill's equations
- Twiss matrix
- Twiss parameters
- Betatron functions
- Phase advance
- Machine tunes

$$\beta(s)$$

$$\alpha(s)$$

$$\gamma(s)$$



11. Transverse Beam Dynamics

- Twiss parameters

Since Matrix M depends on the starting values, Twiss parameters are functions of s

$$\left.\begin{array}{l}\beta(s)\\ \alpha(s)\\ \gamma(s)\end{array}\right\}$$

Optical functions,
Betatron functions,
Amplitude functions,
Lattice functions

Goal: describe machine!

μ – independent of *s*, machine parameter defined by matrix *M* to 2π

Phase advance of $\beta(s)$ per revolution

11. Transverse Beam Dynamics

11.3 Solution of Hill's equations

$$y'' + k_y(s)y = 0$$

(6.24) in Hinterberger

$$y(s) = a\sqrt{\beta(s)}\cos[\Psi(s) + \Psi_0] \qquad (2)$$

$a, \ \Psi_0$ are defined for each particle, which are the amplitude and the phase of oscillations, respectively

$a\sqrt{\beta(s)}$ Variable amplitude along $\vec{s}$

$$\frac{d\Psi}{ds} = \frac{1}{\beta(s)}$$

Variable wave number ($\lambda(s) = 2\pi\beta(s)$ - wavelength)

11. Transverse Beam Dynamics

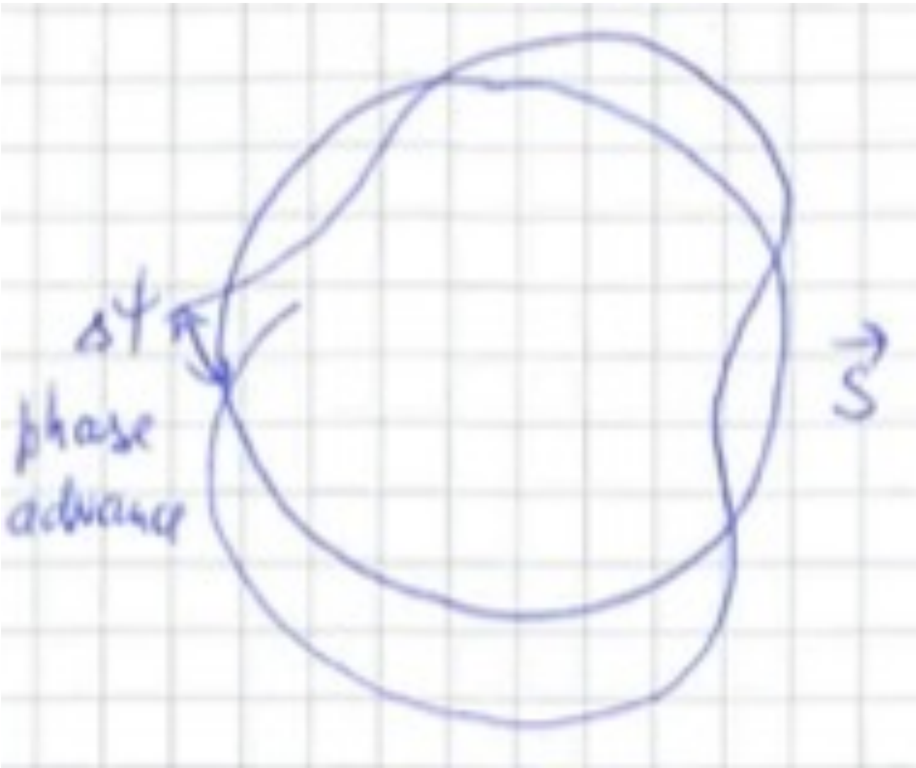
11.4 Phase shift / phase advance

$$\mu = \int_{s}^{s+C} \frac{d\bar{s}}{\beta(\bar{s})} = \oint \frac{d\bar{s}}{\beta(\bar{s})}$$

Number of betatron oscillations per revolution, betatron tune

$$Q = \frac{\mu}{2\pi} = \frac{1}{2\pi} \oint \frac{d\bar{s}}{\beta(\bar{s})}$$

11.4 Phase shift / phase advance



If $\Delta\Psi = 0$

Particle moves always on the same orbit

!!! RESONANCE !!!

Disturbances will be multiplied

!!! Instability !!!

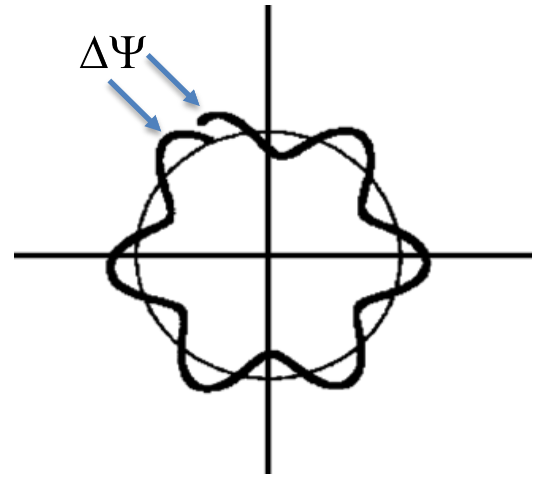
11. Transverse Beam Dynamics

11.4 Phase shift / phase advance

μ – independent of *s*, machine parameter defined by matrix *M* to 2π

Phase advance of $\beta(s)$ per revolution

$$\mu = \int_s^{s+C} \frac{d\bar{s}}{\beta(\bar{s})} = \oint \frac{d\bar{s}}{\beta(\bar{s})}$$



Number of betatron oscillations per revolution, betatron tune

$$Q = \frac{\mu}{2\pi} = \frac{1}{2\pi} \oint \frac{d\bar{s}}{\beta(\bar{s})}$$

If $\Delta\Psi = 0$

Particle moves always on the same orbit

!!! RESONANCE !!!

Disturbances will be multiplied

!!! Instability !!!

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

11.5 Courant-Snyder Invariant

$$\frac{y^2}{\beta} + \frac{(\alpha y + \beta y')^2}{\beta} = a^2 = \epsilon$$

$$\gamma y^2 + 2\alpha y y' + \beta y'^2 = a^2 = \epsilon$$

11.5 Courant-Snyder Invariant

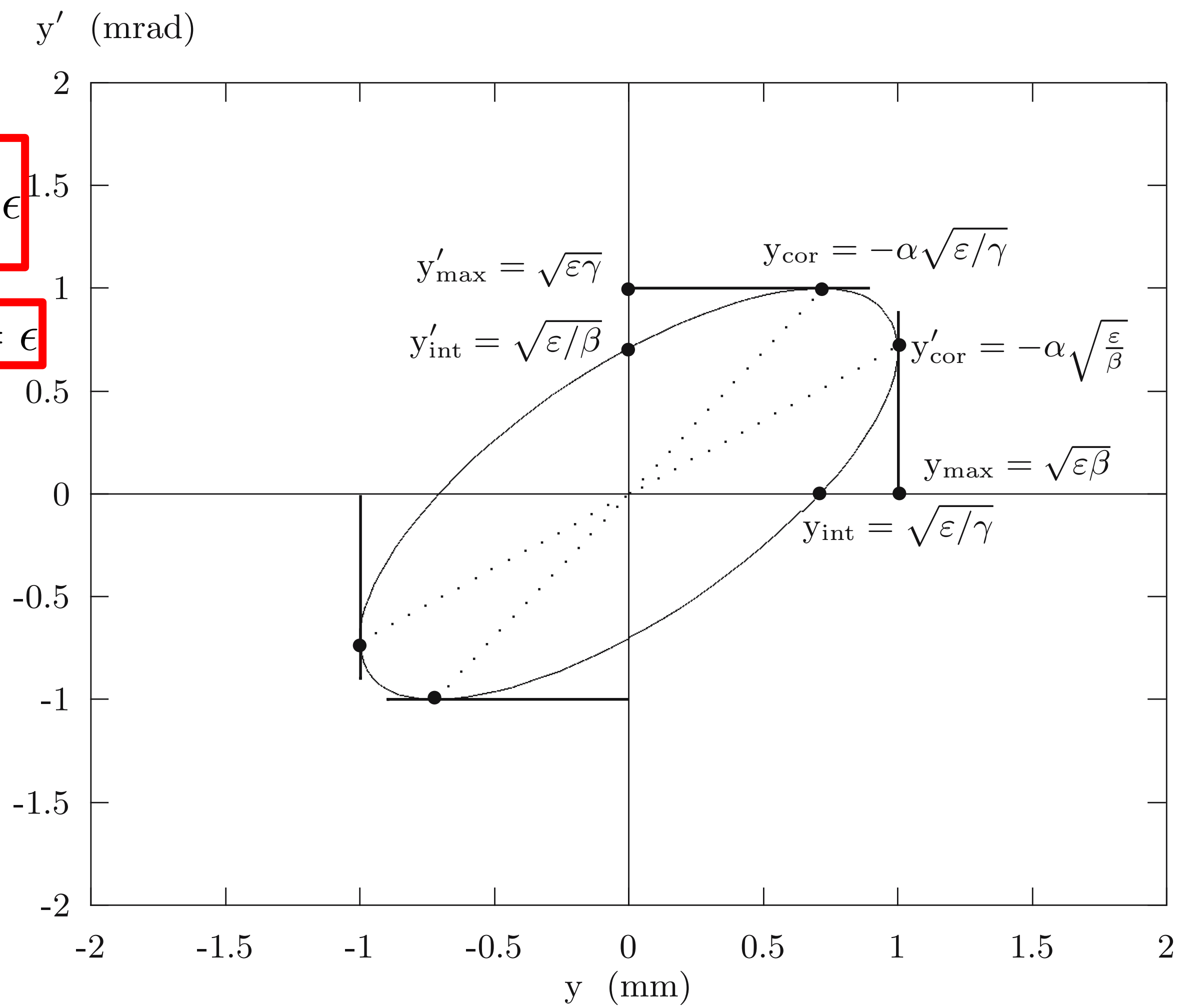
$$\frac{y^2}{\beta} + \frac{(\alpha y + \beta y')^2}{\beta} = a^2 = \epsilon$$

$$\gamma y^2 + 2\alpha y y' + \beta y'^2 = a^2 = \epsilon$$

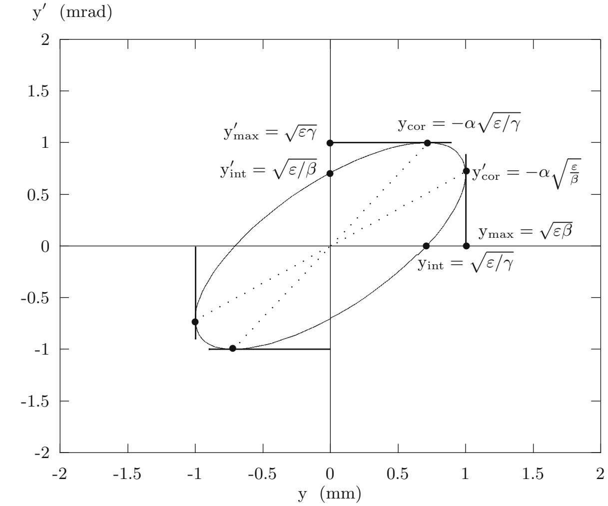
Machine Ellipse

Area

$$E = \pi a^2 = \pi \epsilon = const$$



11. Transverse Beam Dynamics



11.5 Courant-Snyder Invariant

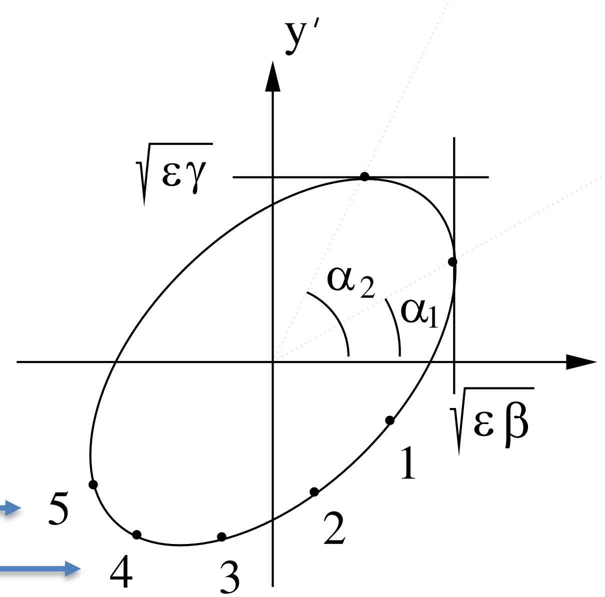
1. A particle with coordinates *(y,y')* propagates along a changing ellipse

