Introduction to Accelerator Physics

Yuri A. Litvinov
y.litvinov@gsi.de

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Physikalisches Institut der Universität Heidelberg

Lecture Dates

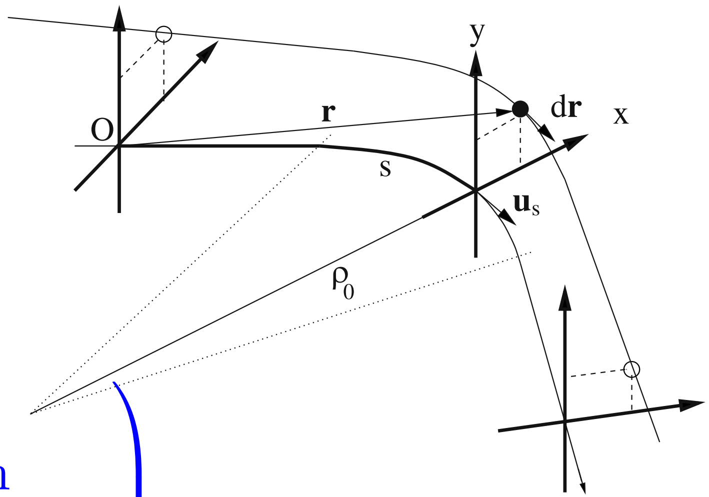
https://uebungen.physik.uni-heidelberg.de/vorlesung/20222/1611/lecture

Date	Topic
19.10.2022	Introduction and basic definitions
26.10.2022	Accelerating structures
02.11.2022	Accelerator Components
09.11.2022	Optics with magnets (1)
16.11.2022	Optics with magnets (2)
23.11.2022	Equations of motion
30.11.2022	Phase ellipses and magneto-optical system / Transverse beam dynamics
07.12.2022	Transverse beam dynamics, beam stability / Longitudinal beam dynamics
14.12.2023	Phase space and beam cooling (Invitation)
11.01.2023	Space charge and beam-beam dynamics
18.01.2023	Physics at Storage Rings
25.01.2023	Physics at Colliders
01.02.2023	New accelerator technologies
08.02.2023	Student seminar
15.02.2023	reserve
22.02.2023	reserve

Wednesdays, 14:15-16:00

Summary of last lecture

Curvilinear coordinate system



Relative coordinates of each particle can be described with a six-dimensional vector

$$\mathbf{x}(s) = \begin{pmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{pmatrix} = \begin{pmatrix} \text{radial orbit deviation} \\ \text{radial direction deviation} \\ \text{axial orbit deviation} \\ \text{axial direction deviation} \\ \text{longitudinal deviation} \\ \text{longitudinal momentum deviation} \end{pmatrix}$$

Since $x,\ x',\ y,\ y',\ l,\ \delta l$ are small $\Longrightarrow$ units are [mm], [mrad], [promil]

$$\boxed{1\ \text{mrad} = 1\ \text{mm}/1\ \text{m}}$$

Linear approximation: x and y planes can be treated independently

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

Summary of the lecture

Curvilinear coordinate system

$$x'' + k_x x = h\delta$$
$$y'' + k_y y = 0$$

Transfer / transport / R-matrix

Equation of motion with/without dispersion (Hill equations)

$$x'' + k_x x = 0$$
$$y'' + k_y y = 0$$

Solution of equation of motion with/without dispersion

Transfer Matrix for

Drift

Qudrupole magnet

Dipole magnet (sector, weak and strong focusing)

Edge focusing

4. Solution of the Equation of Motion
(linear approximation)

Characteristic solutions for transverse motion (drift, quadrupole, dipole)

