Introduction to Accelerator Physics

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Physikalisches Institut der Universität Heidelberg

Lecture 7

Lecture Dates

https://uebungen.physik.uni-heidelberg.de/vorlesung/20222/1611/lecture

Date	Topic
19.10.2022	Introduction and basic definitions
26.10.2022	Accelerating structures
02.11.2022	Accelerator Components
09.11.2022	Optics with magnets (1)
16.11.2022	Optics with magnets (2)
23.11.2022	Equations of motion
30.11.2022	Phase ellipses and magneto-optical system / Transverse beam dynamics
07.12.2022	Transverse beam dynamics, beam stability / Longitudinal beam dynamics
14.12.2023	Phase space and beam cooling (Invitation)
11.01.2023	Space charge and beam-beam dynamics
18.01.2023	Physics at Storage Rings
25.01.2023	Physics at Colliders
01.02.2023	New accelerator technologies
08.02.2023	Student seminar
15.02.2023	reserve
22.02.2023	reserve

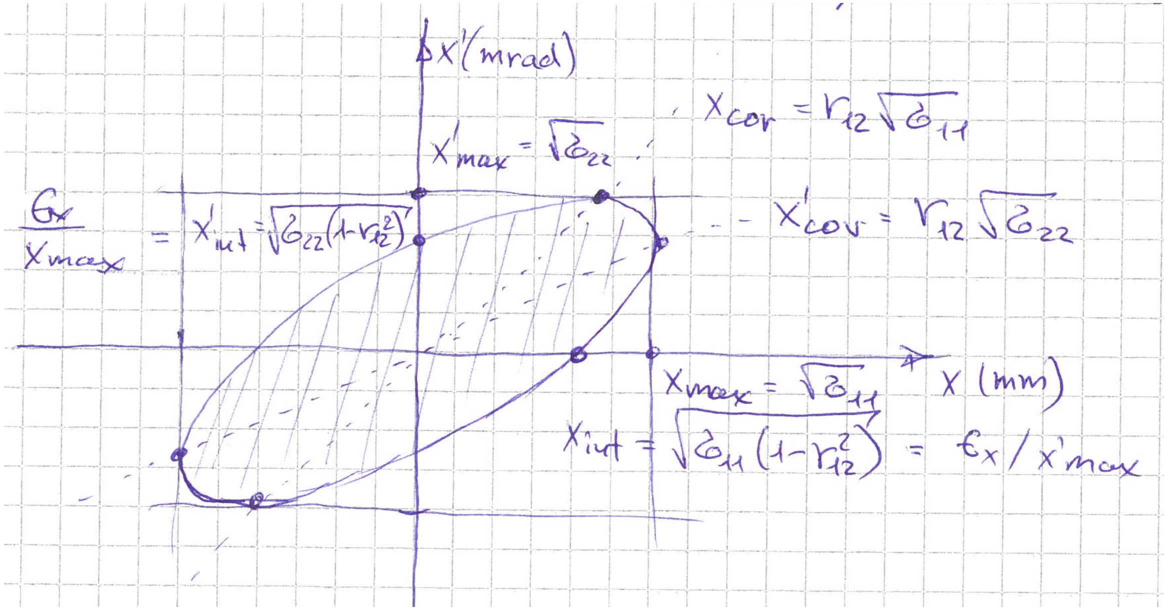
Wednesdays, 14:15-16:00

Summary of last lecture

Geometric optics
- **thin lense**
- **thick lense**

Beam Properties
- **Emittance**
- **Beam profile**
- **Beam waist**

Transformation of beam ellipses



6. Rotations

Positive rotation – clockwise in the s-direction

Coupling of x & y planes (not wanted in most cases)

Rotation by angle (α)

$$\mathbf{R}(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 & 0 & 0 \\ 0 & -\sin\alpha & 0 & \cos\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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Tilted magnets (dipoles, quadrupoles, sextupoles, …

6. Beam Rotator

- **Quadrupole tilted by angle (α)**

$$\mathbf{R} = R(-\alpha) R_Q R(\alpha)$$

- **Vertical beam deflector (90°)**

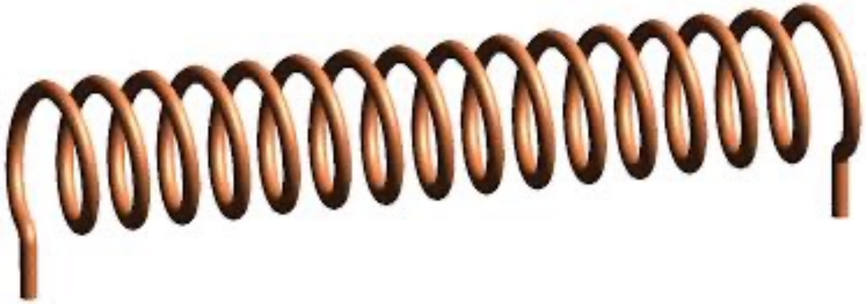
Upwards:

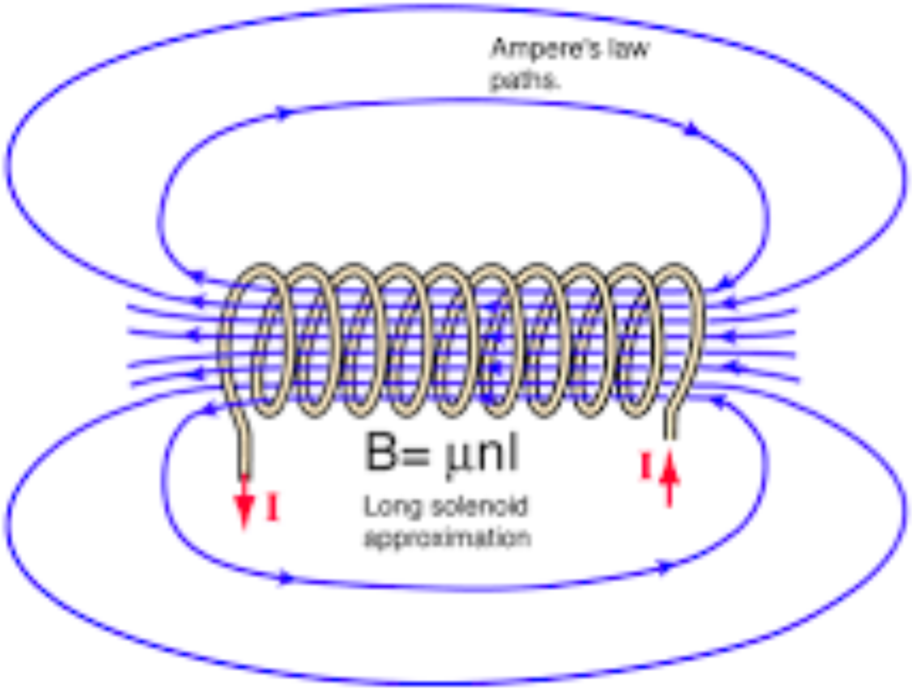
$$\mathbf{R} = R(+90^\circ) R_D R(-90^\circ)$$

Downwards:

$$\mathbf{R} = R(-90^\circ) R_D R(+90^\circ)$$

6. Solenoid





The magnetic field is concentrated into a nearly uniform field in the center of a long solenoid. The field outside is weaker and the lines representing the magnetic field are further apart.

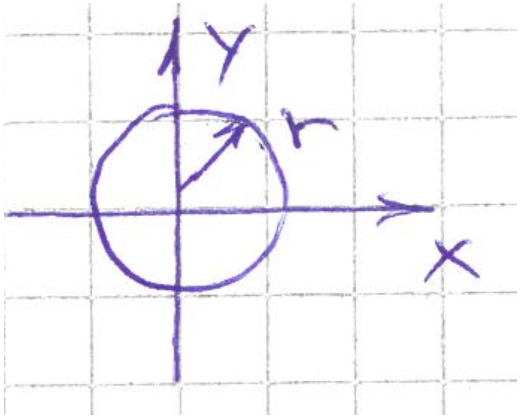
6. Solenoid

Homogeneous **Entrance/Exit**

2 regions:

① ②

6. Solenoid



Parallel to magnetic field: v_t=0

Radius: $r = \frac{|qB|}{\gamma m v_t} = \frac{|qB|}{p_t}$

Transverse velocity-component

length

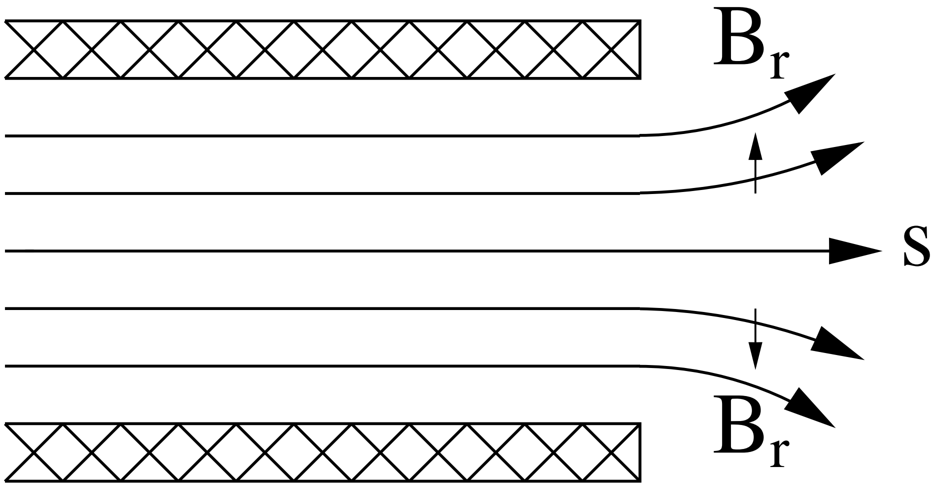
Rotation:

$$\alpha = -\frac{qB_s}{\gamma m}\frac{L}{v_z} = -\frac{qB_s L}{|q(B\rho)_0|}$$

Longitudinal velocity-component

Helix-like trajectory

6. Solenoid



$$B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z}$$

$$B_x = \frac{x}{r}B_r = -\frac{x}{2}\frac{\partial B_z}{\partial z}$$

$$B_y = \frac{y}{r}B_y = -\frac{y}{2}\frac{\partial B_z}{\partial z}$$

Detailed discussion - Hinterberger

① ②

6. Solenoid

Define:
$$K = \frac{B_s}{2(B\rho)_0}$$

$$C = \cos(KL) = \cos(\alpha/2)$$
$$S = \sin(KL) = -\sin(\alpha/2)$$

$$\mathbf{R} = \begin{bmatrix} C^2 & SC/K & SC & S^2/K & 0 & 0 \\ -KSC & C^2 & -KS^2 & SC & 0 & 0 \\ -SC & -S2/K & C^2 & SC/K & 0 & 0 \\ KS^2 & -SC & -KSC & C^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Reminder:

6. Rotations

Positive rotation – clockwise in the s-direction

Coupling of x & y planes (not wanted in most cases)

Rotation by angle (α)

$$\mathbf{R}(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 & 0 & 0 \\ 0 & -\sin\alpha & 0 & \cos\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

6. Solenoid

$$\mathbf{R}(\alpha/\mathbf{2})\mathbf{R} = \begin{bmatrix} C & S/K & 0 & 0 & 0 & 0 \\ -KS & C & 0 & 0 & 0 & 0 \\ 0 & 0 & C & S/K & 0 & 0 \\ 0 & 0 & -KS & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Focusing in both planes
Focusing in independend from the direction of B_s

Positive particle flying in the direction of $B_{s:}$ Rotation counter-clockwise

Major parameters

$$K = \frac{B_s}{2(B\rho_0)} \quad \frac{\alpha}{2} = -KL$$

① ②

6. Solenoid

$$\left|\frac{\alpha}{2}\right| = \frac{\pi}{2} \Rightarrow \quad C = 0 \qquad\qquad \left|\frac{\alpha}{2}\right| = \pi \Rightarrow \quad S = 0$$

$$\mathbf{R}(\alpha/\mathbf{2})\mathbf{R} = \begin{bmatrix} C & S/K & 0 & 0 & 0 & 0 \\ -KS & C & 0 & 0 & 0 & 0 \\ 0 & 0 & C & S/K & 0 & 0 \\ 0 & 0 & -KS & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Transfer matrix

(linear approximation)