2. The area of the ellipse is constant and is defined by ***a***

3. The shape of the ellipse is defined by the machine itself via $\alpha(s)$, $\beta(s)$, $\gamma(s)$ functions (machine ellipse)

4. Plotting *(y,y')* after each revolution gives an ellipse

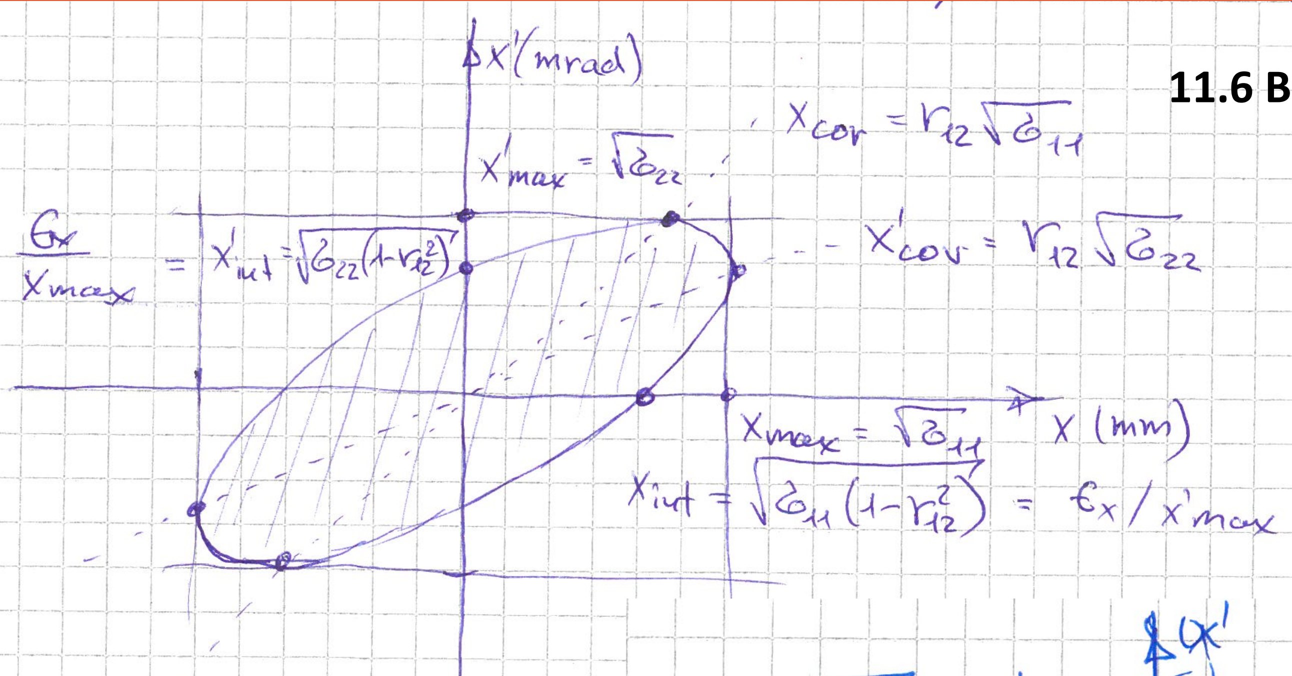
5. All particles with smaller ***a*** are enclosed in the ellipse

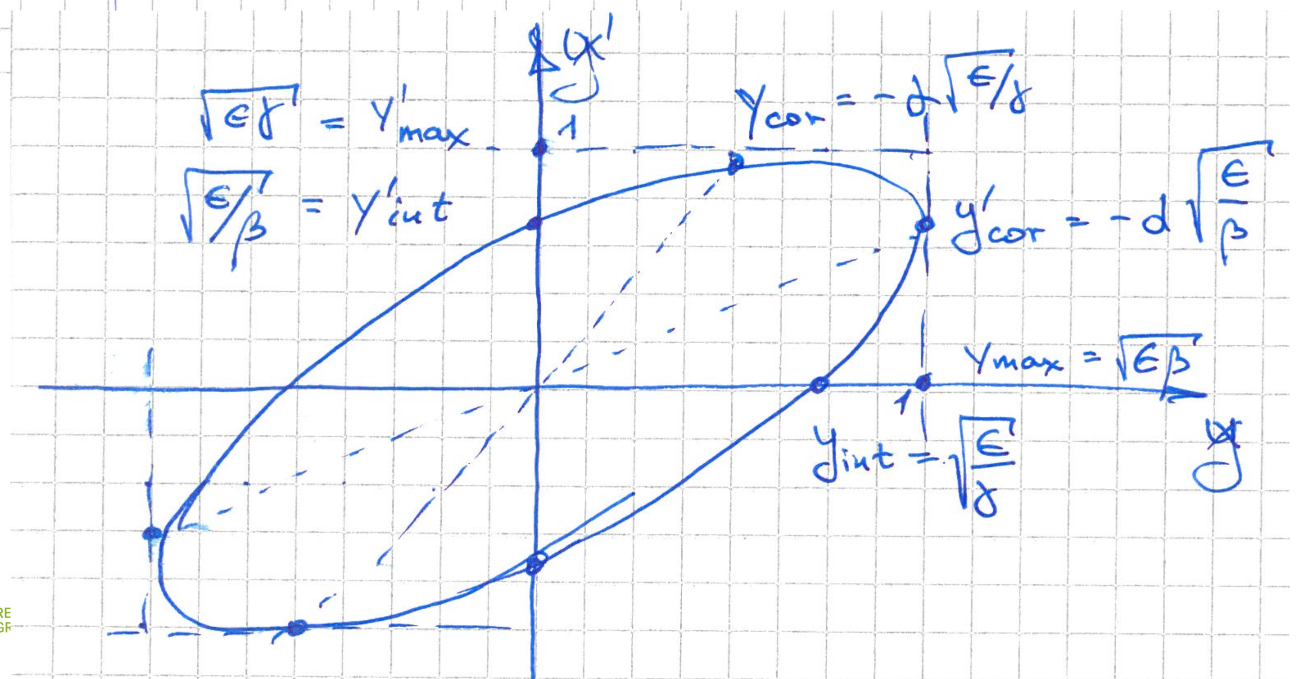


Revolution number

11. Transverse Beam Dynamics

11.6 Beam ellipse & machine ellipse





11. Transverse Beam Dynamics

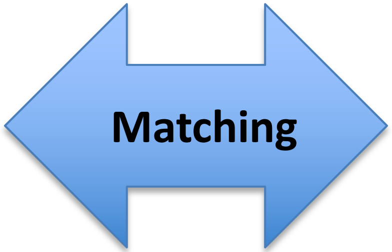
11.6 Beam ellipse & machine ellipse

Machine ellipse

Defined by the machine (lattice, ion-optical settings, apertures)

Beam ellipse

Can be very different from machine ellipse (e.g. injection)



Phase ellipse

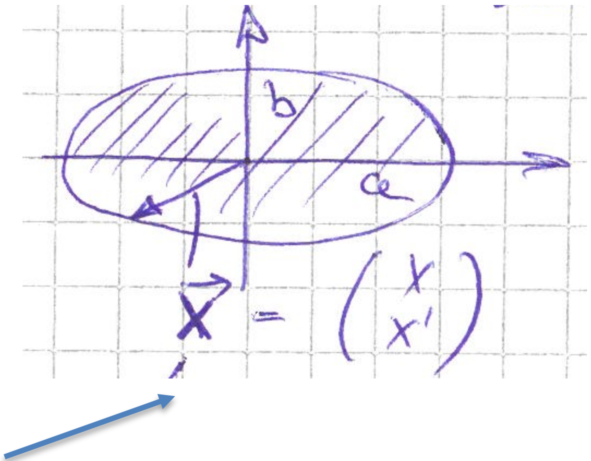
Density distribution in *(x,x')* plane $\rho(x, x')$ **can typically presented with an ellipse**

$$\sigma_x = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

$$\sigma_{12} = \sigma_{21}$$
$$\det(\sigma_x) > 0$$

Phase ellipse:

$$\mathbf{X^T} \sigma_x^{-1} \mathbf{X} = 1 \quad (1)$$



Vector from origin to ellipse boundary

$$\sigma_x^{-1} = \frac{1}{\det(\sigma_x)} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}$$

$$\frac{x^2}{a^2} + \frac{x'^2}{b^2} = 1$$

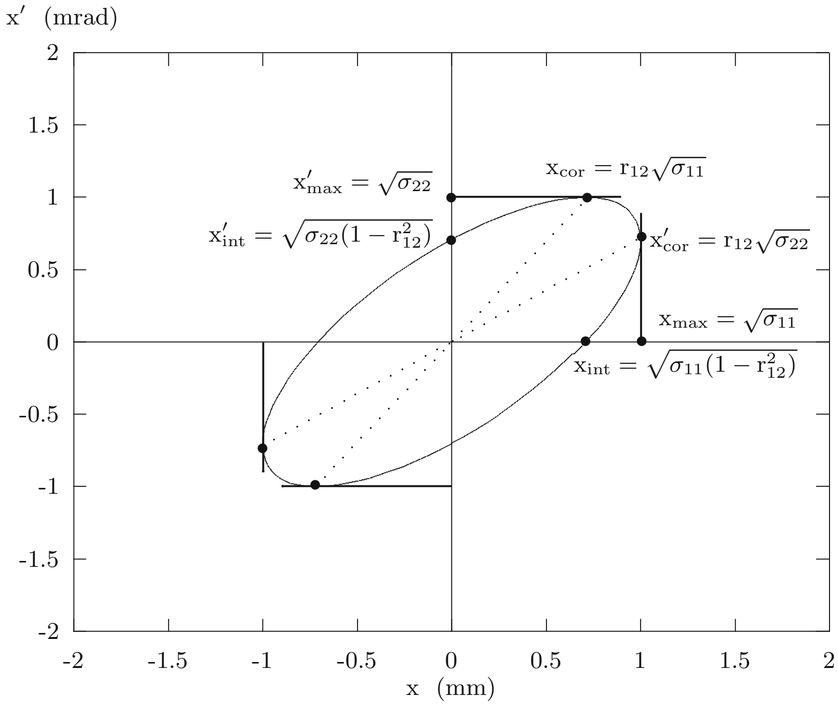
11. Transverse Beam Dynamics

11.6 Beam ellipse & machine ellipse

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \epsilon_x \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} = \begin{pmatrix} \epsilon_x\beta_x & -\epsilon_x\alpha_x \\ -\epsilon_x\alpha_x & \epsilon_x\gamma_x \end{pmatrix}$$