$k_x(s) > 0$ & $k_y(s) > 0$	$k_x(s) = k_y(s) = 0$	$k_x(s) < 0$ & $k_y(s) < 0$
$C_x(s) = \cos(\sqrt{k_x} s)$	$C_x(s) = 0$	$C_x(s) = \cosh(\sqrt{\lvert k_x\rvert} s)$
$S_x(s) = \frac{\sin(\sqrt{k_x} s)}{\sqrt{k_x}}$	$S_x(s) = s$	$S_x(s) = \frac{\sinh(\sqrt{\lvert k_x\rvert} s)}{\sqrt{\lvert k_x\rvert}}$
$d_x(s) = \frac{h}{k_x}[1 - \cos(\sqrt{k_x} s)]$	$d_x(s) = 0$	$d_x(s) = \frac{h}{\lvert k_x\rvert}[\cosh(\sqrt{\lvert k_x\rvert} s) - 1]$
$C_y(s) = \cos(\sqrt{k_y} s)$	$C_y(s) = 1$	$C_y(s) = \cosh(\sqrt{\lvert k_y\rvert} s)$
$S_y(s) = \frac{\sin(\sqrt{k_y} s)}{\sqrt{k_y}}$	$S_y(s) = s$	$S_y(s) = \frac{\sinh(\sqrt{\lvert k_y\rvert} s)}{\sqrt{\lvert k_y\rvert}}$

periodic

$$\cosh(x) = \frac{1}{2}(e^{x} + e^{-x})$$

$$\sinh(x) = \frac{1}{2}(e^{x} - e^{-x})$$

Transfer matrix (drfit, QPs, dipole)

Transfer matrix / transport matrix / R-matrix

$$\vec{x}(s) = \mathbf{R}(s)\vec{x}(0)$$

$$det(\mathbf{R}) = 1$$ (Liouville‘s theorem)

Accelerator structure: $$\mathbf{R} = \prod_i R_i$$

$$\mathbf{R_x} = \begin{bmatrix} (x|x) & (x|x') & 0 & 0 & 0 & (x|\delta) \\ (x'|x) & (x'|x') & 0 & 0 & 0 & (x'|\delta) \\ 0 & 0 & (y|y) & (y|y') & 0 & 0 \\ 0 & 0 & (y'|y) & (y'|y') & 0 & 0 \\ (l|x) & (l|x') & 0 & 0 & (l|l) & (l|\delta) \\ 0 & 0 & 0 & 0 & 0 & (\delta|\delta) \end{bmatrix}$$

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4. Solution of the Equation of Motion

(linear approximation)

Characteristic solutions for longitudinal motion

The corresponding Transfer matrix elements:

$$R_{51}(s) = (l|x_0) = -\int_0^s h(\bar{s}C_x(\bar{s})d\bar{s}$$

$$R_{52}(s) = (l|x_0') = -\int_0^s h(\bar{s}S_x(\bar{s})d\bar{s}$$

$$R_{55}(s) = (l|l_0) = 1$$

$$R_{56}(s) = (l|\delta) = -\int_0^s h(\bar{s}d_x(\bar{s})d\bar{s} + s/\gamma^2$$