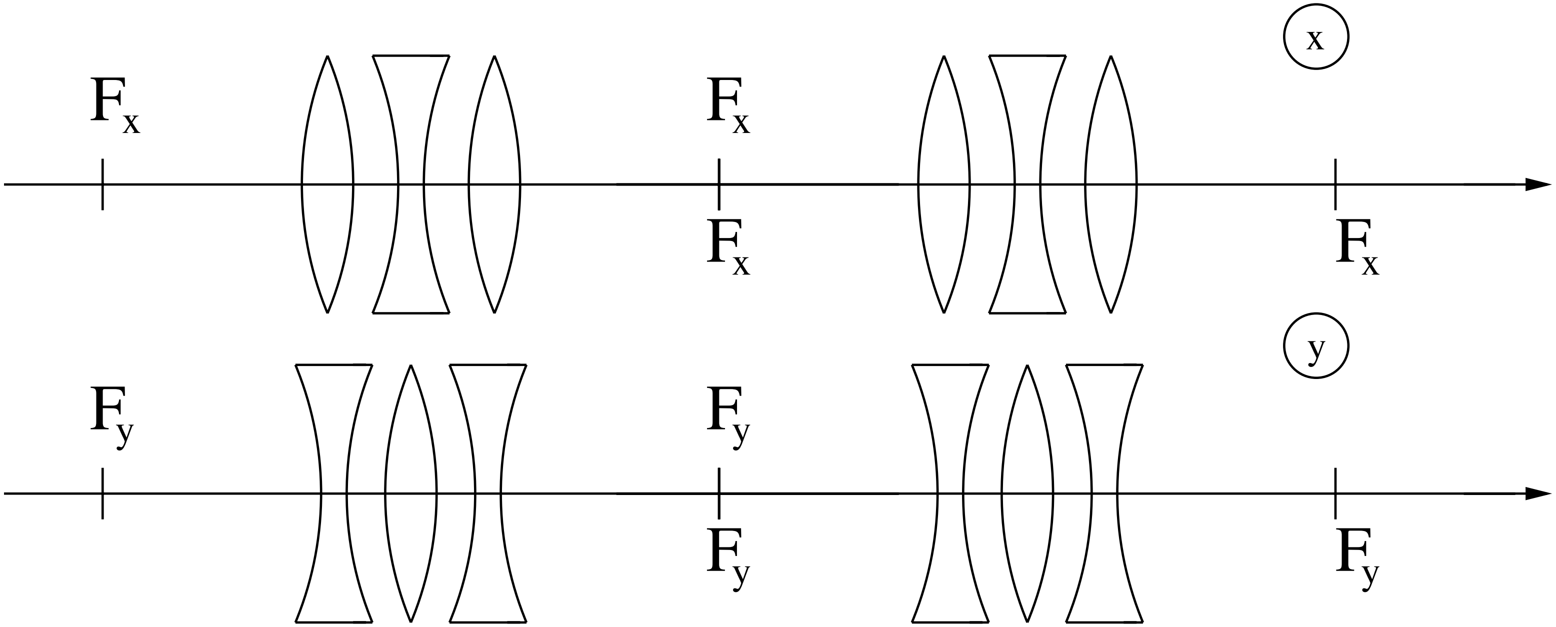
Abbildung	radial	axial
Punkt-zu-Punkt	$R_{12} = (x\|x') = 0$	$R_{34} = (y\|y') = 0$
Punkt-zu-Parallel	$R_{22} = (x'\|x') = 0$	$R_{44} = (y'\|y') = 0$
Parallel-zu-Punkt	$R_{11} = (x\|x) = 0$	$R_{33} = (y\|y) = 0$
Parallel-zu-Parallel	$R_{21} = (x'\|x) = 0$	$R_{43} = (y'\|y) = 0$
Ortsdispersion $= 0$	$R_{16} = (x\|\delta) = 0$	
Winkeldispersion $= 0$	$R_{26} = (x'\|\delta) = 0$	

From Hinterberger

9. Ion Optical Systems

9.3 Double telescope

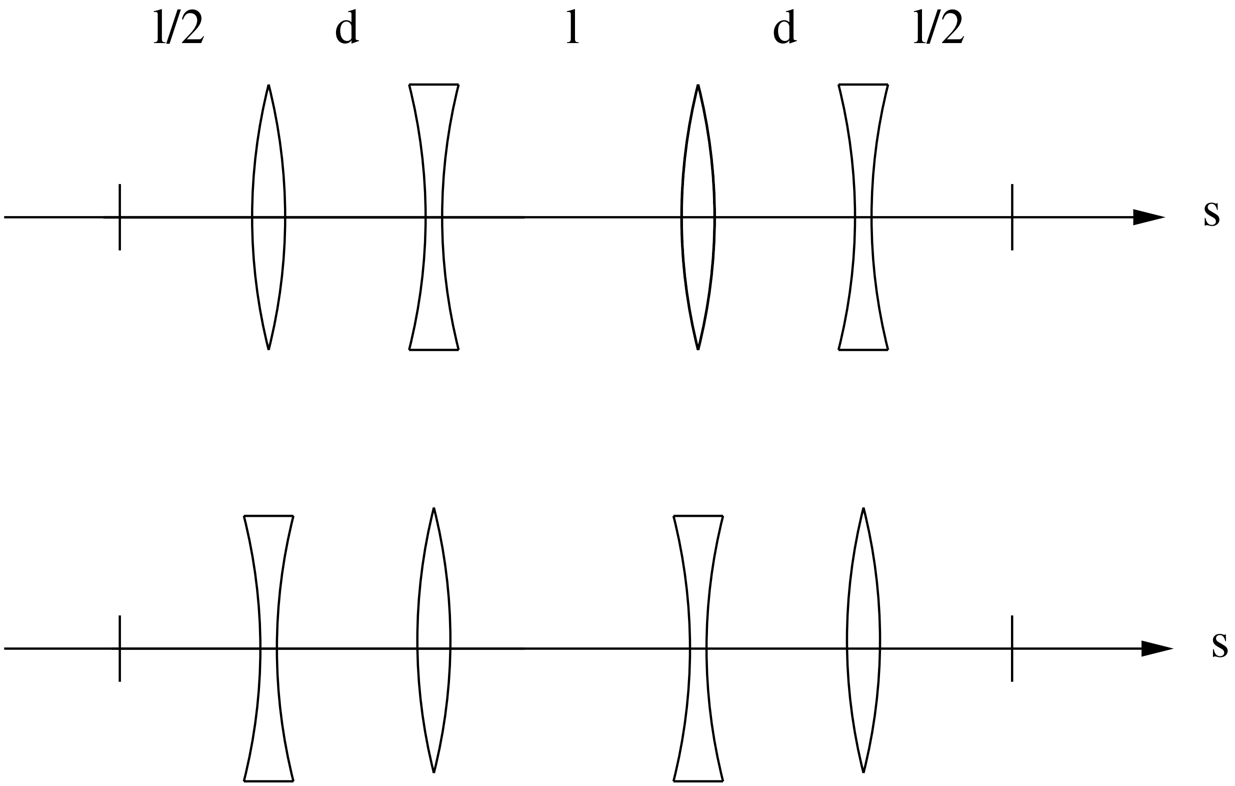
Two quadrupole triplets



9. Ion Optical Systems

9.3 Double telescope

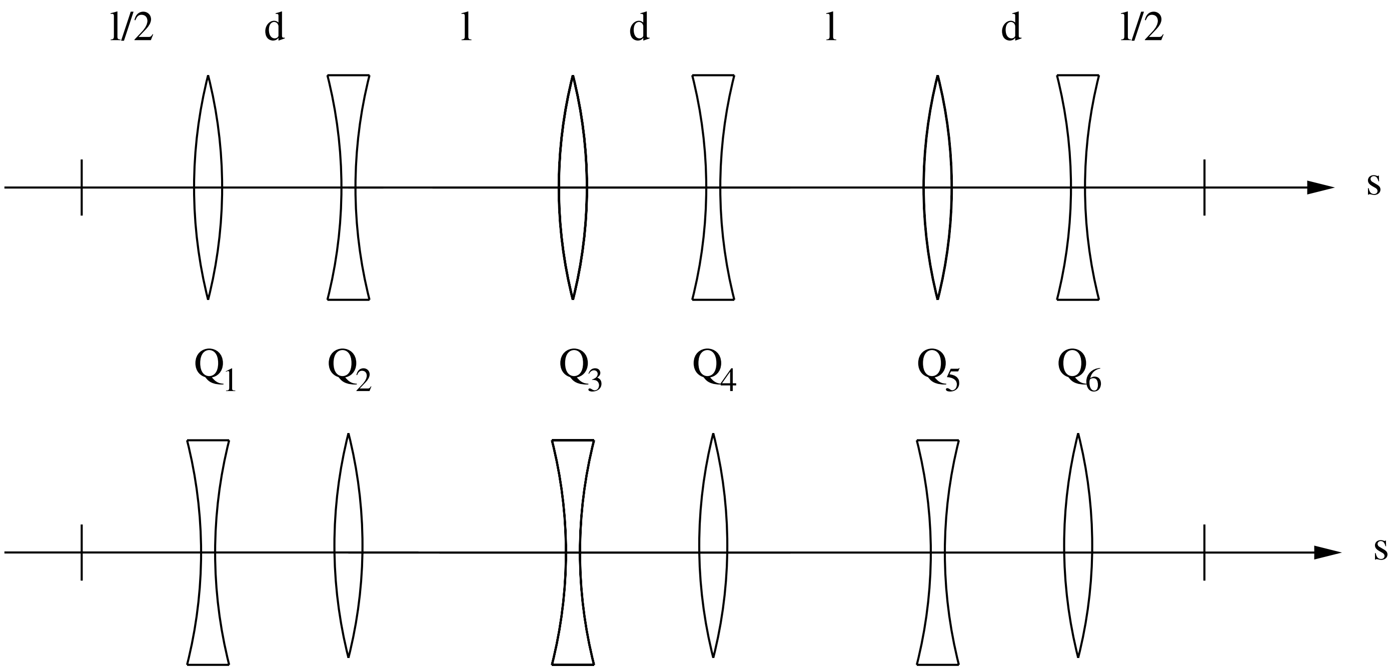
Four quadrupole magnets



9. Ion Optical Systems

9.3 Double telescope

Six quadrupole magnets



$$|f| = \sqrt{ld} : R_x = R_y = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|f| = \sqrt{ld/3} : R_x = R_y = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}$$

9. Ion Optical Systems

Accelerator

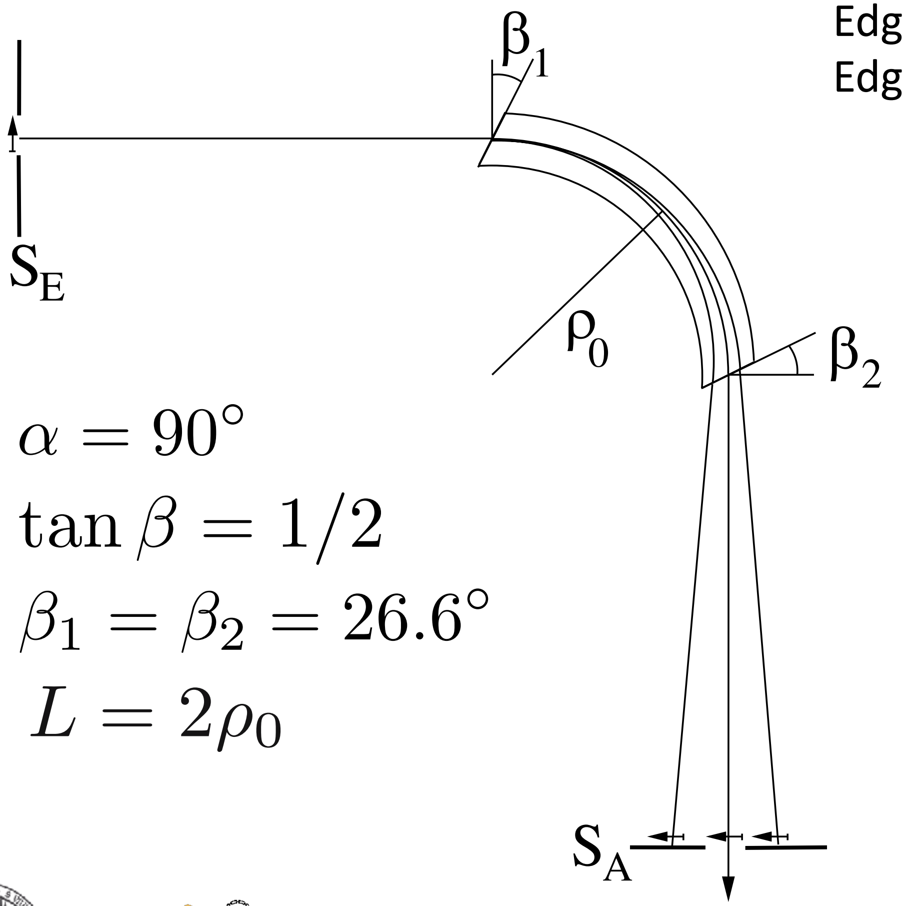
=

repetitive structures of telescopes

9. Ion Optical Systems

9.4 Beam analysator / Monochromator

Dispersion



$$\alpha = 90^\circ$$

$$\tan\beta = 1/2$$

$$\beta_1 = \beta_2 = 26.6^\circ$$

$$L = 2\rho_0$$

Edge focusing in y-plane

Edge defocusing in x-plane

Point-2-Point

$$R_x = \begin{pmatrix} -1 & \textcircled{0} & 4\rho_0 \\ -0{,}75/\rho_0 & -1 & 1{,}5 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_y = \begin{pmatrix} -1 & \textcircled{0} \\ -0{,}61/\rho_0 & -1 \end{pmatrix}.$$

Symmetric QP-triplett with

$$\frac{1}{f_x} = \frac{0.75}{\rho_0} \qquad \frac{1}{f_y} = \frac{0.61}{\rho_0}$$

$$M_x = M_y = -1$$

5. Edge focusing

de-focusing

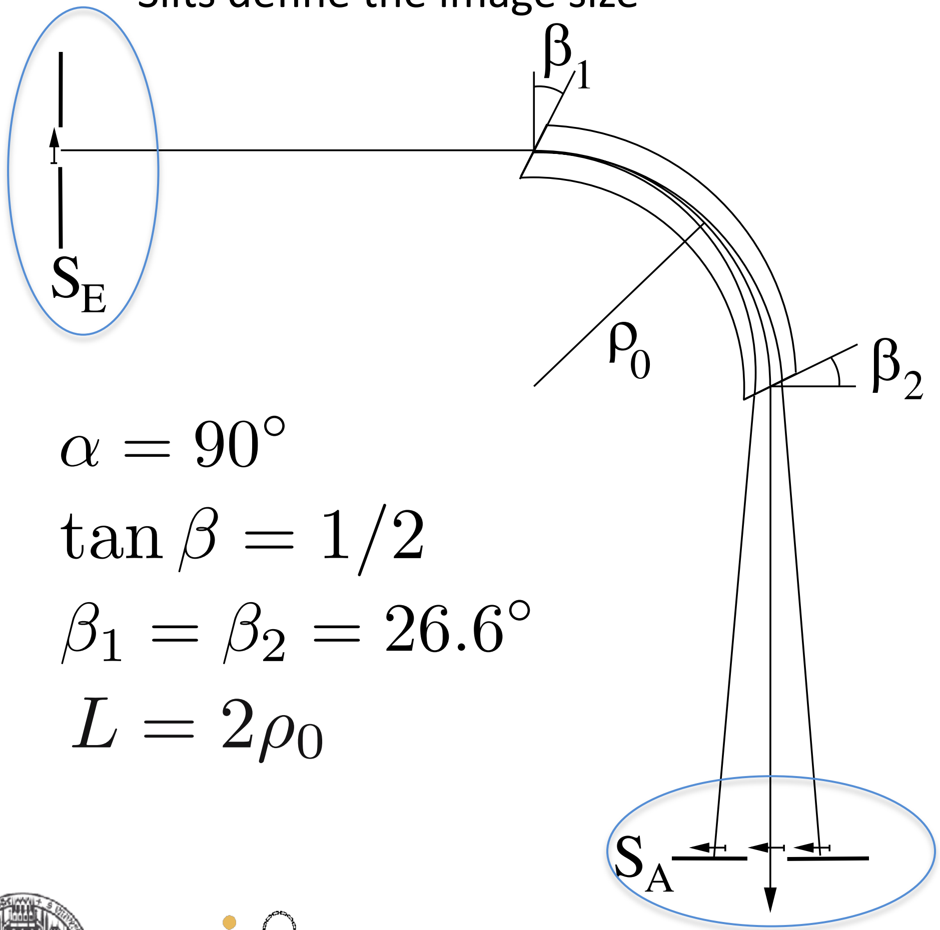
focusing

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\tan(\beta)}{\rho_0} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\tan(\beta_{\mathrm{eff}})}{\rho_0} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