$\sqrt{\epsilon_x \beta_x}$ Maximal spatial extension

$\sqrt{\epsilon_x \gamma_x}$ Maximal angular extension



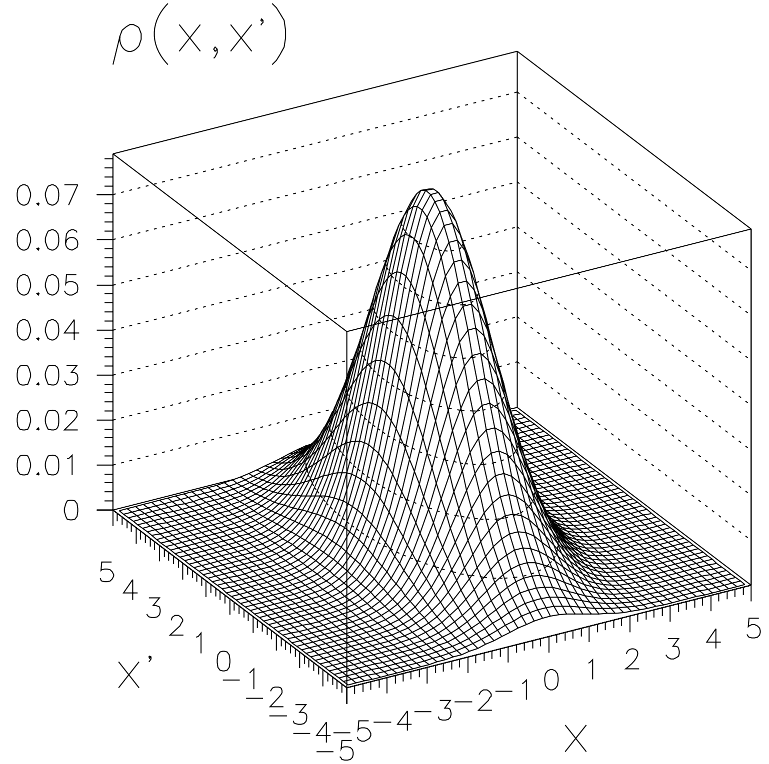
11. Transverse Beam Dynamics

11.7 RMS Emittance

Definition:

$$\epsilon_x^{1\sigma} = \sqrt{\langle x^2 \rangle \, \langle x'^2 \rangle - \langle x x' \rangle^2}$$

$$\epsilon_y^{1\sigma} = \sqrt{\langle y^2 \rangle \, \langle y'^2 \rangle - \langle y y' \rangle^2}$$



Expressed in Twiss parameteres

$$\epsilon_x^{1\sigma} = \frac{1}{N} \sum_i \epsilon_{x,i} = \frac{1}{N} \sum_i \gamma_x x_i^2 + 2\alpha_x x_i x_i' + \beta_x {x_i}'^2$$

$$\epsilon_y^{1\sigma} = \frac{1}{N} \sum_i \epsilon_{y,i} = \frac{1}{N} \sum_i \gamma_y y_i^2 + 2\alpha_y y_i y_i' + \beta_y {y_i}'^2$$

Machine parameters

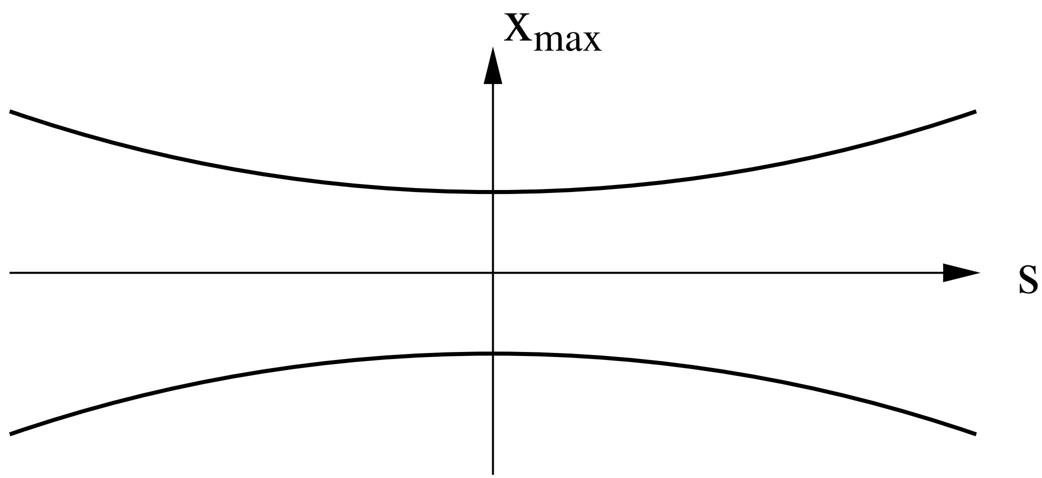
11.8 Beam envelope (RMS envelope)

$$x_{\max}(s) = \sqrt{\sigma_{11}(s)}$$

$$x_{\max}(s) = \sqrt{\epsilon}\sqrt{\beta(s)}$$

$$x'_{\max}(s) = \sqrt{\epsilon}\sqrt{\gamma(s)}$$

Beam parameter (ϵ) — Machine parameter (β, γ)



Beam waist („Strahltaille“) / focus

$r_{12} < 0$ $\qquad$ $r_{12} > 0$

11.9 Machine Acceptance

Maximum beam emittance which can be transmitted through the machine

$$\epsilon_{\text{max}} = \frac{x_{\text{max}}^2}{\beta}$$

Acceptance/Admittance $A = \pi \epsilon_{\text{max}}$

11. Transverse Beam Dynamics

11.8 Machine ellipse

Ellipses defined by α, β, γ functions are nothing else than eigenellipses σ_e of matrix M at easch s!

Gordon,M.M.: Orbit properties of the isochronous cyclotron ring with radial sectors, Annals of Physics **3** (1968) 571

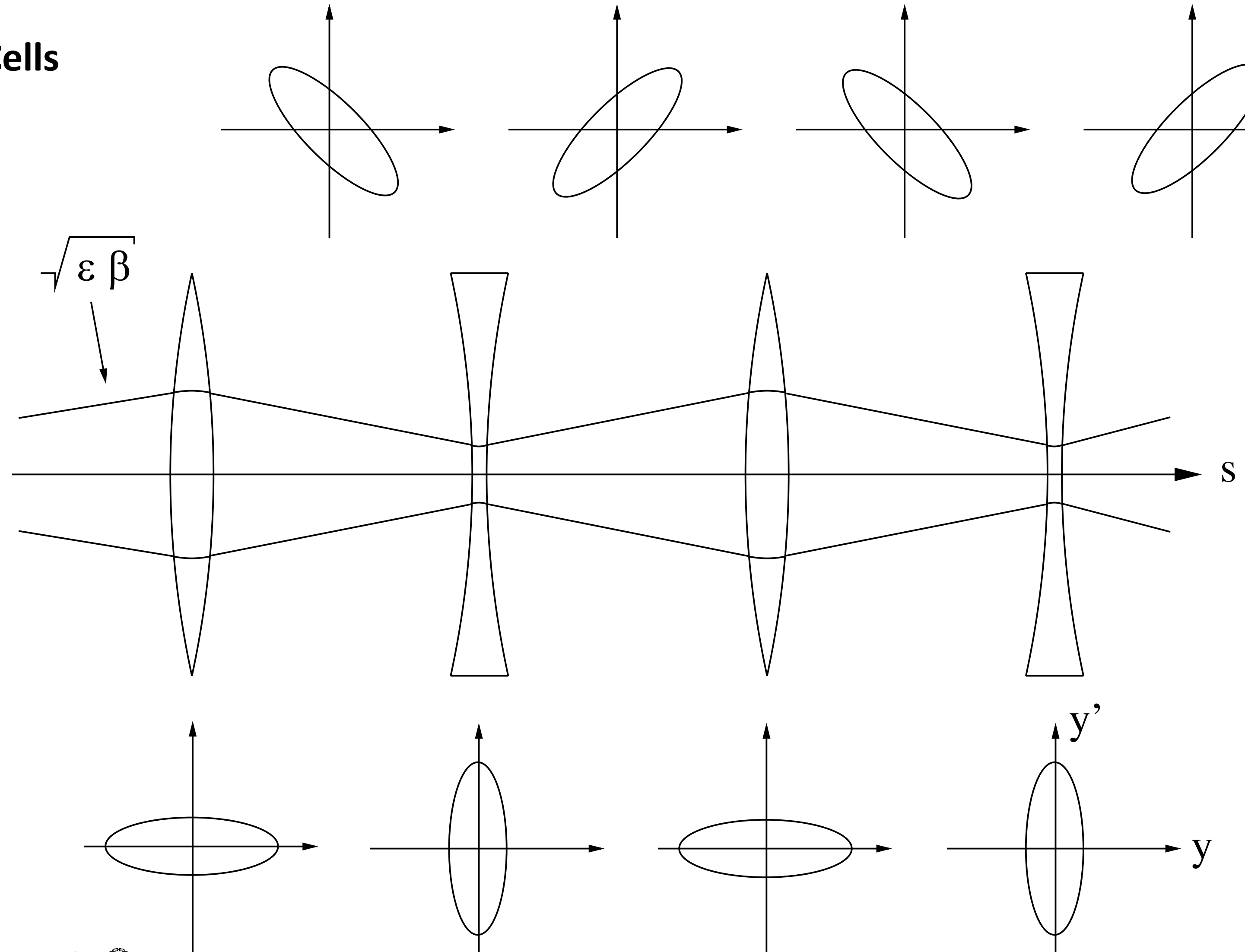
$$\sigma_e = M \sigma_e M^T$$

$$\sigma_e(s + C) = \sigma_e(s)$$

Eigenellipses are defined at each s via Twiss matrix

$$\sigma_\mathrm{e}(s) = \epsilon \begin{pmatrix} \beta(s) & -\alpha(s) \\ -\alpha(s) & \gamma(s) \end{pmatrix}$$

FODO Cells



11. Transverse Beam Dynamics

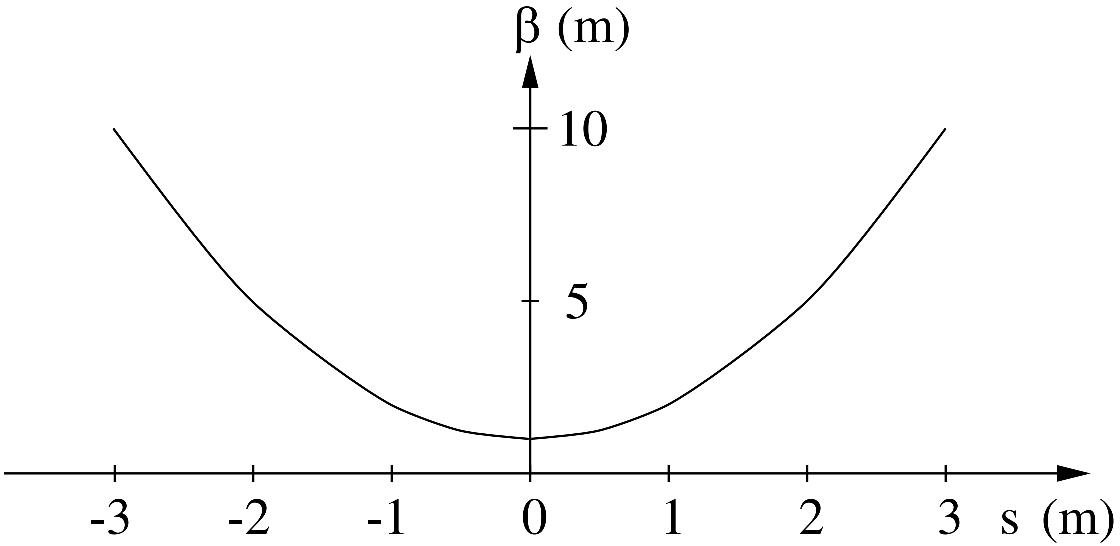
11.8 Transformation of Twiss parameters

Example: Drift

Beam waist

$$\alpha_0 = 0$$

$$\gamma_0 = 1/\beta_0$$

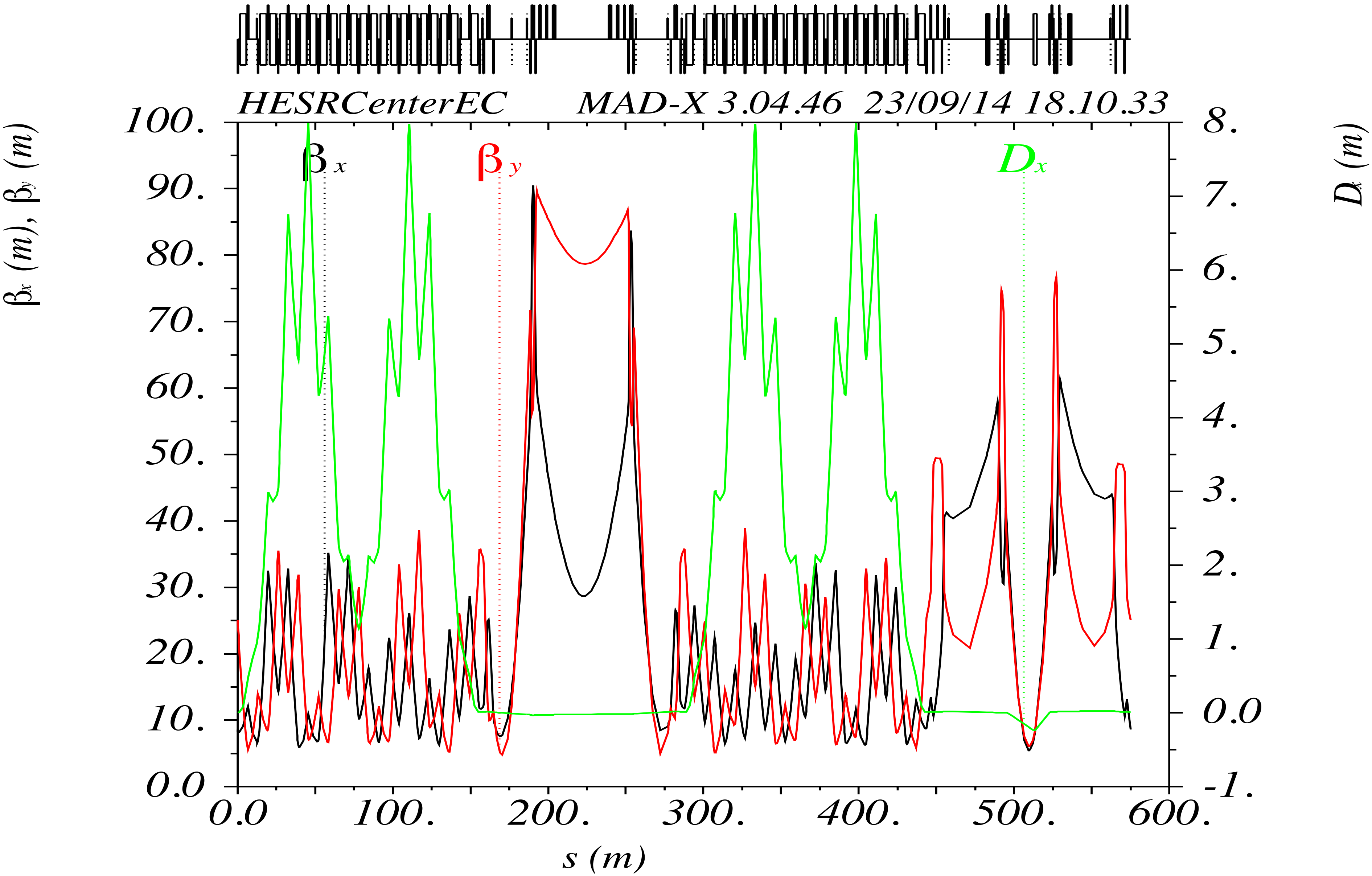


$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

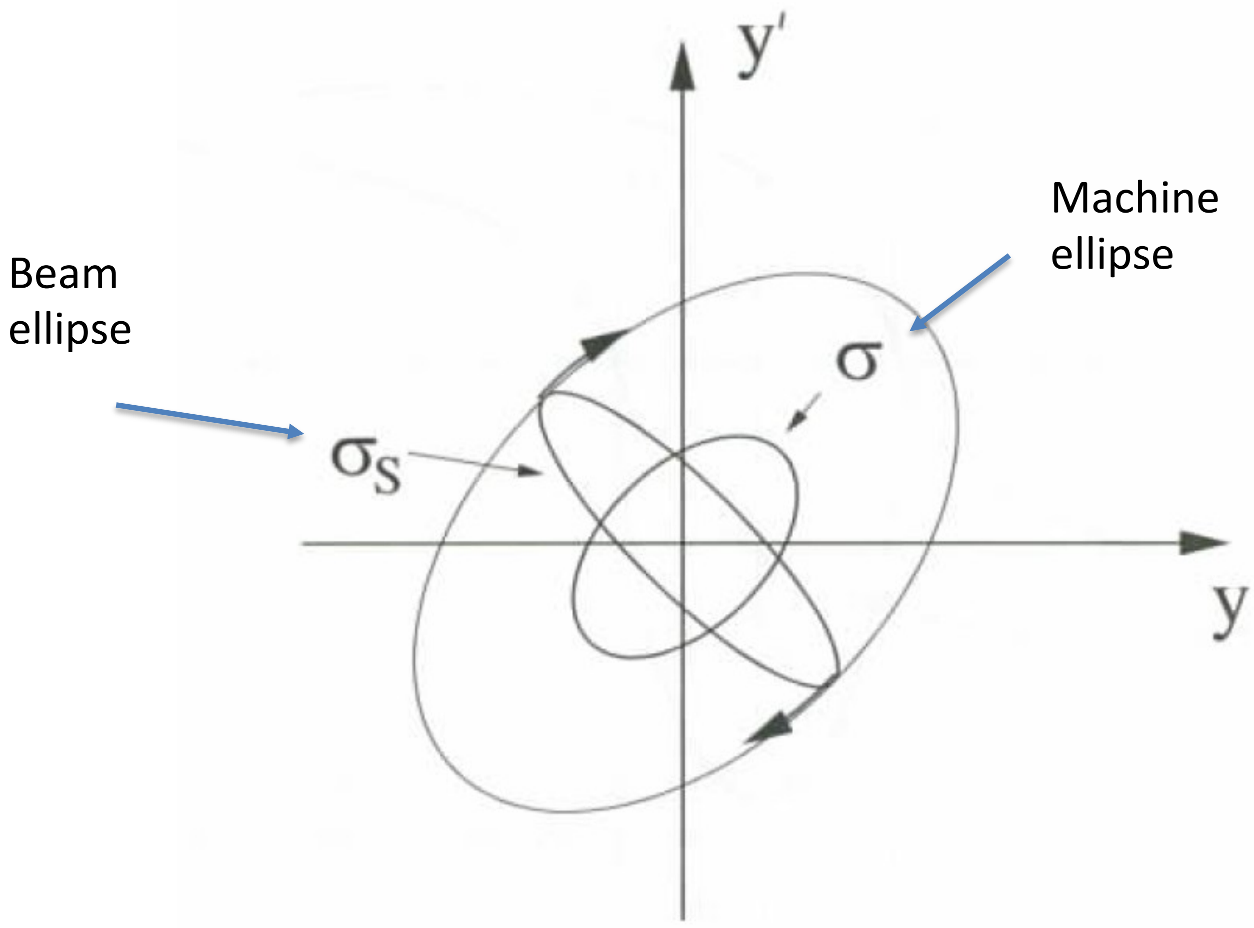
In general:

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

Typical example (HESR at FAIR)