9. Ion Optical Systems

9.4 Beam analysator / Monochromator

Slits define the image size

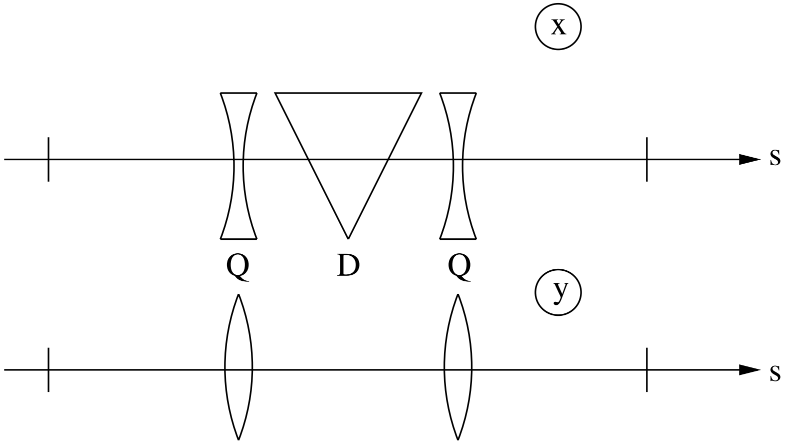


$$\alpha = 90^\circ$$

$$\tan\beta = 1/2$$

$$\beta_1 = \beta_2 = 26.6^\circ$$

$$L = 2\rho_0$$



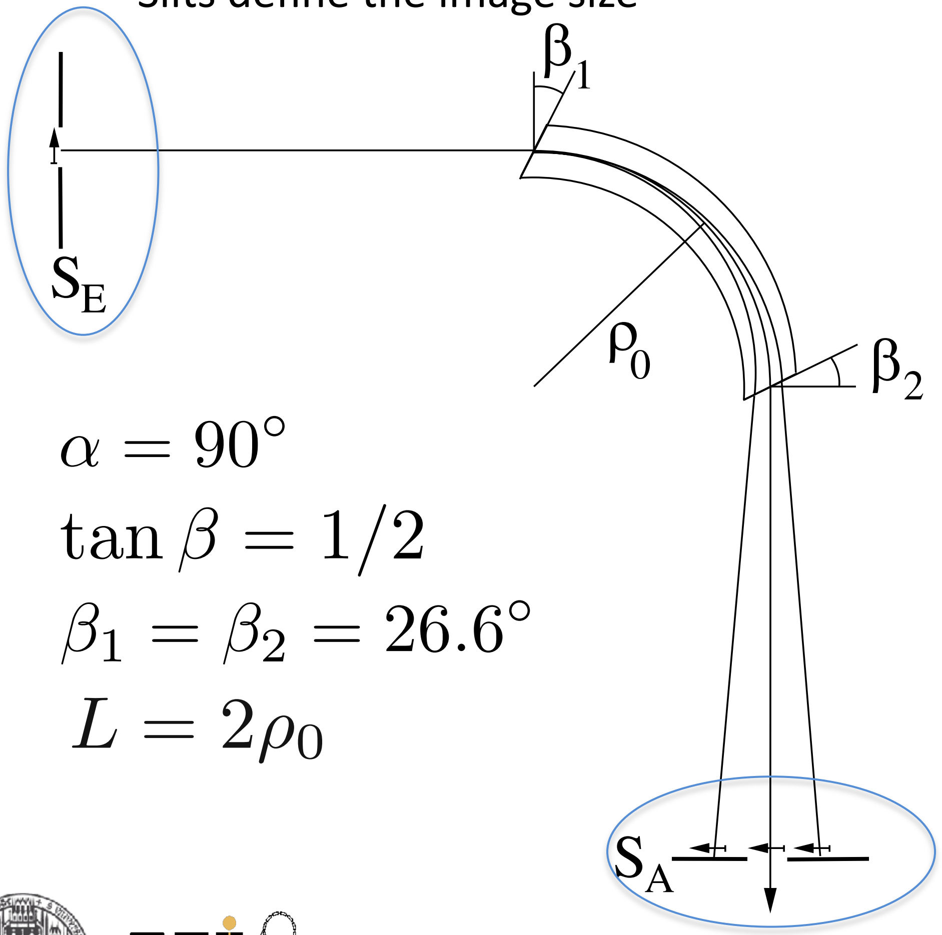
$$\delta_{\mathrm{FWHM}} = |M_x \Delta x_{\mathrm{E}} / D_x|$$

Slits define the
momentum spread

9. Ion Optical Systems

9.4 Beam analysator / Monochromator

Slits define the image size



Slits define the momentum spread

$$\rho_0 = 1 \text{ m}$$
$$\Delta x_E = 1 \text{ mm}$$

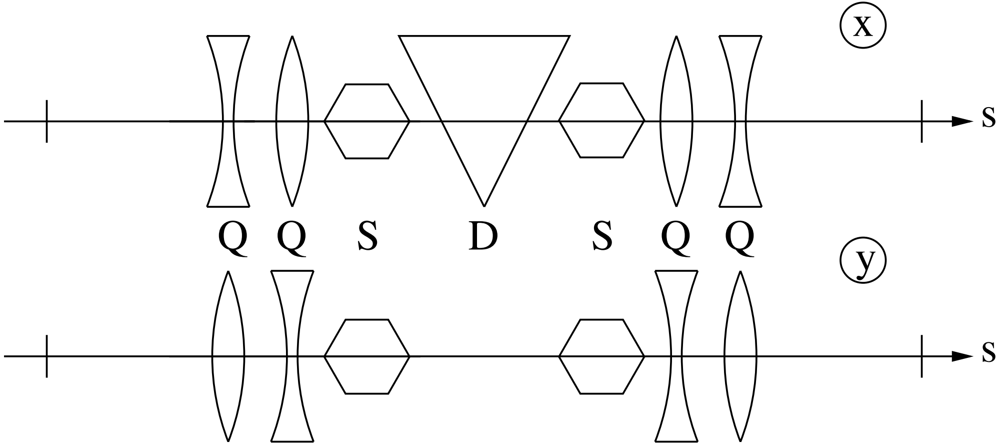
$$|D_x/M_x| = 4 \text{ m} = 4(\text{mm}/\text{promille})$$
$$\delta_{\text{FWHM}} = 2{,}5 \cdot 10^{-4} = 1/4000\,,$$
$$A_{\text{FWHM}} = 4000\,.$$

$$\alpha = 90^\circ$$
$$\tan\beta = 1/2$$
$$\beta_1 = \beta_2 = 26.6^\circ$$
$$L = 2\rho_0$$