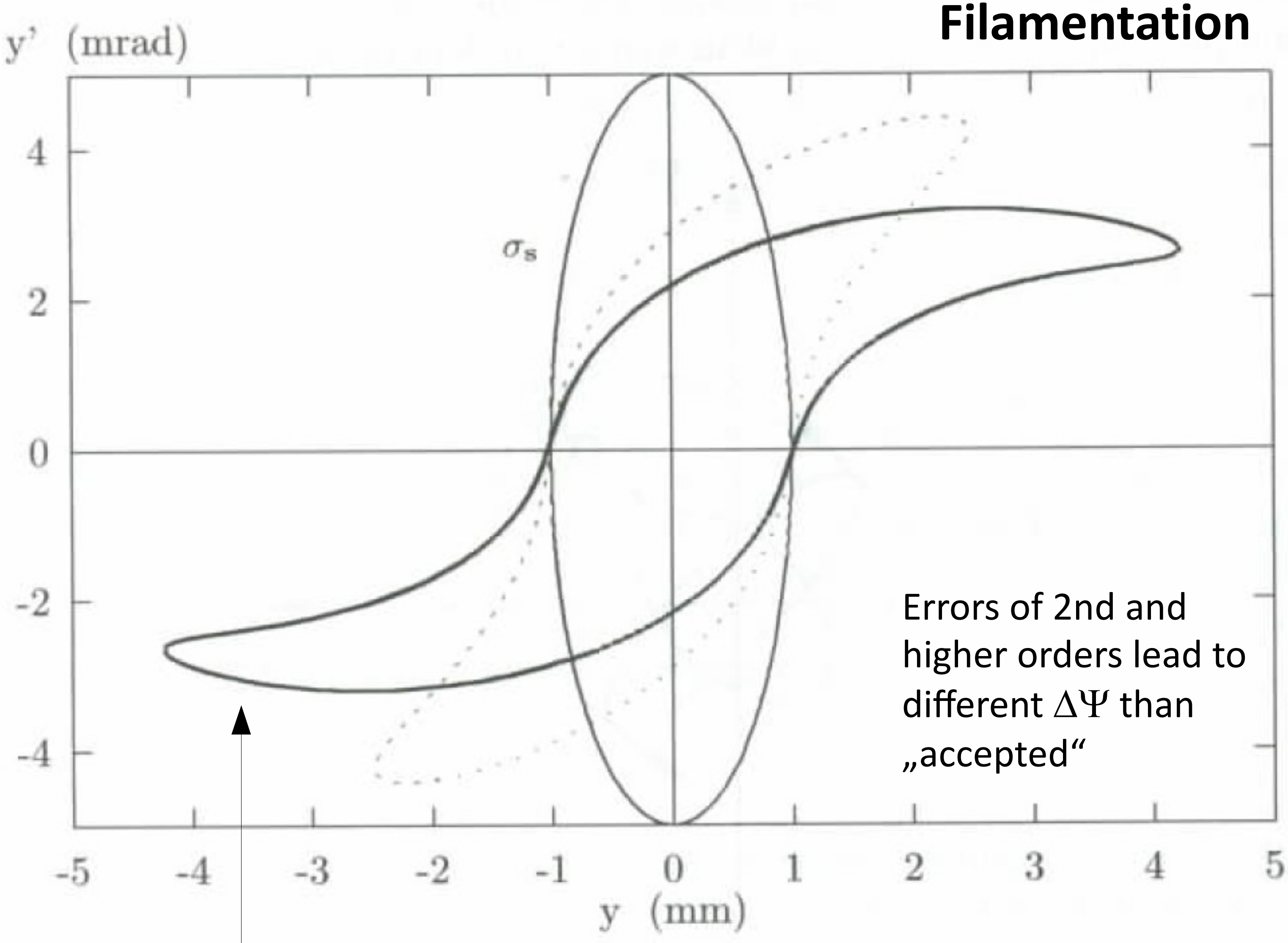


11.9 Matching beam and machine



11.9 Matching beam and machine

Filamentation

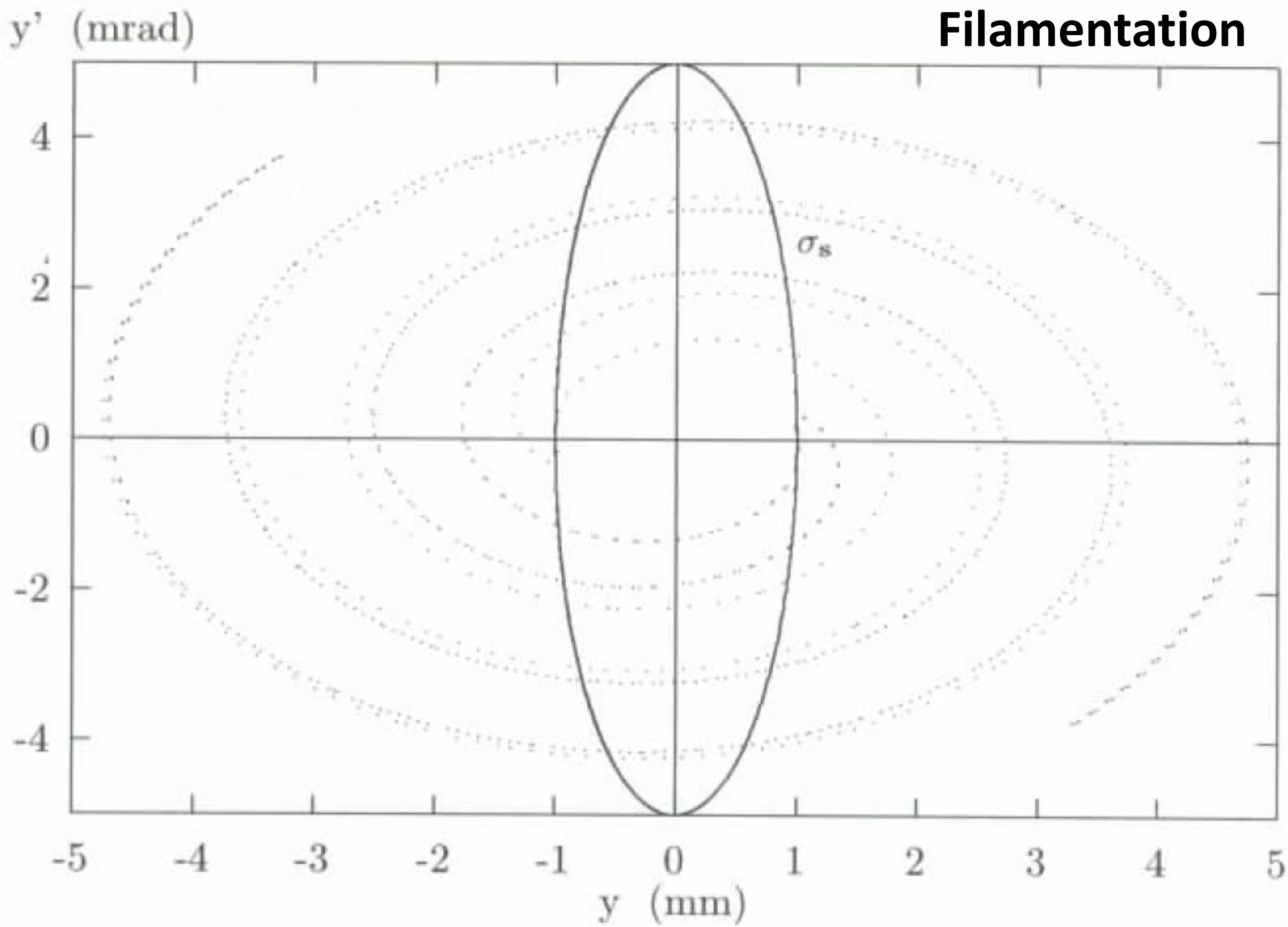


Errors of 2nd and higher orders lead to different $\Delta\Psi$ than „accepted"

11. Transverse Beam Dynamics

11.9 Matching beam and machine

Filamentation



Spiralling

12. Transverse Beam Dynamics with Dispersion

12.1 Equations of motion

1

$$x'' + k_x(s)x = h(s)\delta$$

$$h(s) = \frac{1}{\rho_0(s)}$$

$$\delta = \Delta p / p_0$$

$k_x(s)$ - Perdiodic coefficients

$$k_x(s+C) = k_x(s)$$

$$h(s+C) = h(s)$$

δ leads to a modified equillibrium trajectory $x_D(s) = \delta D(s)$

D(s) – dispersion function, periodic function of the machine

12. Transverse Beam Dynamics with Dispersion

12.2 Solution of equations of motion

$$x_\delta(s) = x(s) + \delta D(s)$$ (2)

Homogeneous solution (δ=0) ↑ $x(s)$

Insert (2) into (1)

$$D'' + k_x(s)D = h_x(s)$$

Periodic conditions:

$$D(s+C) = D(s)$$
$$D'(s+C) = D'(s)$$

Set starting point which can be any **s**

12.2 Solution of equations of motion

General solution

$$D(s) = D_0 C(s) + D_0' S(s) + d(s)$$

Coefficients

Cos/Sin – like solutions, basis solutions of homogeneous equations of motions

Special solution (4th lecture)

4. Solution of the Equation of Motion
(linear approximation)

- **If particles are NOT monoenergetic** $(\delta \neq 0) \Longrightarrow$ Inhomogenious equation

$$x'' + k_x(s)x = h(s)\delta$$ (dipole magnet)

Solution: solution of homogenious differential equation plus a particular solution of inhomogenious equation

$$x(s) = x_0 C_x(s) + x'_0 S_x(s) + \delta d_x(s)$$

Dispersion function

Particles are sorted dependent on field index
Correlation of δ and x - dispersion

4. Solution of the Equation of Motion
(linear approximation)

$$d_x(s) = \int_0^s h(\bar{s}) G_x(s, \bar{s}) d\bar{s}$$

With Green function

$$G_x(s, \bar{s}) = S_x(s) C_x(\bar{s}) - C_x(s) S_x(\bar{s})$$

And initial conditions:

$$d_x(0) = 0, \;\; d_x'(0) = 0$$

12. Transverse Beam Dynamics with Dispersion

12.2 Solution of equations of motion

After lengthy calculations

6.67 In Hinterberger

$$D(s) = \frac{\sqrt{\beta}}{2\sin(\mu/2)} \int_s^{s+C} h(\bar{s})\sqrt{\beta(\bar{s})}\cos[\Psi(\bar{s}) - \Psi(s) - \mu/2]d\bar{s}$$

- Dispersion is an integral effect of ALL bending magnets!
- Perturbations in *h(s)* contribute with $\sqrt{\beta(s)}$
- Contribution of an individual bending magnet is $\propto \quad h = 1/\rho_0$
- If $\mu/2 = 0 \quad \Rightarrow \quad D \to \infty \quad \equiv$ Resonance catastrophy!!!

$$\mu = (mod)2\pi \quad \Rightarrow \quad Q \in N$$

Tune, number of betatron oscillations

12. Transverse Beam Dynamics with Dispersion

12.3 Calculation of D(s)

- Either numerically
- From Twiss matrix

$$M(s) = R(s + C)$$

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{16} \\ M_{21} & M_{22} & M_{26} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}$$

$$D = \frac{M_{12}M_{26} + (1 - M_{22})M_{16}}{(1 - M_{11})(1 - M_{22}) - M_{12}M_{21}} = \frac{M_{12}M_{26} + (1 - M_{22})M_{16}}{4\sin^2 \mu/2}$$

$$D' = \frac{M_{21}M_{16} + (1 - M_{11})M_{26}}{(1 - M_{11})(1 - M_{22}) - M_{12}M_{21}} = \frac{M_{21}M_{16} + (1 - M_{11})M_{26}}{4\sin^2 \mu/2}$$

12.3 Calculation of D(s)

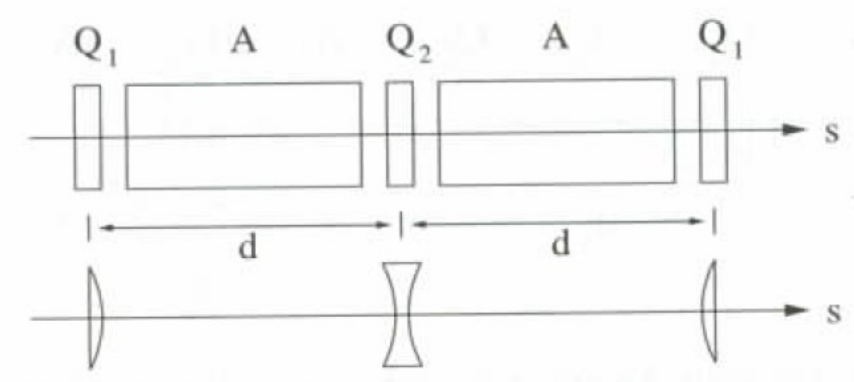
- Either numerically
- From Twiss matrix $$M(s) = R(s + C)$$

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{16} \\ M_{21} & M_{22} & M_{26} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}$$

- In each accelerator, due to symmetry conditions, there are points where $D'(s) = 0$