$$\delta_{\text{FWHM}} = |M_x \Delta x_{\text{E}}/D_x|$$

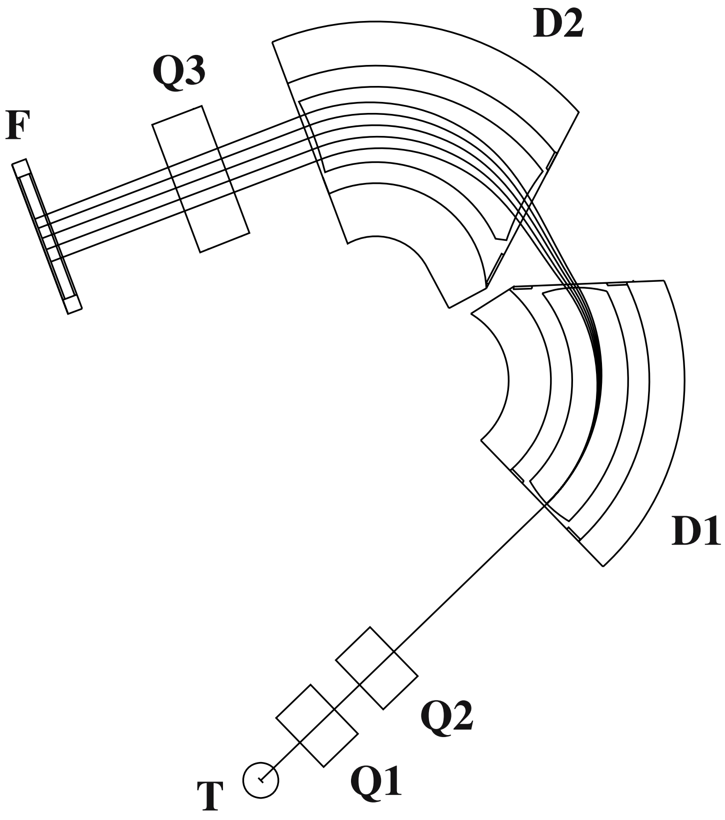
9. Ion Optical Systems

9.5 QQDQQ

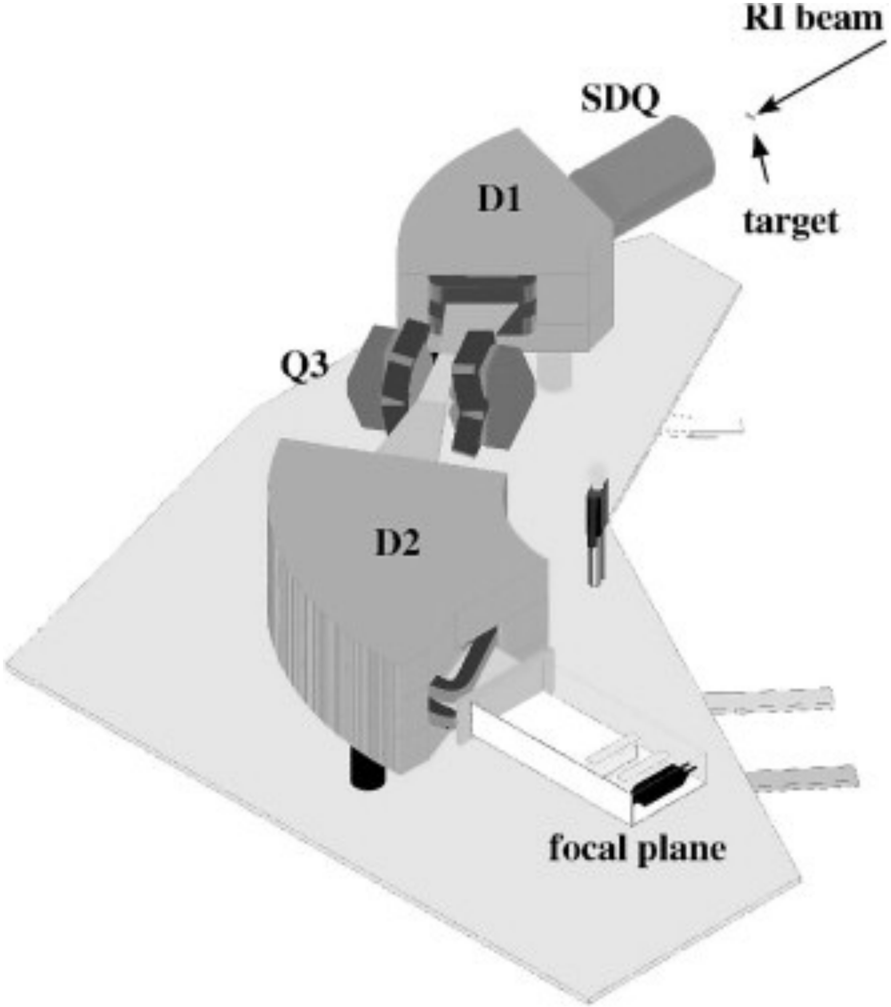


9. Ion Optical Systems

9.6 High-resolution spectrographs



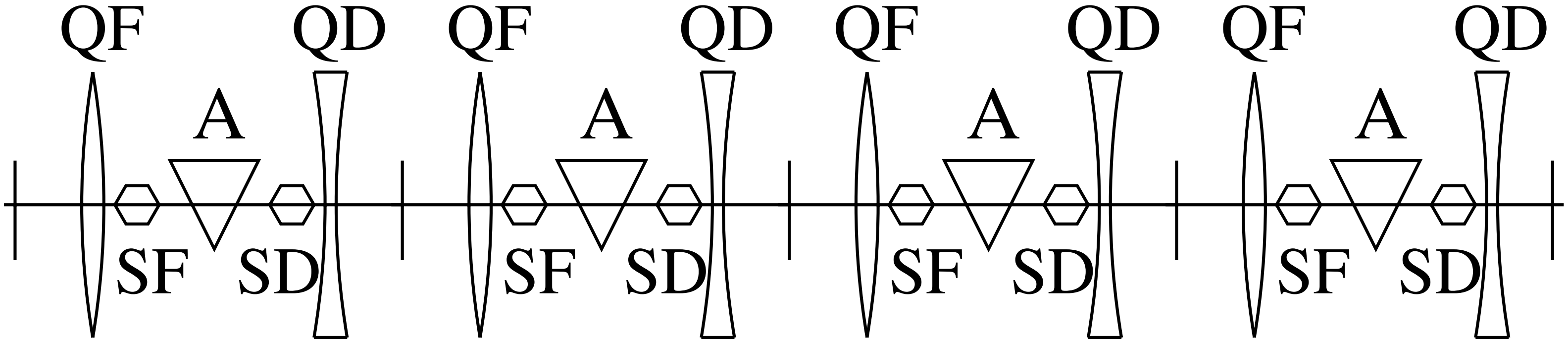
Big Karl at COSY, Juelich



SHARAQ at RIKEN, Saitama

9. Ion Optical Systems

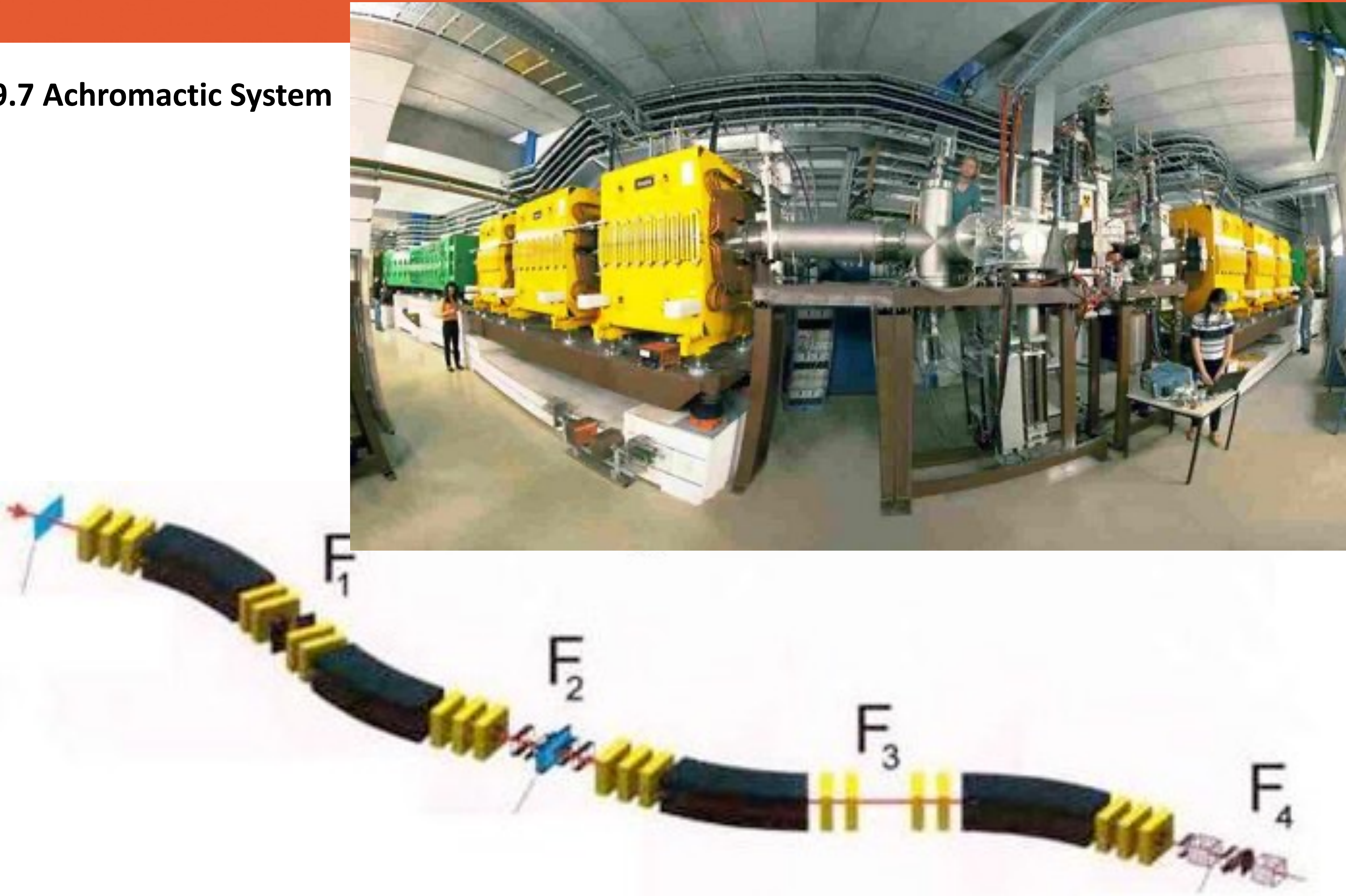
9.7 Achromactic System



Second-order achromatic system

9. Ion Optical Systems

9.7 Achromactic System

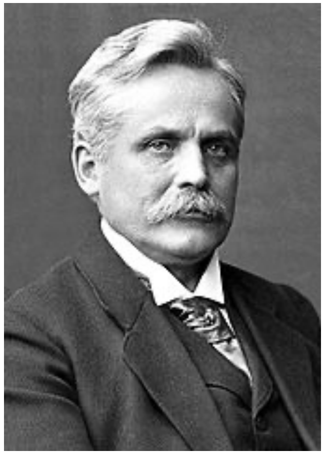


9. Ion Optical Systems

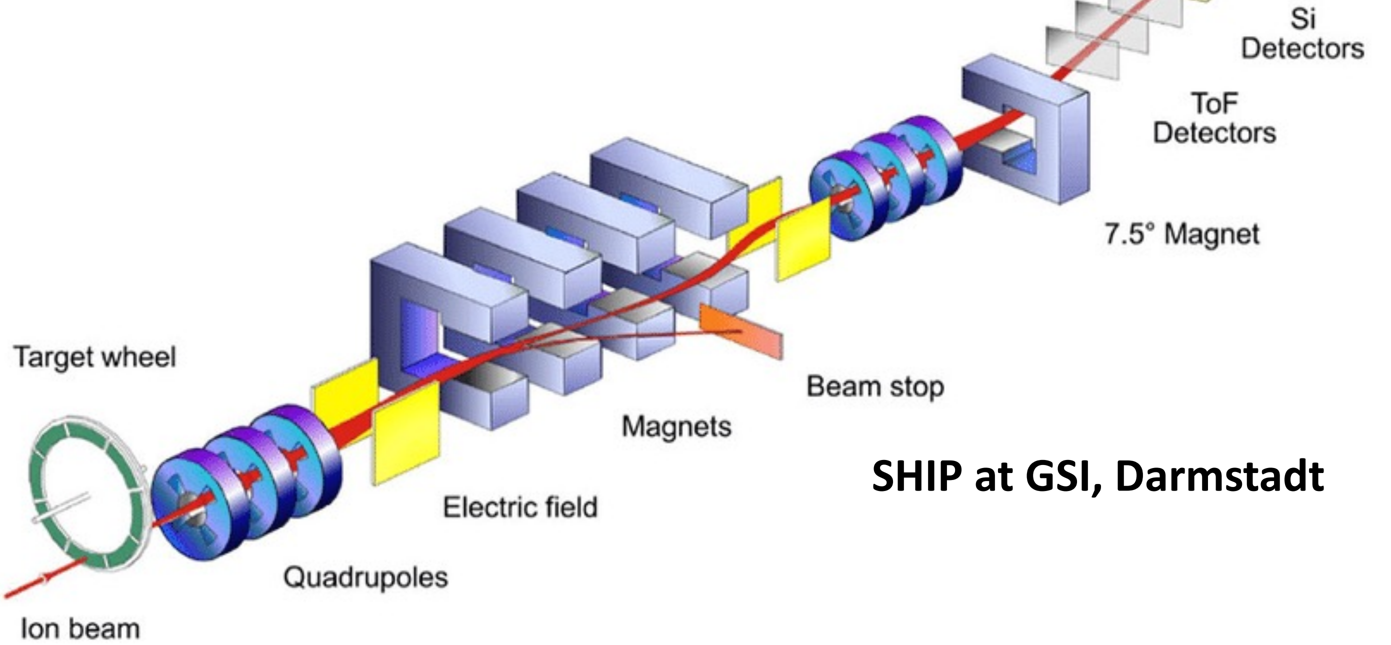
9.7 Velocity Filter / Wien filter

Topic:
Superheavy elements

Perpendicular arrangement of electric and magnetic fields



Wilhelm Wien
(1864-1928)

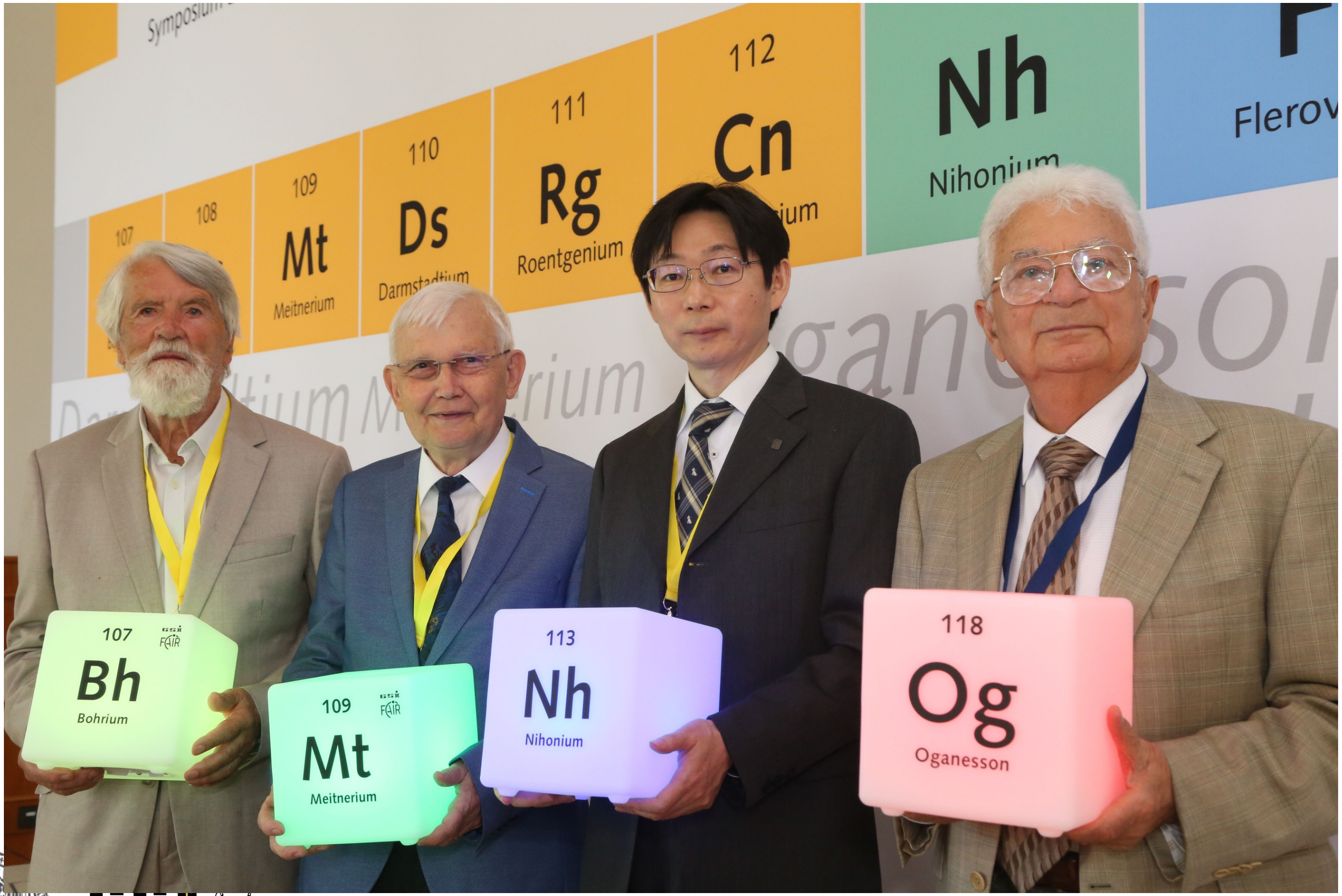


SHIP at GSI, Darmstadt

9. Ion Optical Systems

9.7 Velocity Filter / Wien filter

Topic:
Superheavy elements



10. Electrostatic Elements

Lenses, deflectors, drifts, accelerator sections ….

- Very similar description as for magnetic elements

Details – see Hinterberger

11. Transverse Beam Dynamics

11.2 Hill's equations (linear approximation)

$$\boxed{\begin{aligned} \frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + k_x(s)x &= \frac{1}{\rho_0(s)}\frac{\Delta p}{p_0}\,, \\ \frac{\mathrm{d}^2 y}{\mathrm{d}s^2} + k_y(s)y &= 0\,. \end{aligned}}$$

①

First we assume $\frac{\Delta p}{p} = 0$ - no dispersion

Monochromatic/monoenergetic beam

Result: Oscillation along the reference orbit $\vec{s}$ with variable amplitude $a\sqrt{\beta(s)}$

Wave number: $1/\beta(s)$

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

11. Transverse Beam Dynamics

11.2 Twiss matrix

Formal solution of ①

Motion of a particle obviously depends on the start values *[x(s), x'(s), y(s), y'(s)]* at a given *s*

Let us assume motion in one plane only *[y(s), y'(s)]*

- Special matrix

$$\boxed{M = R(s + C)}$$

Circumference

11. Transverse Beam Dynamics

11.2 Twiss matrix

- **General form**

$$M = \begin{bmatrix} \cos(\mu) + \alpha\sin(\mu) & \beta\sin(\mu) \\ -\gamma\sin(\mu) & \cos(\mu) - \alpha\sin(\mu) \end{bmatrix}$$

$$\Downarrow$$

$$\det(M) = 1$$

11. Transverse Beam Dynamics

11.2 Twiss matrix

- **General form**

$$M = \cos(\mu) \times \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} + \sin(\mu) \times \underbrace{\begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}}_{J}$$

$\alpha, \ \beta, \ \gamma$ - Twiss parameters

11. Transverse Beam Dynamics

11.2 Twiss matrix

$$\det(M) = 1 \quad \Rightarrow$$

$$\det(J) = \beta\gamma - \alpha^2 = 1; \quad J^2 = J \times J = -I$$

- **Stability criterion**

  For many turnns N

$$\boxed{\begin{aligned} M^N = (I \cdot \cos(\mu) + J \cdot \sin(\mu))^N = \\ = I \cdot \cos(N\mu) + J \sin(N\mu) \end{aligned}}$$

Similar to Moivre formula:

$$(\cos(\mu) + i \sin(\mu))^N = \cos(N\mu) + i \sin(N\mu)$$

11. Transverse Beam Dynamics

- **Stability criterion**

  For many turnns N

$$\boxed{\begin{aligned} M^N &= (I\cdot\cos(\mu) + J\cdot\sin(\mu))^N = \\ &= I\cdot\cos(N\mu) + J\sin(N\mu) \end{aligned}}$$

$$\Downarrow$$

$$\mu \text{ must be real}$$

$$|\cos(\mu)| \leq 1$$

$$|Tr(M)| \leq 2$$

Trace-theorem

11. Transverse Beam Dynamics

- Twiss parameters

Assume Tr(M)<2

$$\cos(\mu) = \frac{1}{2}Tr(M) = \frac{1}{2}(M_{11} + M_{22})$$

$$\sin(\mu) = sign(M_{12})\sqrt{(1 - \cos^2(\mu))}$$

$$\text{Real} \left\{ \begin{aligned} \beta &= \frac{M_{12}}{\sin(\mu)} \\ \alpha &= \frac{M_{11} - M_{22}}{2\sin(\mu)} \\ \gamma &= -\frac{M_{21}}{\sin(\mu)} \end{aligned} \right.$$

11. Transverse Beam Dynamics

- Twiss parameters

Since Matrix M depends on the starting values, Twiss parameters are functions of s

$$\left.\begin{array}{l}\beta(s)\\ \alpha(s)\\ \gamma(s)\end{array}\right\}$$

Optical functions,
Betatron functions,
Amplitude functions,
Lattice functions

Goal: describe machine!

μ – independent of *s*, machine parameter defined by matrix *M* to 2π

Phase advance of $\beta(s)$ per revolution

11. Transverse Beam Dynamics

11.3 Solution of Hill‘s equations

$$y'' + k_y(s)y = 0$$

(6.24) in Hinterberger

$$y(s) = a\sqrt{\beta(s)}\cos[\Psi(s) + \Psi_0]$$ ②

$a, \quad \Psi_0$ are defined for each particle, which are the amplitude and the phase of oscillations, respectively

$a\sqrt{\beta(s)}$ Variable amplitude along $\vec{s}$

$\frac{d\Psi}{ds} = \frac{1}{\beta(s)}$ Variable wave number ($\lambda(s) = 2\pi\beta(s)$ - wavelength)

11. Transverse Beam Dynamics

11.4 Phase shift / phase advance

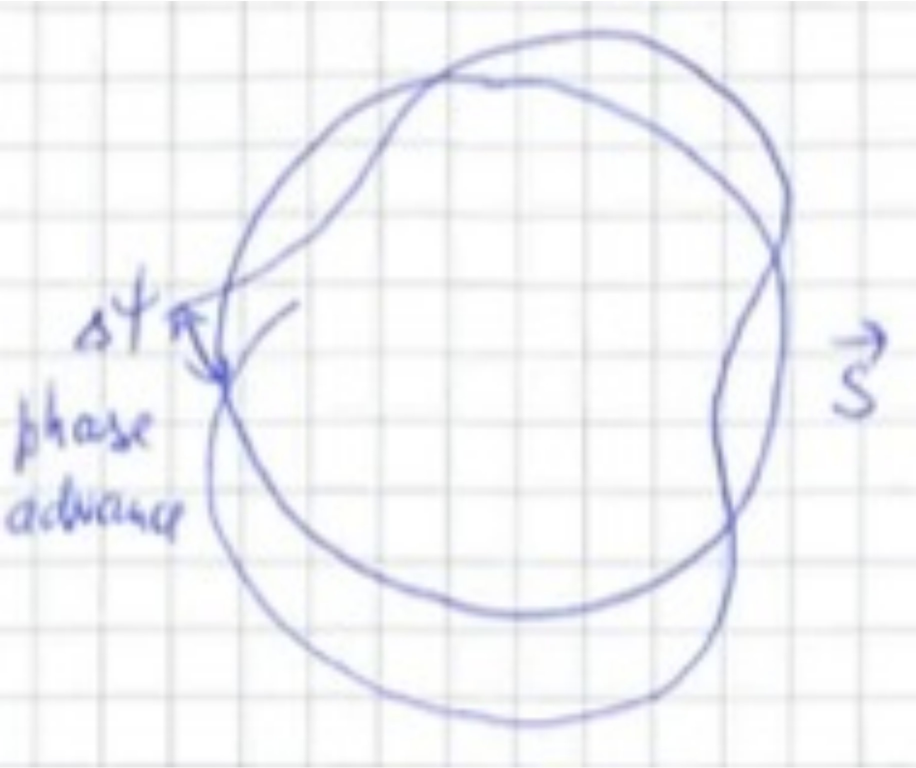
$$\mu = \int_s^{s+C} \frac{d\bar{s}}{\beta(\bar{s})} = \oint \frac{d\bar{s}}{\beta(\bar{s})}$$

Number of betatron oscillations per revolution, betatron tune

$$Q = \frac{\mu}{2\pi} = \frac{1}{2\pi} \oint \frac{d\bar{s}}{\beta(\bar{s})}$$

11. Transverse Beam Dynamics

11.4 Phase shift / phase advance



If $\Delta\Psi = 0$

Particle moves always on the same orbit

!!! RESONANCE !!!

Disturbances will be multiplied

!!! Instability !!!

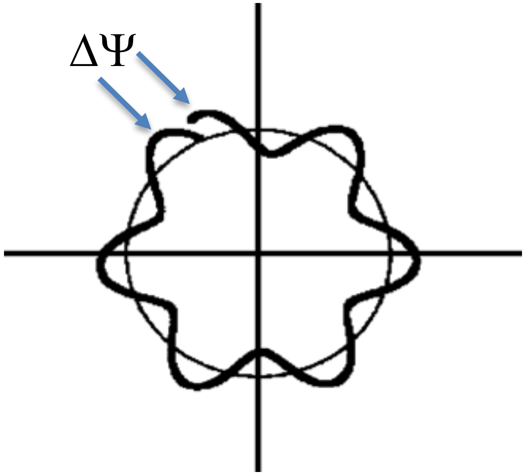
11. Transverse Beam Dynamics

11.4 Phase shift / phase advance

μ – independent of *s*, machine parameter defined by matrix *M* to 2π

Phase advance of $\beta(s)$ per revolution

$$\mu = \int_s^{s+C} \frac{d\bar{s}}{\beta(\bar{s})} = \oint \frac{d\bar{s}}{\beta(\bar{s})}$$



Number of betatron oscillations per revolution, betatron tune

$$Q = \frac{\mu}{2\pi} = \frac{1}{2\pi} \oint \frac{d\bar{s}}{\beta(\bar{s})}$$

If $\Delta\Psi = 0$

Particle moves always on the same orbit

!!! RESONANCE !!!

Disturbances will be multiplied

!!! Instability !!!

11. Transverse Beam Dynamics

11.5 Courant-Snyder Invariant

$$y(s) = a\sqrt{\beta(s)}\cos[\Psi(s) + \Psi_0] \qquad ②$$

What is **a**?

After lengthy and tedious mathematical efforts one can rewrite ② as:

$$\boxed{\frac{y^2}{\beta} + \frac{(\alpha y + \beta y')^2}{\beta} = a^2 = \epsilon}$$

Courant-Snyder Invariant

11. Transverse Beam Dynamics

11.5 Courant-Snyder Invariant

$$\boxed{\frac{y^2}{\beta} + \frac{(\alpha y + \beta y')^2}{\beta} = a^2 = \epsilon}$$

$$\boxed{\gamma y^2 + 2\alpha y y' + \beta y'^2 = a^2 = \epsilon}$$

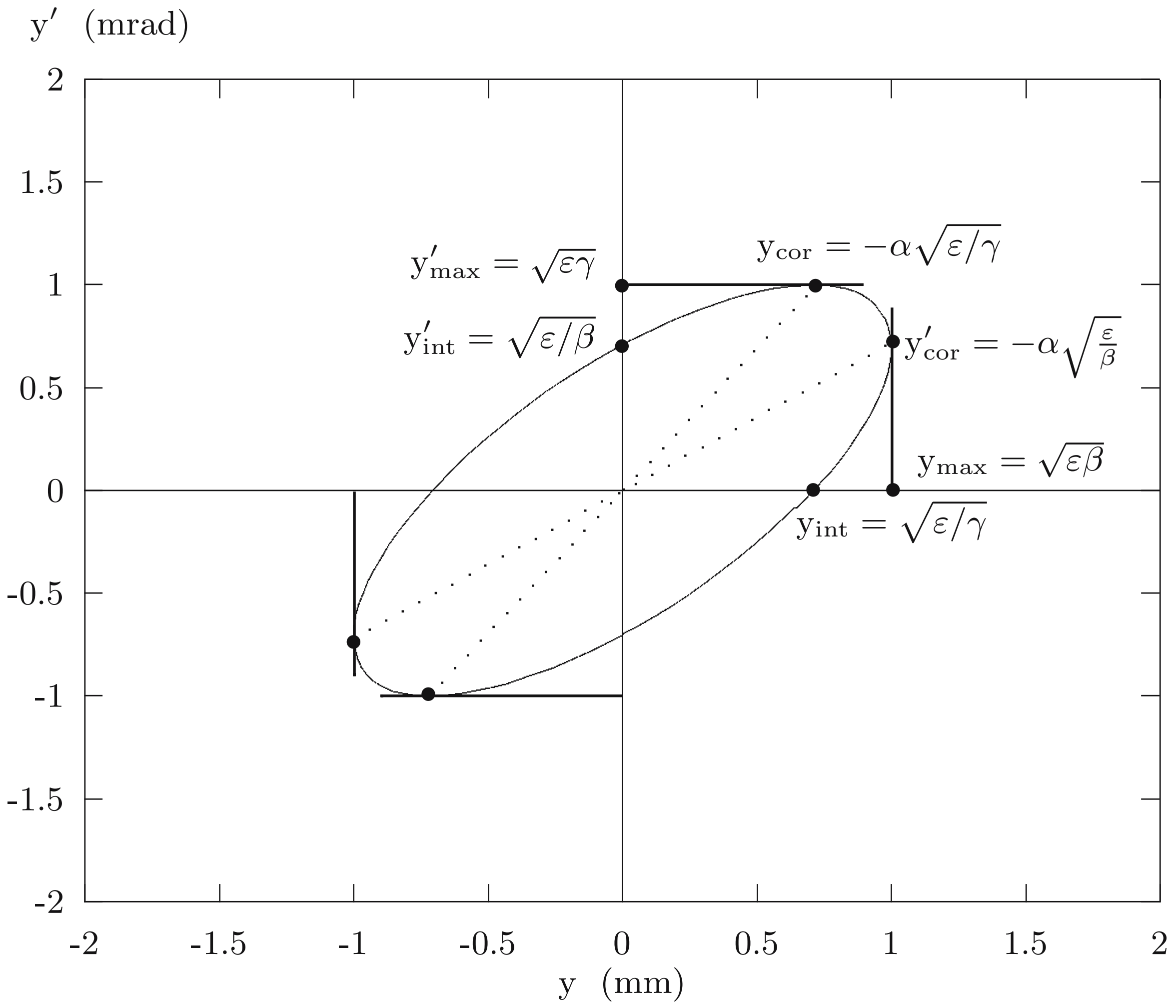
11. Transverse Beam Dynamics

11.5 Courant-Snyder Invariant

Machine Ellipse

Area

$$E = \pi a^2 = \pi \epsilon = const$$



11. Transverse Beam Dynamics

11.5 Courant-Snyder Invariant

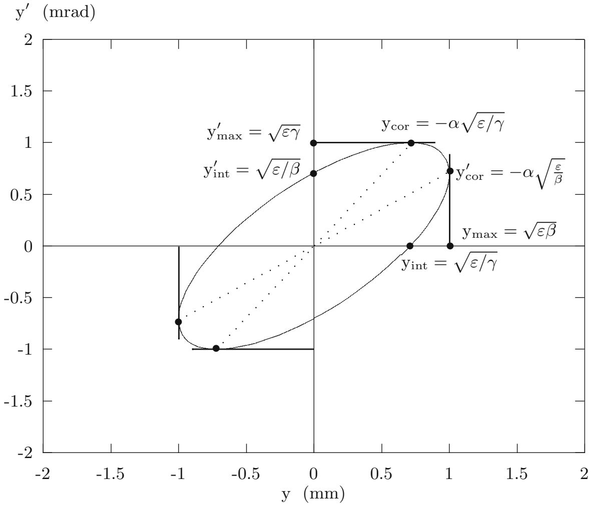
1. A particle with coordinates *(y,y‘)* propagates along a changing ellipse