$$D = \frac{M_{16}}{1 - M_{11}}$$

12.3 Example: Periodic accelrator built of FODO structures / Transport beam-lines



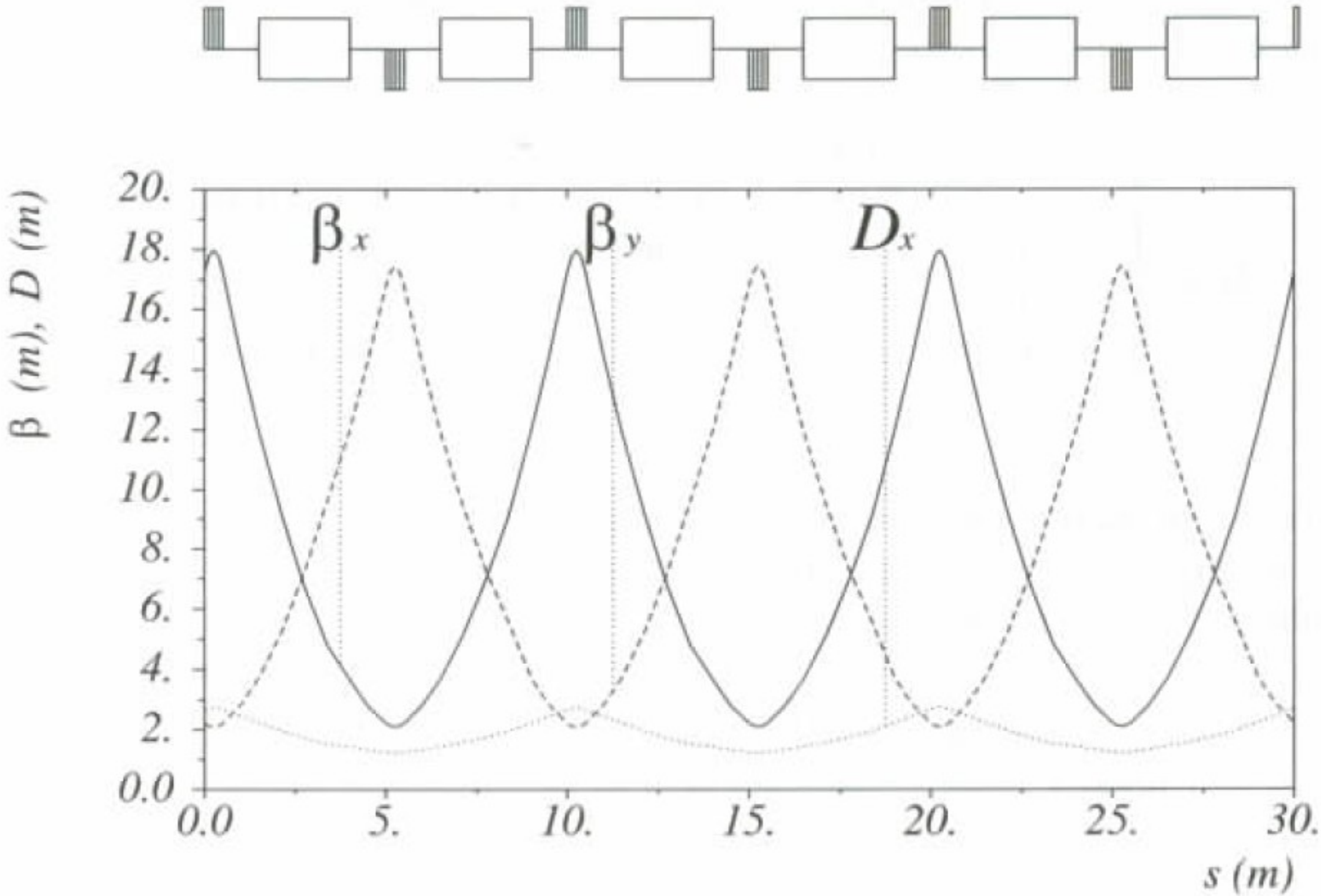
1/2F-0-D-0-1/2F

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_1} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 - d\left(\frac{1}{f_1} + \frac{1}{f_2}\right) + \frac{d^2}{2f_1 f_2} & 2d - \frac{d^2}{f_2} \\ -\left(\frac{1}{f_1} + \frac{1}{f_2}\right) + \frac{d}{2f_1}\left(\frac{1}{f_1} + \frac{2}{f_2}\right) - \frac{d^2}{4f_1^2 f_2} & 1 - d\left(\frac{1}{f_1} + \frac{1}{f_2}\right) + \frac{d^2}{2f_1 f_2} \end{pmatrix}$$

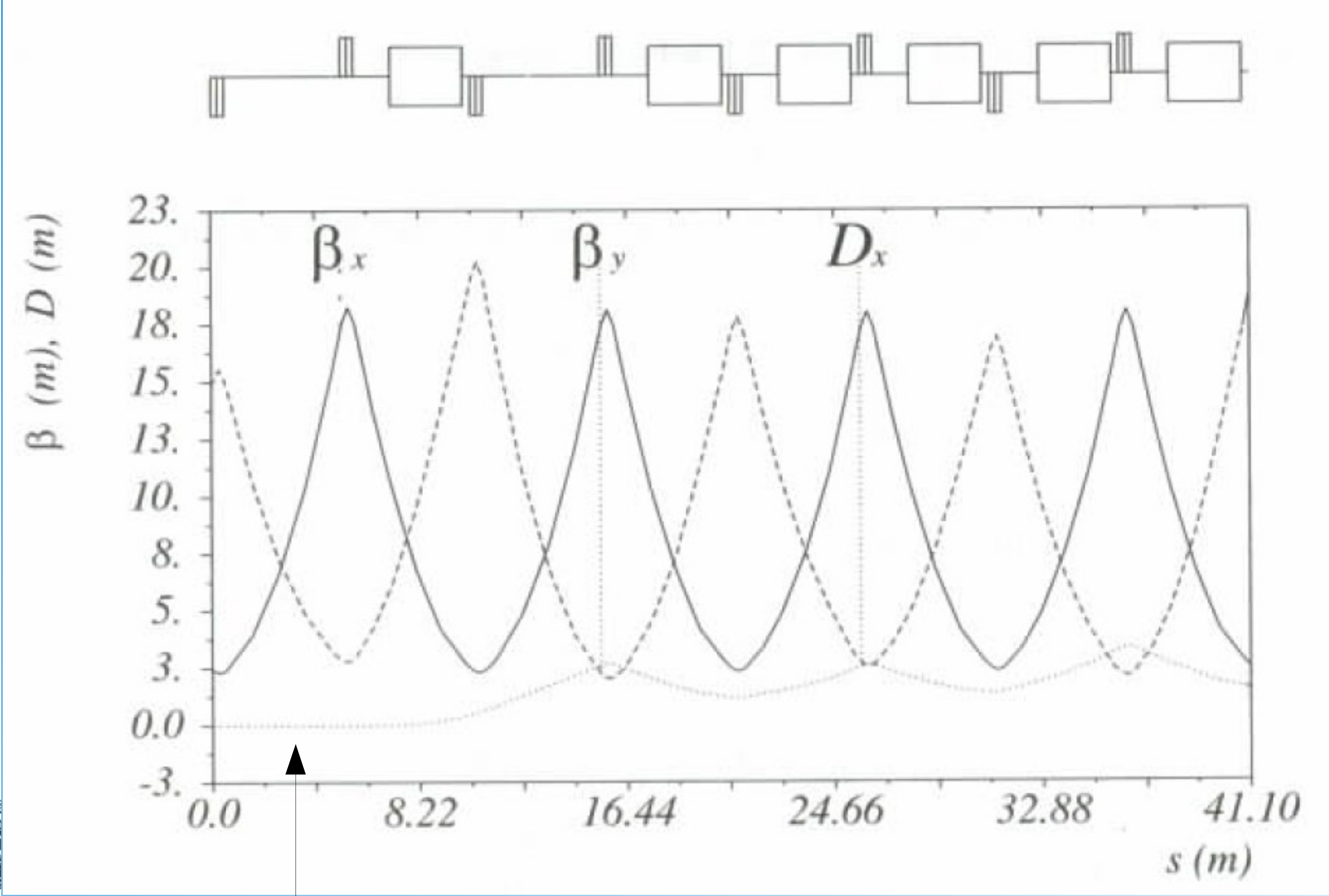
12. Transverse Beam Dynamics with Dispersion

12.3 Example: Model ring

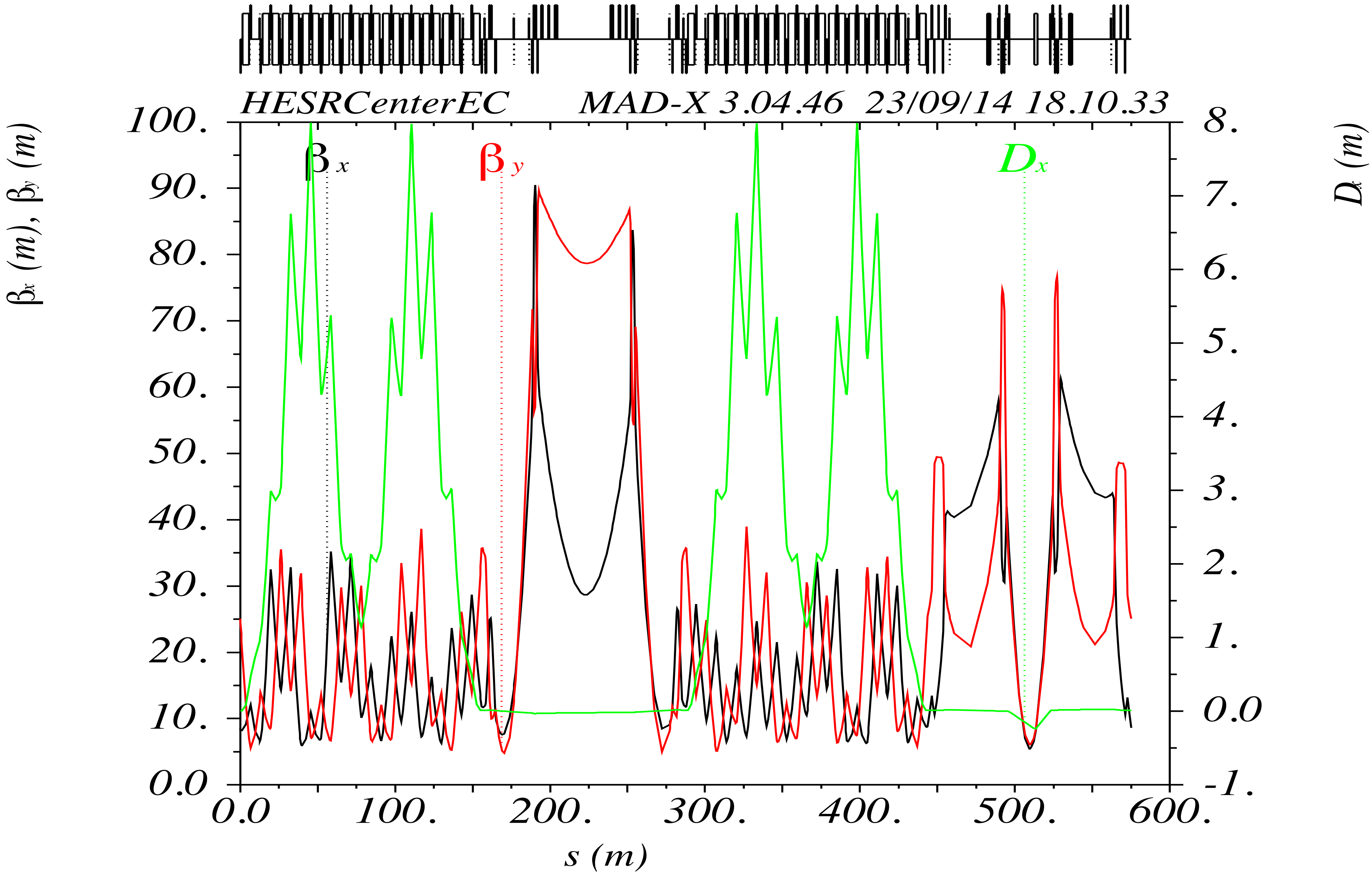


12. Transverse Beam Dynamics with Dispersion

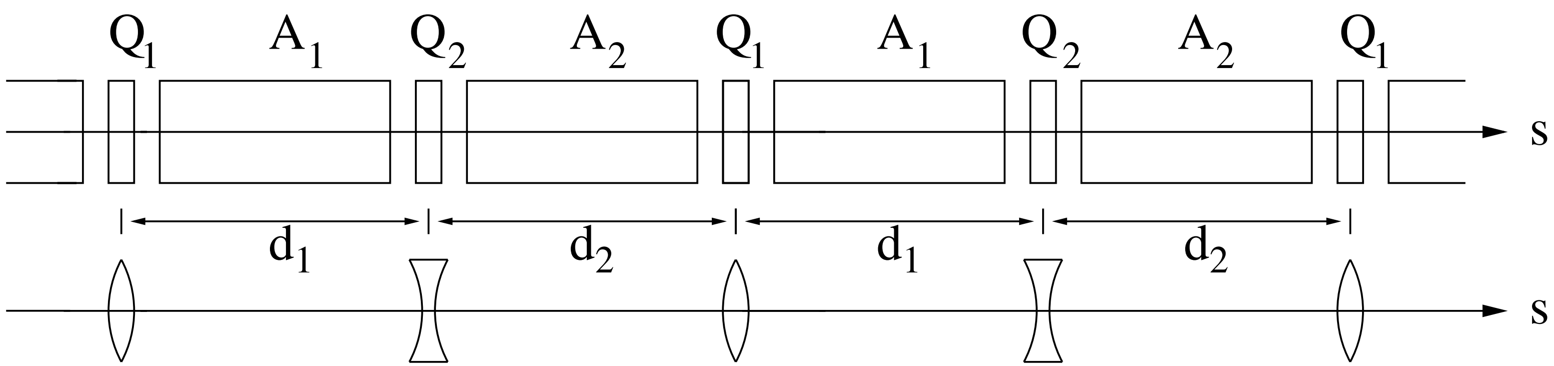
12.3 Example: Electron Stretcher Anlage (ELSA, Bonn)