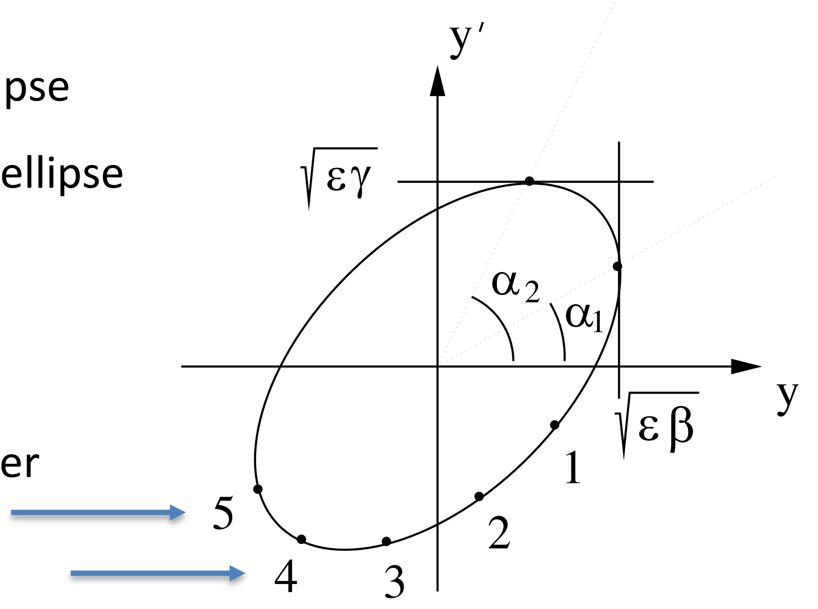
2. The area of the ellipse is constant and is defined by ***a***

3. The shape of the ellipse is defined by the machine itself via $\alpha(s)$, $\beta(s)$, $\gamma(s)$ functions (machine ellipse)

4. Plotting *(y,y‘)* after each revolution gives an ellipse

5. All particles with smaller ***a*** are enclosed in the ellipse

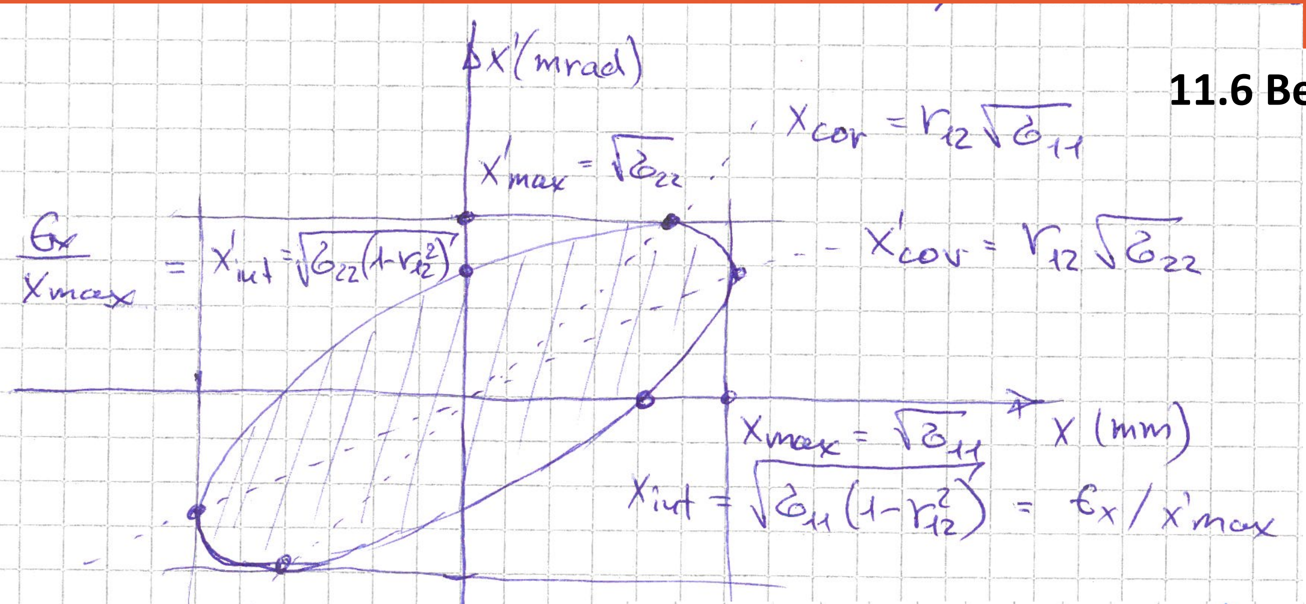


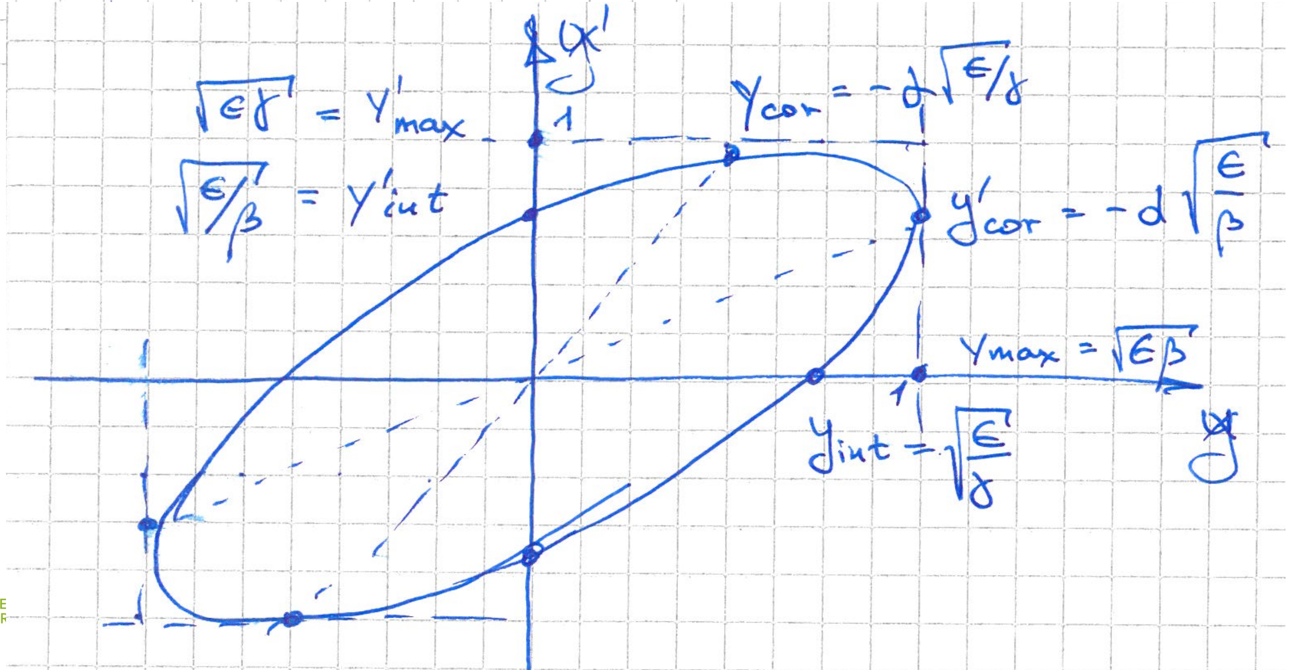


Revolution number

11. Transverse Beam Dynamics

11.6 Beam ellipse & machine ellipse





8. Beam Properties

Phase ellipse

Density distribution in *(x,x‘)* plane $\rho(x, x')$ **can typically presented with an ellipse**

$$\sigma_x = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

$$\sigma_{12} = \sigma_{21}$$

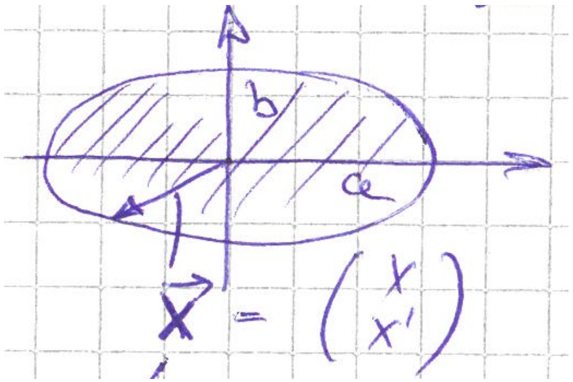
$$\det(\sigma_x) > 0$$

Phase ellipse:

$$\mathbf{X}^{\mathbf{T}} \sigma_x^{-1} \mathbf{X} = 1$$ ①

⇙

$$\sigma_x^{-1} = \frac{1}{\det(\sigma_x)} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}$$



$$\frac{x^2}{a^2} + \frac{x'^2}{b^2} = 1$$

Vector from origin to ellipse boundary

8. Beam Properties

Emittance

① $$\det(\sigma_x) = \sigma_{22}x_1^2 - 2\sigma_{12}x_1x_2 + \sigma_{11}x_2^2 = \epsilon_x^2$$

Area of the ellipse

Emittance: $$E_x = \pi\epsilon_x = \pi\sqrt{\det(\sigma_x)} = \pi\sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}$$

$$1\ [\mathrm{mm}]\cdot[\mathrm{mrad}] = 1\cdot 10^{-6}\ [\mathrm{m}]\cdot[\mathrm{rad}]$$

Often this is emittance

Maximal values:

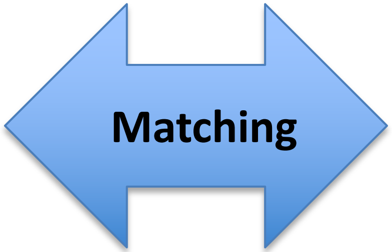
$$x_{\max} = \sqrt{\sigma_{11}} \qquad x'_{\max} = \sqrt{\sigma_{22}}$$

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

11. Transverse Beam Dynamics

11.6 Beam ellipse & machine ellipse

Machine ellipse	Beam ellipse
Defined by the machine (lattice, ion-optical settings, apertures)	Can be very different from machine ellipse (e.g. injection)



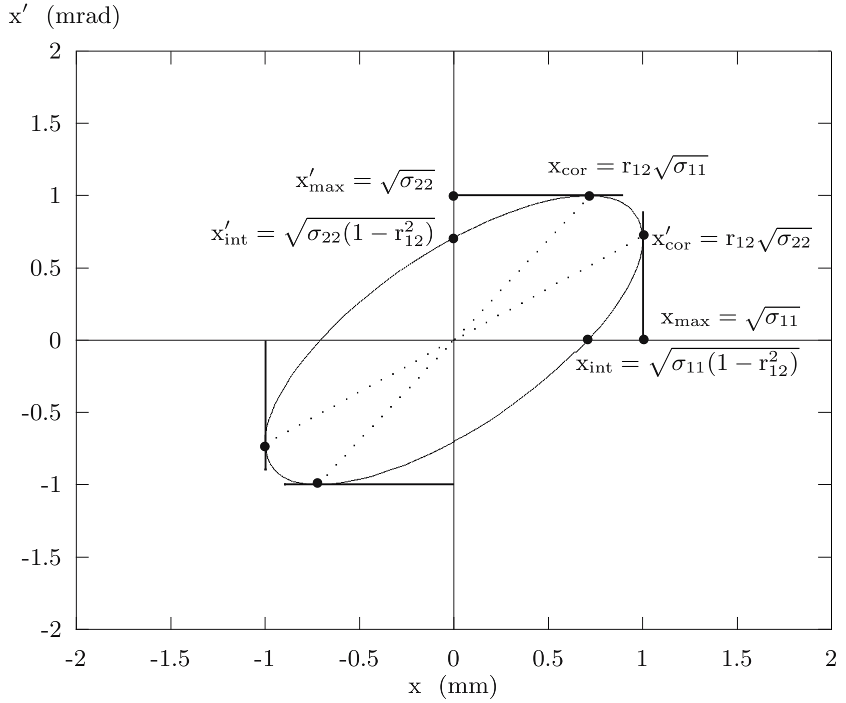
Matching

11. Transverse Beam Dynamics

11.6 Beam ellipse & machine ellipse

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \epsilon_x \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} = \begin{pmatrix} \epsilon_x\beta_x & -\epsilon_x\alpha_x \\ -\epsilon_x\alpha_x & \epsilon_x\gamma_x \end{pmatrix}$$

$\sqrt{\epsilon_x\beta_x}$	Maximal spatial extension
$\sqrt{\epsilon_x\gamma_x}$	Maximal angular extension



11. Transverse Beam Dynamics

11.7 RMS Emittance

Definition:

$$\epsilon_x^{1\sigma} = \sqrt{\langle x^2 \rangle \, \langle {x'}^2 \rangle - \langle x x' \rangle^2}$$

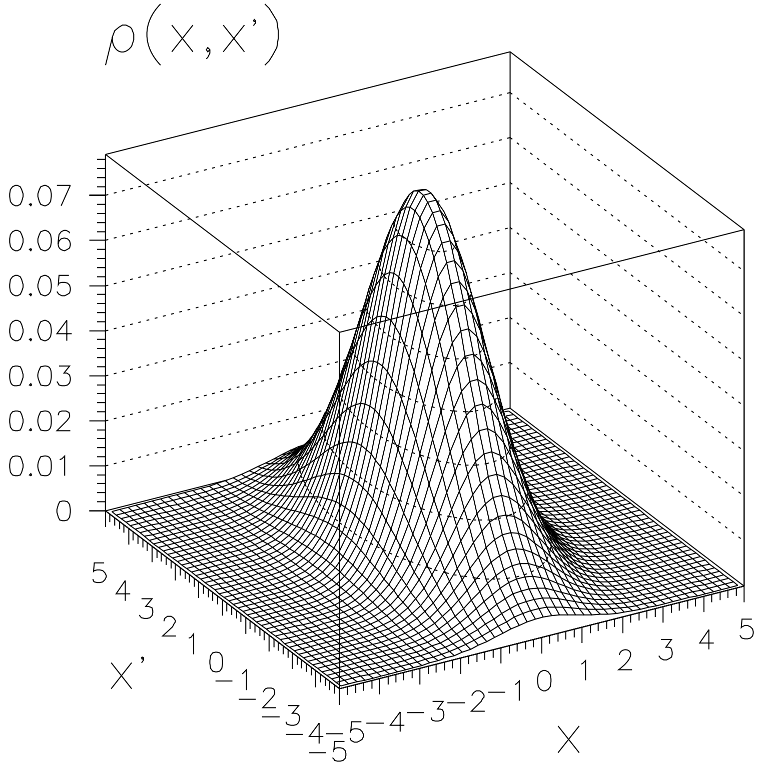
$$\epsilon_y^{1\sigma} = \sqrt{\langle y^2 \rangle \, \langle {y'}^2 \rangle - \langle y y' \rangle^2}$$