Typical example (HESR at FAIR)



12.3 Example: Periodic accelrator built of FODO structures



„Standart“ separate-function machine

SPS/CERN (Super Proton Synchrotron)

108 Cells [Q_F+4D(ρ_0=741.2 m; α=8.445 mrad)+Q_D]
6 Super-periods of 14 Cells + 4 insertions with missing dipoles

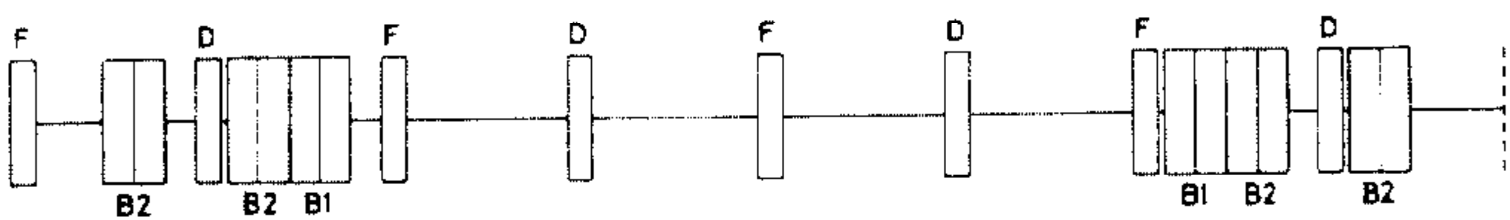


Fig. 3 Long straight section insertion

12.3 Example: Periodic accelrator built of FODO structures

SPS/CERN (Super Proton Synchrotron)

108 Cells [Q_F+4D(ρ_0=741.2 m; α=8.445 mrad)+Q_D]
6 **Superperiods** of 14 Cells + 4 insertions with missing dipoles

Edge focusing ~ 10^{-5} m^{-1} - neglect

$$\mu_C = 91.8620^\circ \quad \text{(per Cell)}$$

$$\mu = 91.8620^\circ \times 108$$
$$= 9921.10^\circ \quad \text{(per revolution)}$$
$$| \Rightarrow \quad Q = 27.559$$

Due to slightly different field gradients in QF and QD (3 ‰)

Q_x = 27.574 and Q_y=27.554

12. SPS

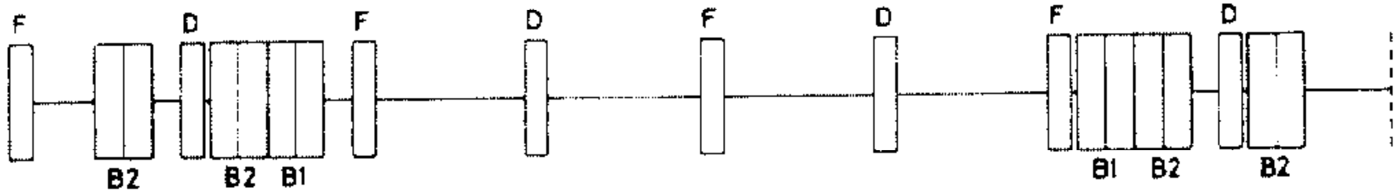
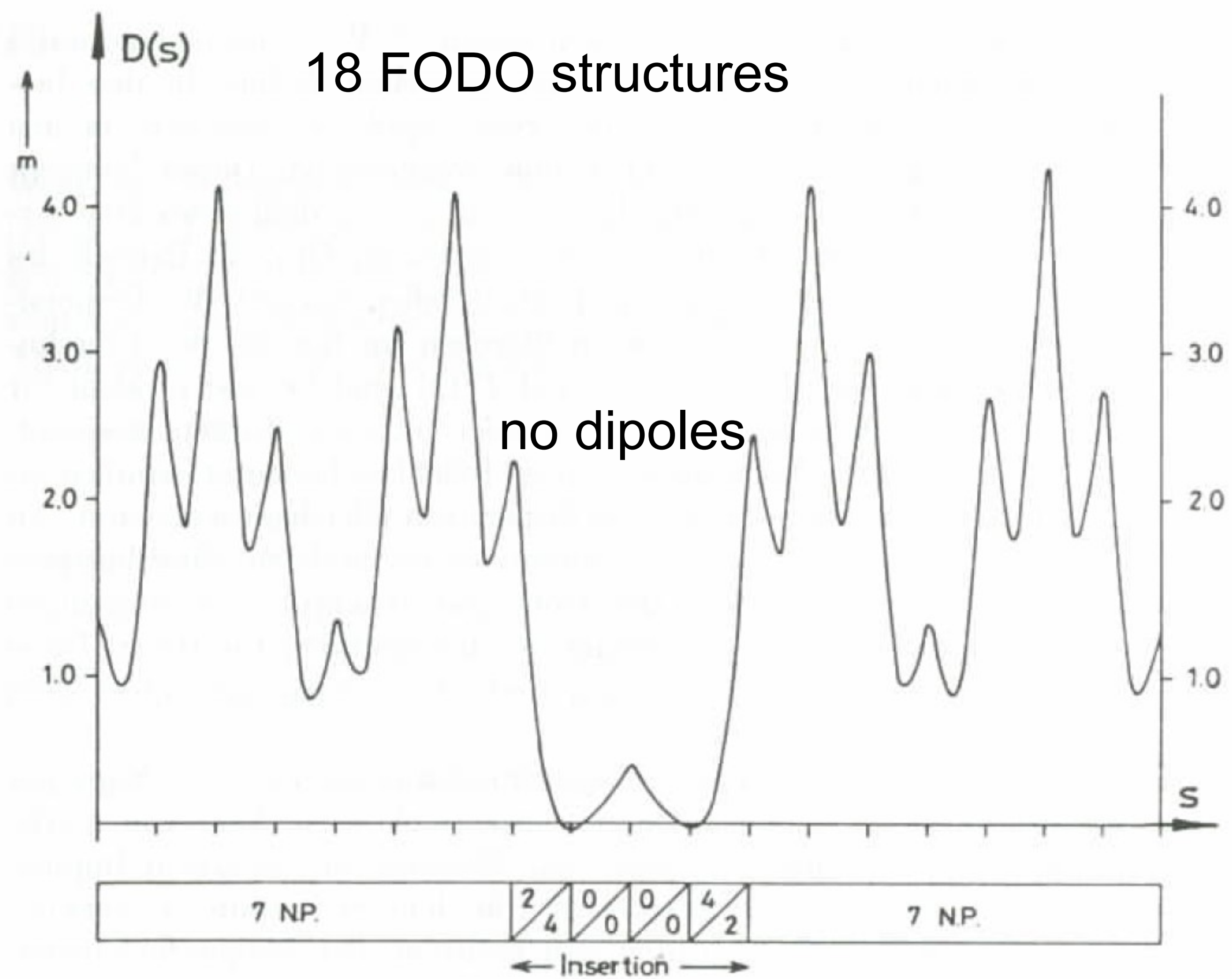
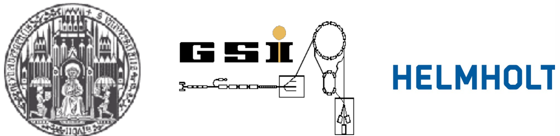


Fig. 3 Long straight section insertion



18 FODO structures

no dipoles



12. SPS

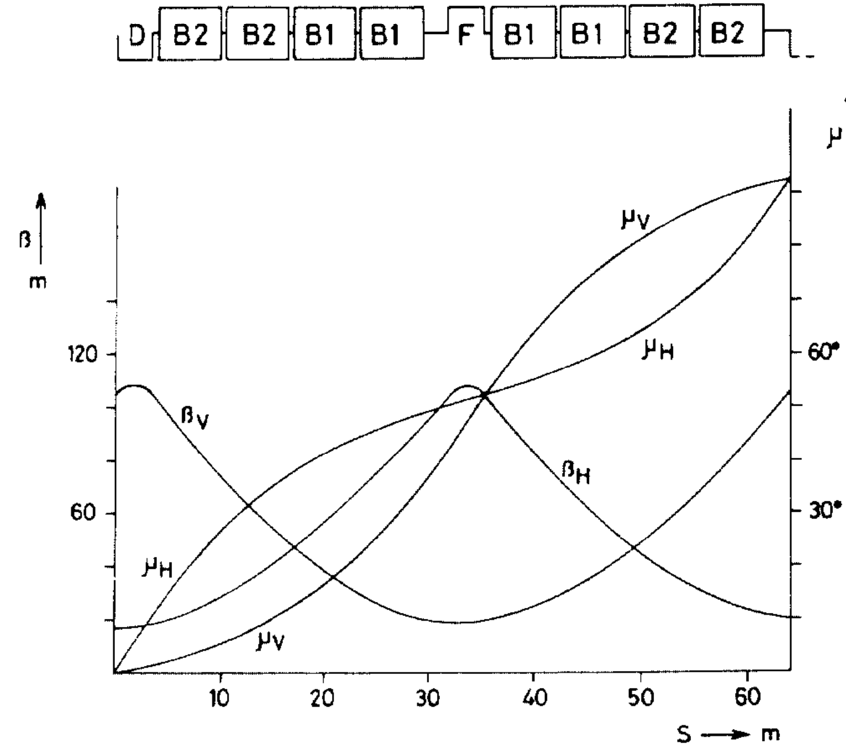


Fig. 2(a) Lattice functions

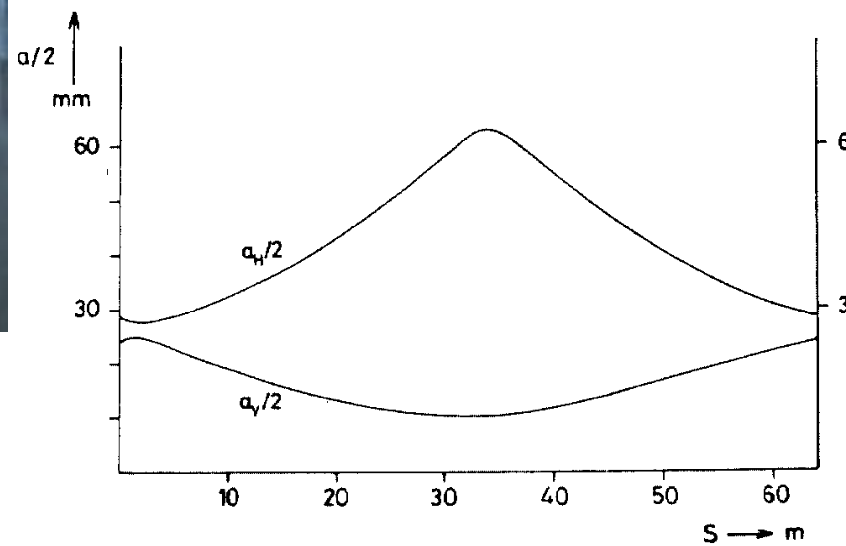


Fig. 2(b) Beam envelope fitting mechanical apertures

12. Transverse Beam Dynamics with Dispersion

12.3 Example: Periodic accelrator built of FODO structures

Q_x = 27.574 and Q_y=27.554

!!! However, the running point is now at 27.4. Why? !!!