Expressed in Twiss parameteres

$$\epsilon_x^{1\sigma} = \frac{1}{N} \sum_i \epsilon_{x,i} = \frac{1}{N} \sum_i \gamma_x x_i^2 + 2\alpha_x x_i x_i' + \beta_x {x_i}'^2$$

$$\epsilon_y^{1\sigma} = \frac{1}{N} \sum_i \epsilon_{y,i} = \frac{1}{N} \sum_i \gamma_y y_i^2 + 2\alpha_y y_i y_i' + \beta_y {y_i}'^2$$

Machine parameters

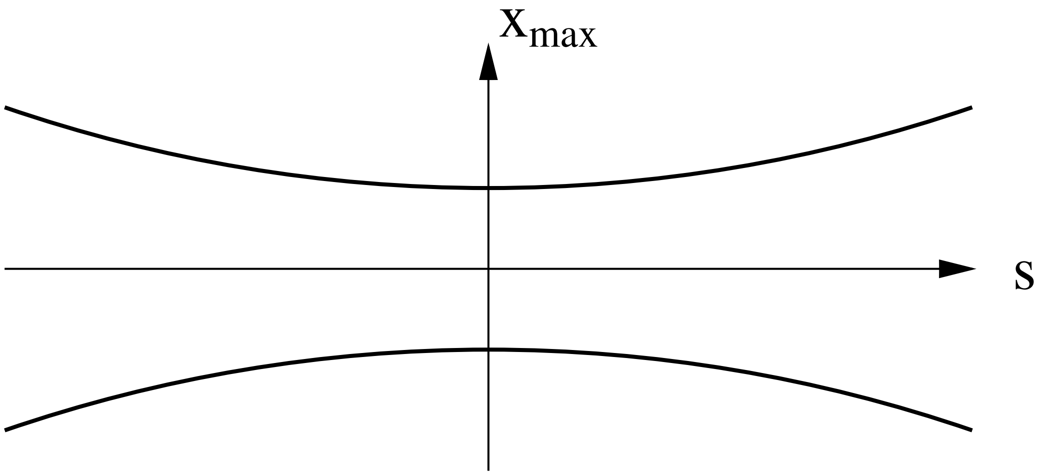


8. Beam Properties

Beam Envelope

Beam envelope (RMS envelope)

$$x_{\mathrm{max}}(s) = \sqrt{\sigma_{11}(s)}$$



Beam waist („Strahltaille“) / focus

$r_{12} < 0$ $r_{12} > 0$

11. Transverse Beam Dynamics

11.8 Beam envelope (RMS envelope)

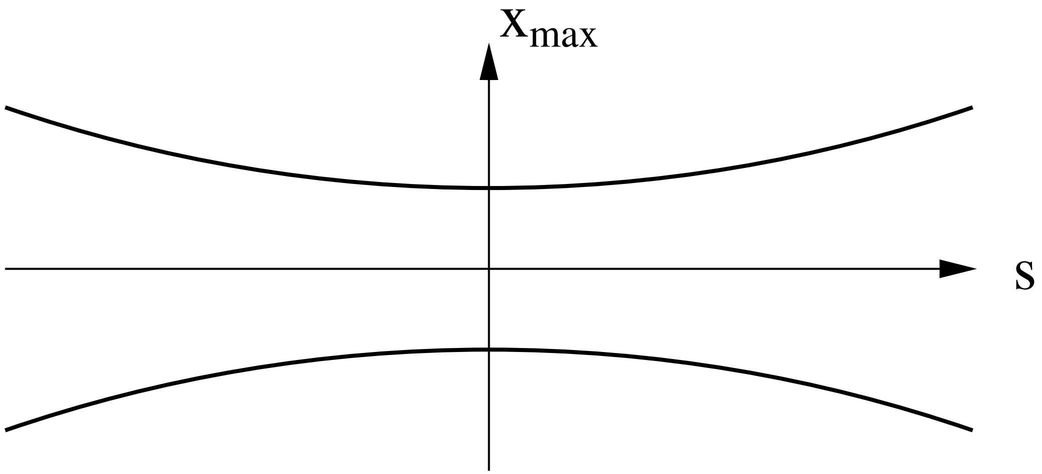
$$x_{\max}(s) = \sqrt{\sigma_{11}(s)}$$

$$x_{\max}(s) = \sqrt{\epsilon}\sqrt{\beta(s)}$$

$$x'_{\max}(s) = \sqrt{\epsilon}\sqrt{\gamma(s)}$$

Beam parameter

Machine parameter



Beam waist („Strahltaille“) / focus

$r_{12} < 0$

$r_{12} > 0$

11. Transverse Beam Dynamics

11.9 Machine Acceptance

Maximum beam emittance which can be transmitted through the machine

$$\epsilon_{\mathrm{max}} = \frac{x_{\mathrm{max}}^2}{\beta}$$

Acceptance/Admittance $A = \pi\epsilon_{\mathrm{max}}$

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

11. Transverse Beam Dynamics

11.8 Machine ellipse

Ellipses defined by α, β, γ functions are nothing else than eigenellipses σ_e of matrix *M* at easch *s*!

Gordon,M.M.: Orbit properties of the isochronous cyclotron ring with radial sectors, Annals of Physics **3** (1968) 571

$$\sigma_e = M \sigma_e M^T$$

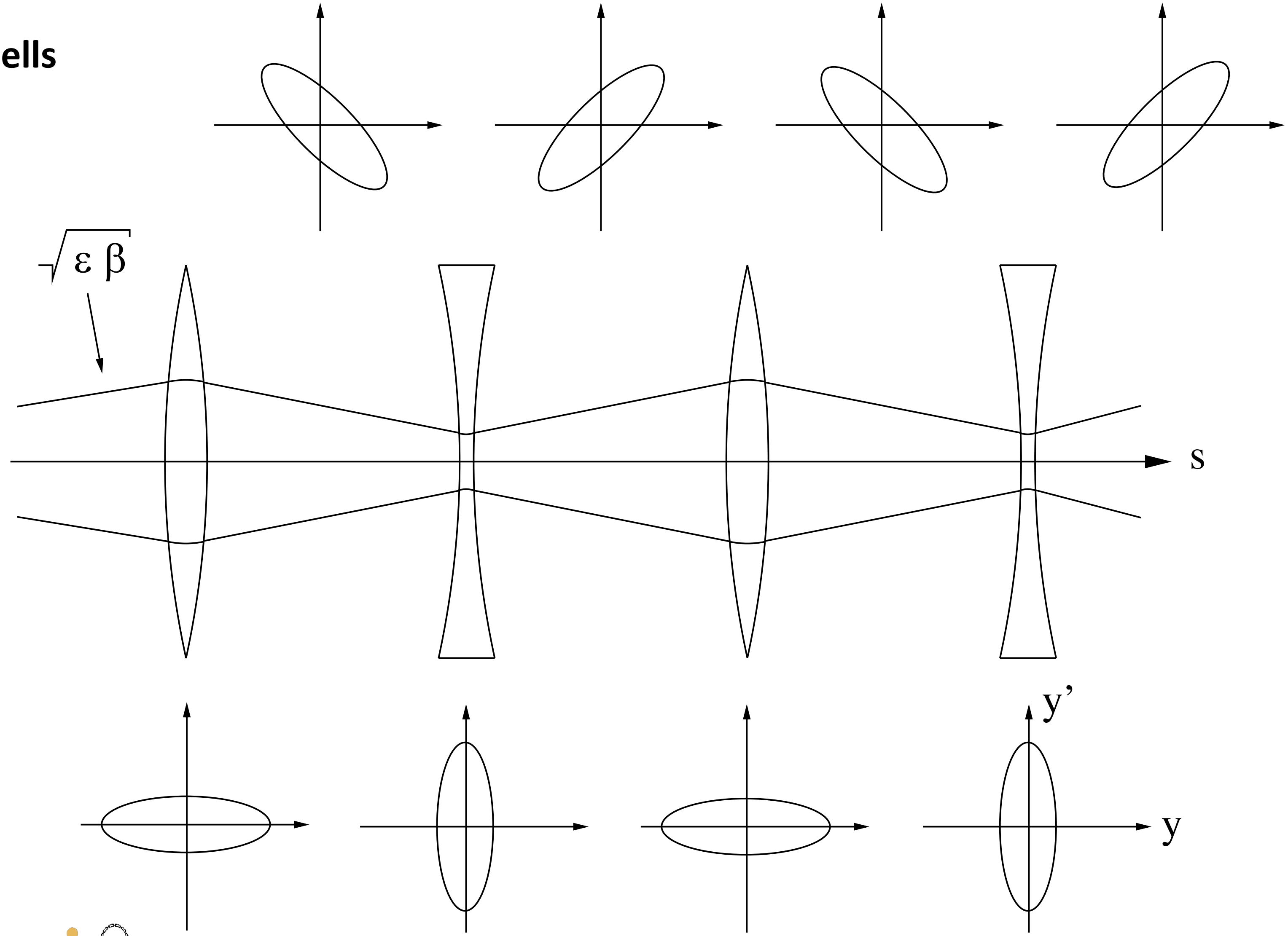
$$\sigma_e(s + C) = \sigma_e(s)$$

Eigenellipses are defined at each s via Twiss matrix

$$\sigma_\mathrm{e}(s) = \epsilon \begin{pmatrix} \beta(s) & -\alpha(s) \\ -\alpha(s) & \gamma(s) \end{pmatrix}$$

11. Transverse Beam Dynamics

FODO Cells



11. Transverse Beam Dynamics

11.8 Transformation of Twiss parameters

$$R = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix}$$

$$\sigma = R\sigma_0 R^{\mathrm{T}}$$

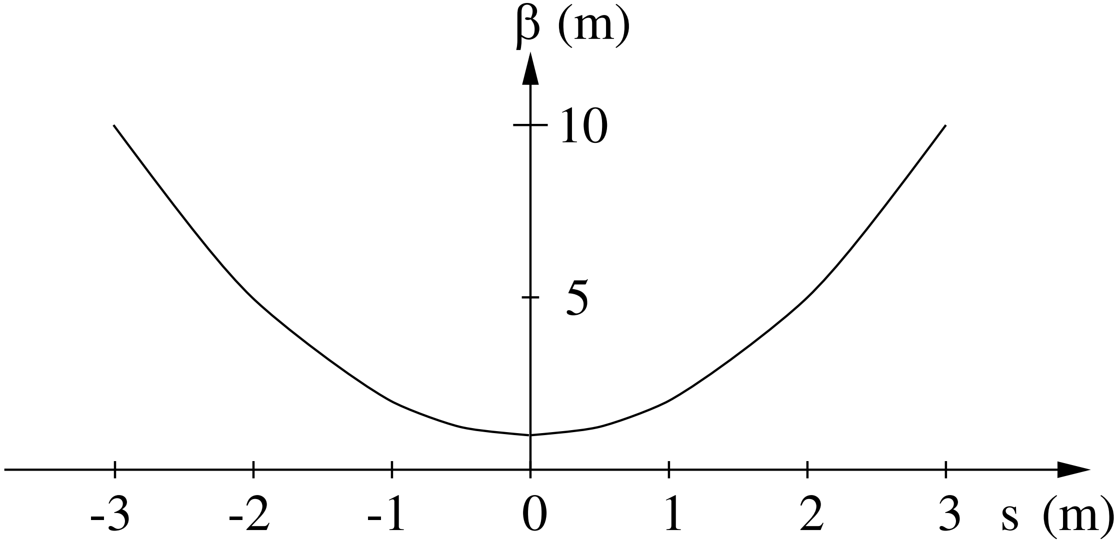
$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = R \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} R^{\mathrm{T}}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

11. Transverse Beam Dynamics

11.8 Transformation of Twiss parameters

Example: Drift



Beam waist

$$\alpha_0 = 0$$

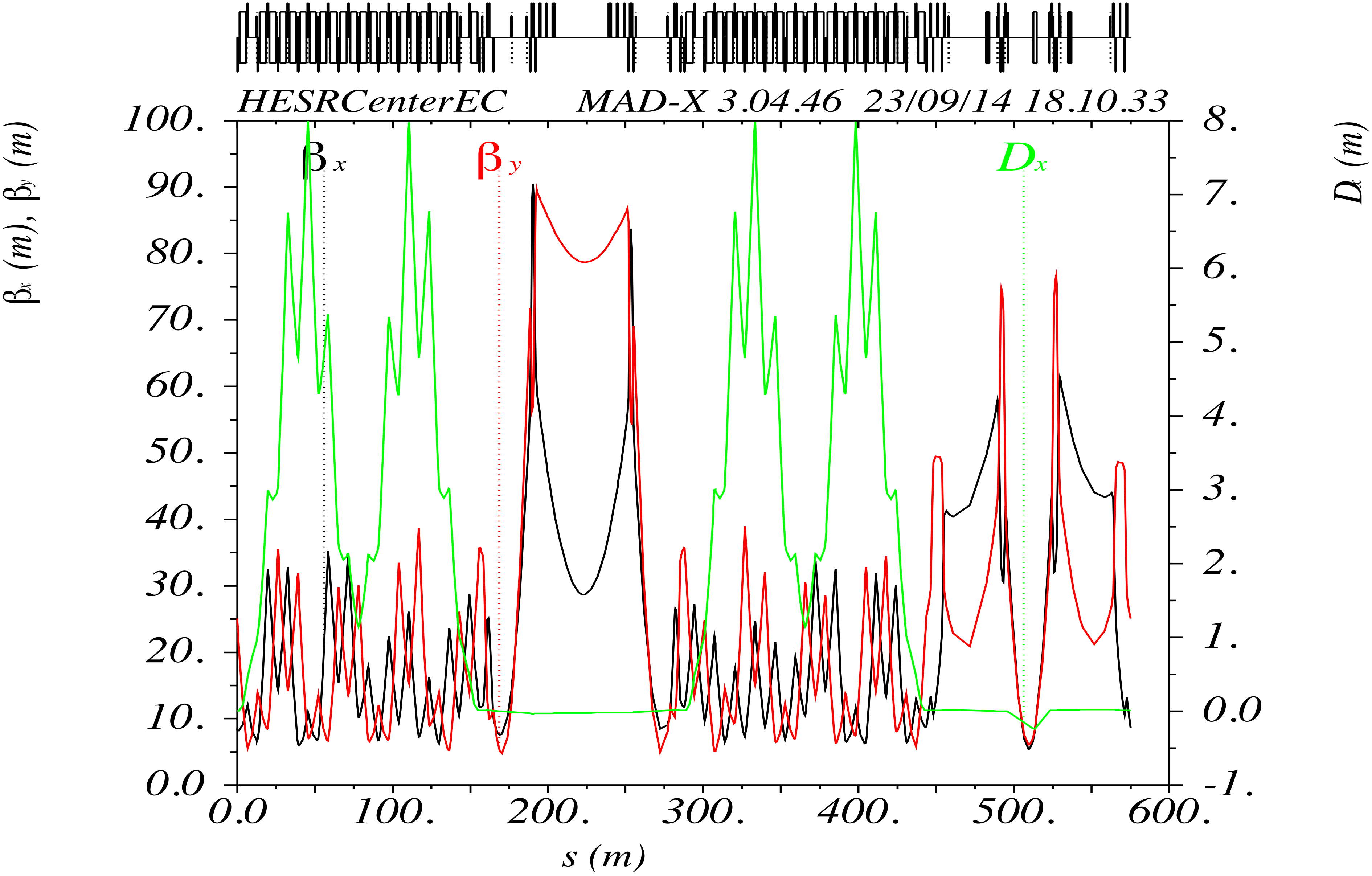
$$\gamma_0 = 1/\beta_0$$

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

In general:

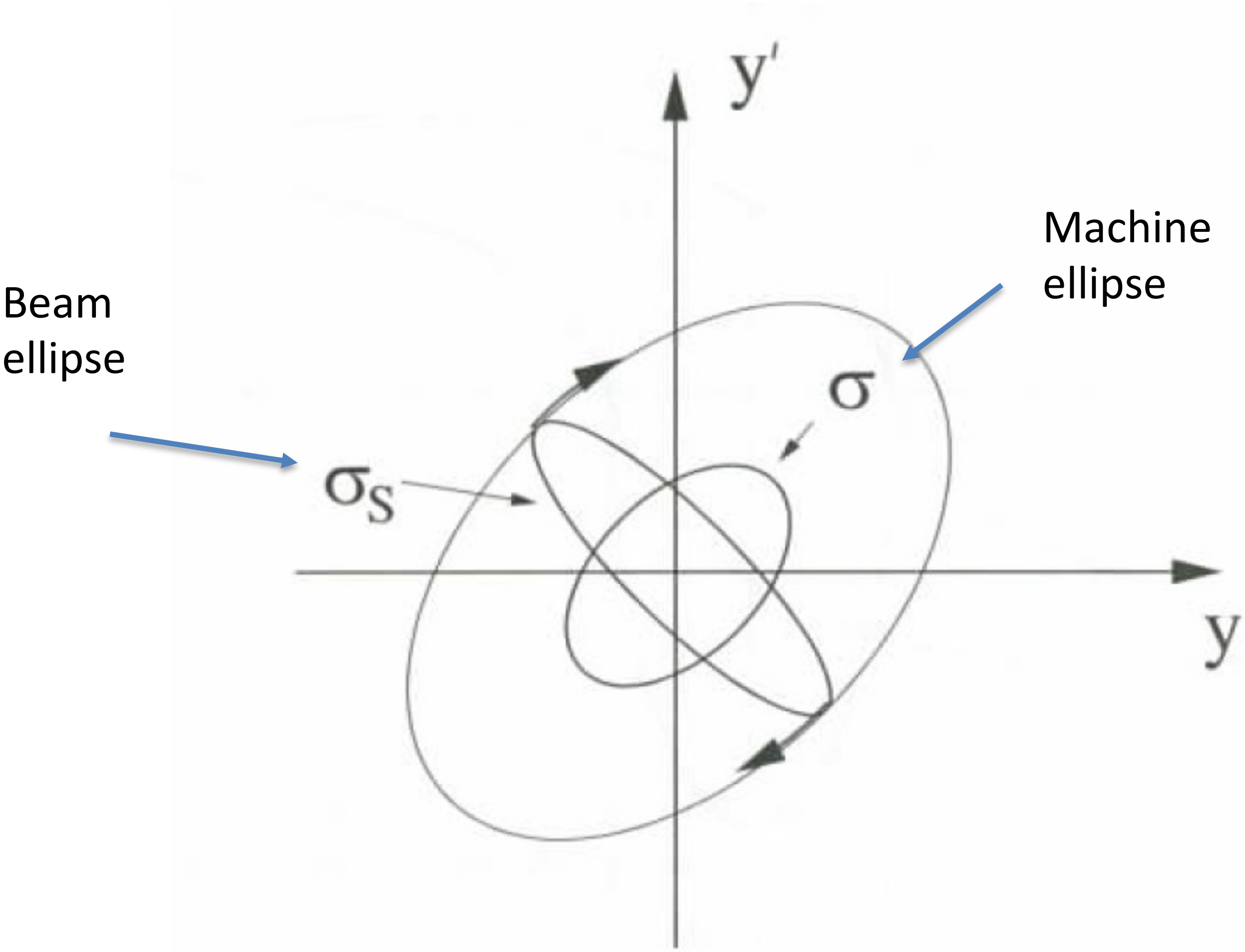
$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

Typical example (HESR at FAIR)



11. Transverse Beam Dynamics

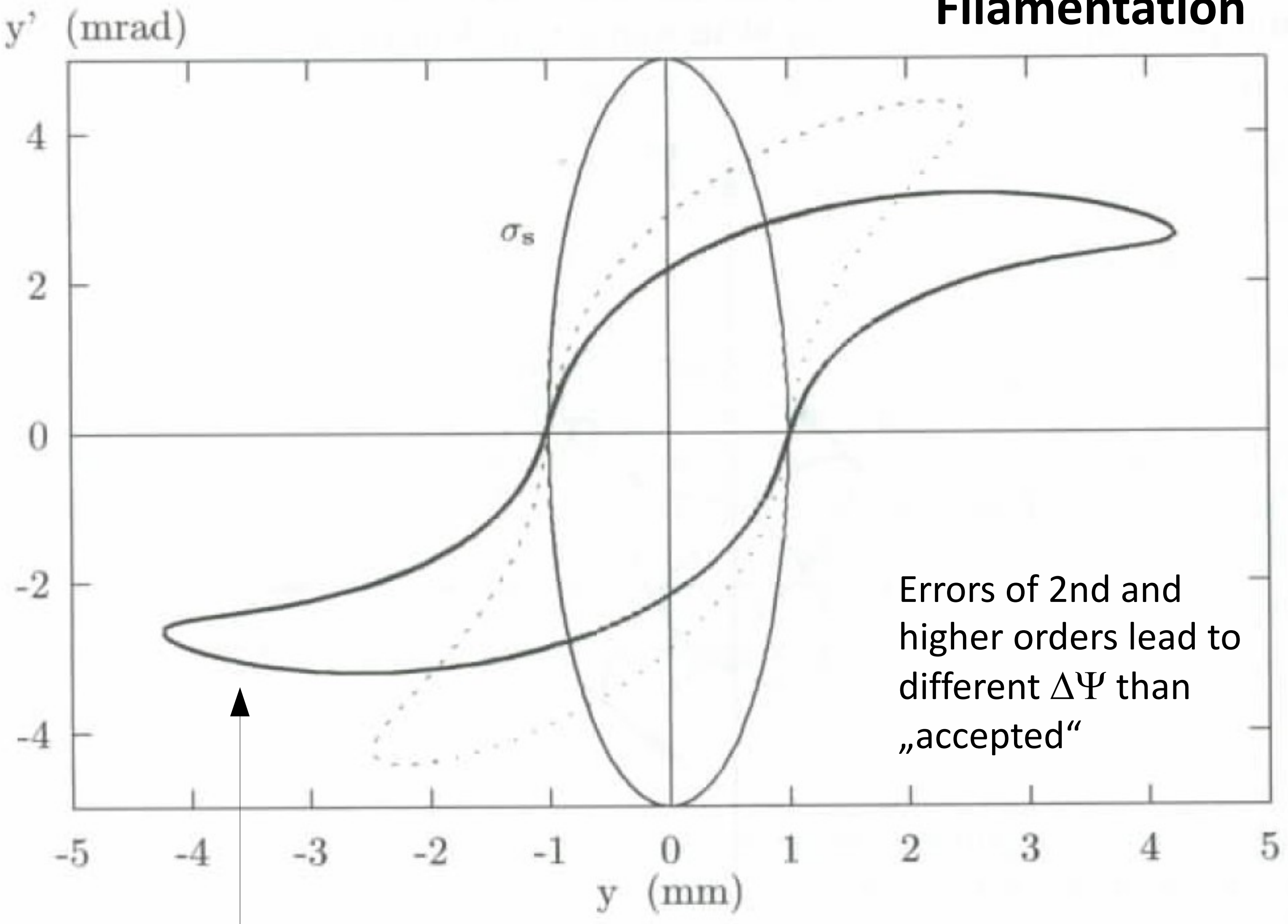
11.9 Matching beam and machine



11. Transverse Beam Dynamics

11.9 Matching beam and machine

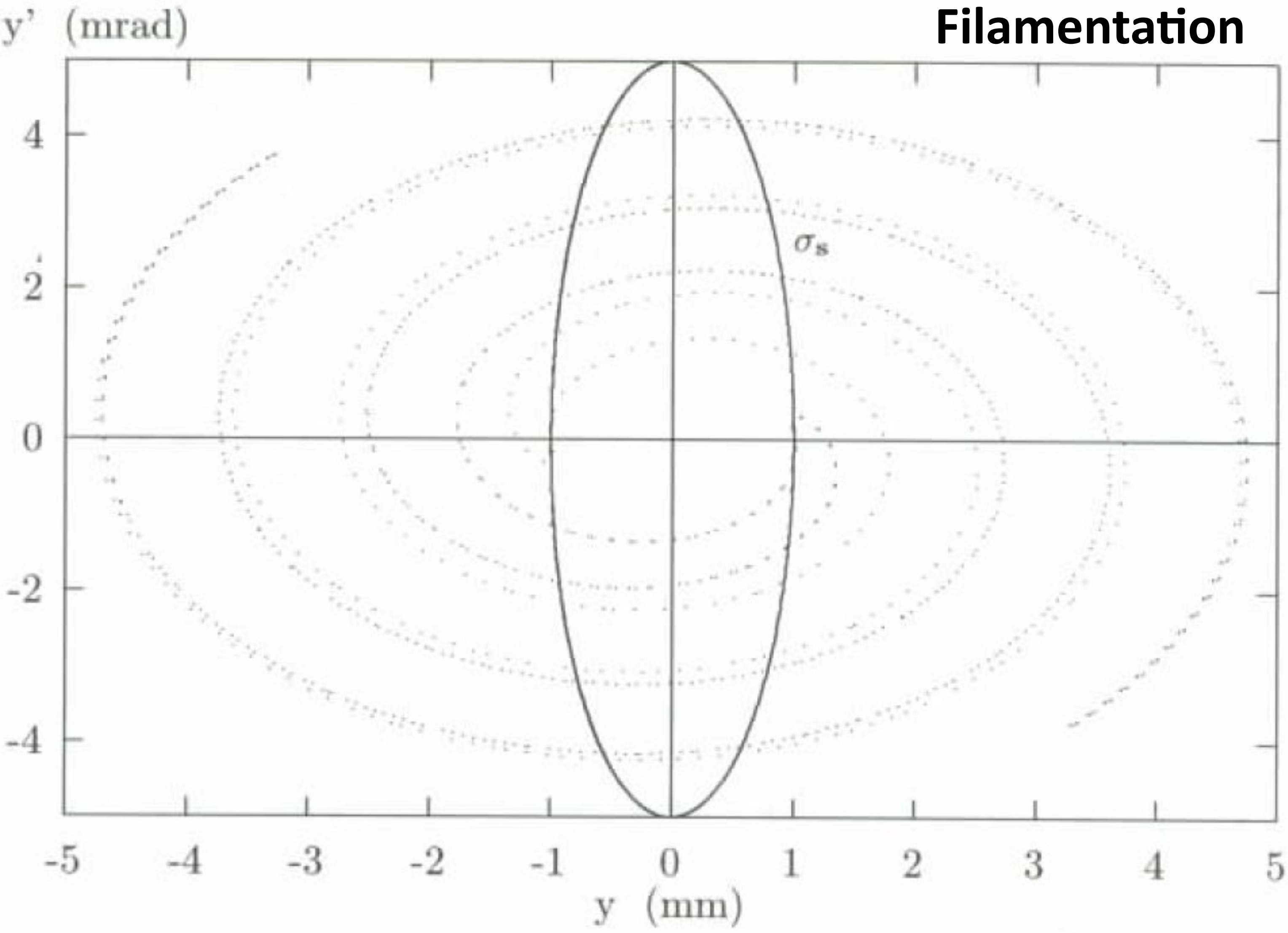
Filamentation



Errors of 2nd and higher orders lead to different $\Delta\Psi$ than „accepted“

11. Transverse Beam Dynamics

11.9 Matching beam and machine



Filamentation

Spiralling