$$5 \cdot 27.6 = 138$$

5th order resonance

With 6 SP ⇒
4.6 betatron oscillations per period

12. Transverse Beam Dynamics with Dispersion

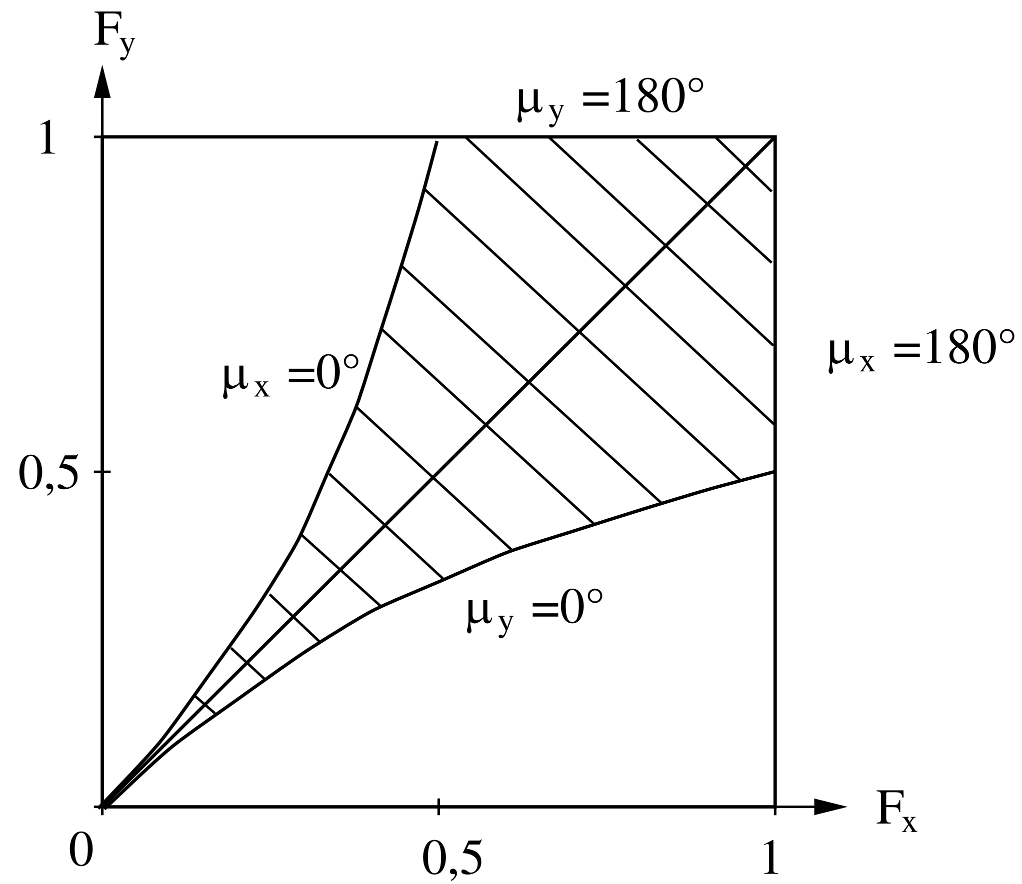
12.3 Example: Periodic accelrator built of FODO structures

Stability criterion

$$-1 < \cos\mu < 1\,, \qquad -1 < 1 - \frac{d}{f_1} - \frac{d}{f_2} + \frac{d^2}{2f_1 f_2} < 1$$

$$0 < \sin^2\frac{\mu}{2} < 1\,, \qquad 0 < \frac{d}{2f_1} + \frac{d}{2f_2} - \frac{d}{2f_1}\frac{d}{2f_2} < 1$$

$$F_x = \left|\frac{d}{2f_1}\right|, \qquad F_y = \left|\frac{d}{2f_2}\right|$$



Necktie diagram

13. Distortions and Resonances

13.2 Dipole errors

- Starting from equilibrium trajectory

$$\Delta x' = \frac{-\,\delta B}{B\rho}\Delta s = F(s_0)\Delta s$$

„Closed orbit distortion“

$$\begin{pmatrix} x_c \\ x'_c \end{pmatrix}_{s_0} = M(s_0)\begin{pmatrix} x_0 \\ x'_0 + \Delta x' \end{pmatrix}_{s_0}$$

Describes the distored reference orbit

13.2 Dipole errors

Solution is analogous to the case of dispersion

Distorted orbit:

$$x_c(s) = \frac{\sqrt{\beta(s)}}{2\sin(Q\pi)} \Delta x' \sqrt{\beta(s_0)} \cos[\Psi(s) - \Psi(s_0) - Q\pi]$$

Floquet transformation:

$$\eta_\mathrm{c} = a_\mathrm{c} \cos \overbrace{[\psi(s) - \psi(s_0) - Q\pi]}^{\psi_\mathrm{c}} \qquad \Delta\left(\frac{\mathrm{d}\eta}{\mathrm{d}\psi}\right) = \sqrt{\beta(s_0)}\Delta x' = 2a_\mathrm{c}\sin Q\pi$$

13.2 Dipole errors

Distorted orbit:

$$x_c(s) = \frac{\sqrt{\beta(s)}}{2\sin(Q\pi)} \Delta x' \sqrt{\beta(s_0)} \cos[\Psi(s) - \Psi(s_0) - Q\pi]$$

In general from (many distortions):

$$x_c(s) = \frac{\sqrt{\beta(s)}}{2\sin(Q\pi)} \int_s^{s+C} F(\bar{s})\sqrt{\beta(s)} \cos[\Psi(\bar{s}) - \Psi(s) - Q\pi] d\bar{s}$$

F(s) – distortion function

Consequences:

$$x_c(s) \propto \Delta x'$$

$$x_c(s) \propto \sqrt{\beta(s_0)}$$

$$x_c(s) \propto \sqrt{\beta(s)}$$

Correction magnets (trial & error)

$$x_c(s) \propto 1/\sin(Q\pi) \Leftarrow Q \text{ can not be integer}$$

13. Distortions and Resonances

13.3 Quadrupole errors (stop-band second-order)

Quadrupole magnet is a lense with focal length *f*

Small distortions -> additional (de)focusing

$$\frac{1}{f} = \delta k \Delta s$$

Twiss matrix:

$$M_0 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

Undistorted matrix

$$M' = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} M_0$$

An additional thin lense

13.3 Quadrupole errors (stop-band second-order)

$$\mu' = \mu_0 + \Delta\mu \quad (\Delta\mu << 1)$$

Stability condition:

$$\cos\mu = \frac{1}{2}Tr(M) = \cos(\mu_0) - \frac{1}{2}\frac{\beta_0}{f}\sin(\mu_0)$$

An additional term

$$\Delta\mu = 2\pi\Delta Q = \frac{1}{2}\frac{\beta_0}{f}$$

$$\Delta Q = \frac{1}{4\pi}\frac{\beta_0}{f}$$

Can be used to measure β_0

13.3 Quadrupole errors (stop-band second-order)

Result: shift of **operation point**

$$\Delta Q = \frac{1}{4\pi} \oint \beta(\bar{s}) \delta K(\bar{s}) d\bar{s}$$

Similar as in the case of the dispersion, but β instead of $\sqrt{\beta}$

$$\Delta\beta(s) = \frac{\beta(s)}{2\sin^2(2Q\pi)} \int_s^{s+C} \delta K(\bar{s}) \beta(\bar{s}) \cos 2[\Psi(\bar{s}) - \Psi(s) - Q\pi] d\bar{s}$$

Important: *2Q*π -> *Q*=half-integer -> Resonance
Stop-band of second order

13.3 Quadrupole errors (stop-band second-order)

Floquet transformation

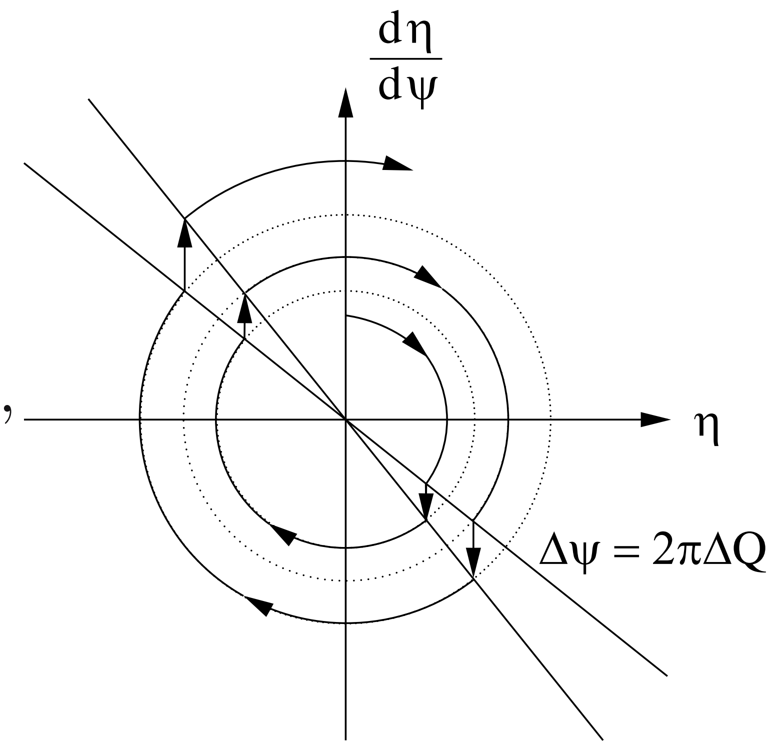
Kick: $$\Delta y' = -\frac{y}{f} = -\frac{1}{f} a\sqrt{\beta}\cos\psi$$

Amplitude enlargement

$$\Delta a = \Delta\left(\frac{d\eta}{d\psi}\right)\sin\psi = -\frac{a\beta}{f}\cos\psi\sin\psi\,,$$

$$\Delta\psi = -\frac{1}{a}\Delta\left(\frac{d\eta}{d\psi}\right)\cos\psi = \frac{\beta}{f}\cos^2\psi\,.$$

Phase shift



13. Distortions and Resonances

13.3 Quadrupole errors (stop-band second-order)

$$\Delta Q = \frac{1}{2\pi}\frac{\beta}{f}\cos^2\psi = \frac{1}{4\pi}(1 - 2\cos(2\Psi))$$

Average shift of the working point by $$\Delta\bar{Q} = \frac{1}{4\pi}\frac{\beta}{f}$$

With superimposed modulation $$\delta\bar{Q} = \frac{1}{4\pi}\frac{\beta}{f}\cos(2\Psi)$$

Modulations and shift are small if β-small and f-large

13. Distortions and Resonances

13.4 Sextupole errors (stop-band third-order)

Similar to quadrupoles

Resonance of *3Q*-integer

Stop-band of 3rd order:

$$\delta Q = \frac{\beta^{3/2}}{16\pi} \left(\frac{\partial^2 B_y}{\partial x^2} \right) \frac{\Delta s}{B\rho} a \cos(3\Psi)$$

Amplitude of betatron oscillations!!!

13. Distortions and Resonances

13.4 Sextupole errors (stop-band third-order)

- Non-linear effect (fast grows: $\Delta a \propto a^2$)
- Dynamic aperture