For drifts, quadrupoles, (sextupoles, octupoles) ***h=0***

$$R_{51}(s) = (l|x_0) = 0$$

$$R_{52}(s) = (l|x_0') = 0$$

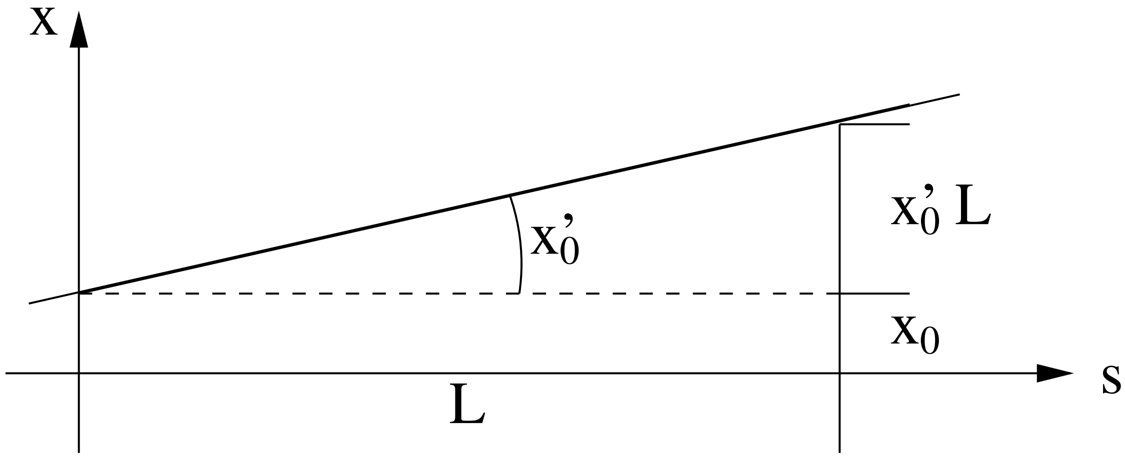
$$R_{55}(s) = (l|l_0) = 1$$

$$R_{56}(s) = (l|\delta) = +s/\gamma^2$$

4. Transfer matrix

- **Drift**

$$x = x_0 + Lx'_0$$
$$x' = x'_0$$
$$y = y_0 + Ly'_0$$
$$y' = y'_0$$
$$l = l_0 + \delta_0 \frac{L}{\gamma^2}$$
$$\delta = \delta_0$$



$$\mathbf{R}_{\mathrm{drift}} = \begin{bmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Transfer matrix

(linear approximation)

Abbildung	radial	axial
Punkt-zu-Punkt	$R_{12} = (x\|x') = 0$	$R_{34} = (y\|y') = 0$
Punkt-zu-Parallel	$R_{22} = (x'\|x') = 0$	$R_{44} = (y'\|y') = 0$
Parallel-zu-Punkt	$R_{11} = (x\|x) = 0$	$R_{33} = (y\|y) = 0$
Parallel-zu-Parallel	$R_{21} = (x'\|x) = 0$	$R_{43} = (y'\|y) = 0$
Ortsdispersion $= 0$	$R_{16} = (x\|\delta) = 0$	
Winkeldispersion $= 0$	$R_{26} = (x'\|\delta) = 0$	

From Hinterberger

7. Geometrical Optics

Thin lense approximation

Change of direction: $$\Delta x' = -\frac{1}{f} x_0$$

Focal length

Matrix equation:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\mathrm{R_x} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

Similar for y-direction

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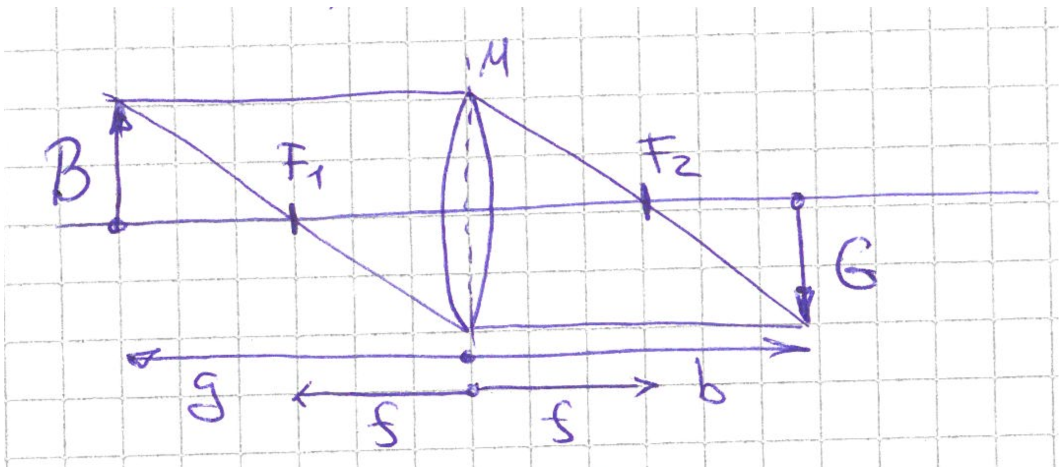
7. Geometrical Optics

Thin lense approximation — Point-2-Point Image

Convex lense

F-focus, image point



Condition: (optics)

$$\frac{1}{g}+\frac{1}{b}=\frac{1}{f}$$

Matrix representation

$$R_x=\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -\frac{1}{f} & 1\end{bmatrix}\begin{bmatrix}1 & g\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-\frac{b}{g} & 0\\ -\frac{1}{f} & -\frac{g}{b}\end{bmatrix}$$

Drift **Lense** **Drift**

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7. Geometrical Optics

Thin lense approximation

Point-2-Point Image

$$R_{12} = (x|x') = 0$$

$$\mathrm{R_x} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{b}{g} & 0 \\ -\frac{1}{f} & -\frac{g}{b} \end{bmatrix}$$

Drift **Lense** **Drift**

Amplification / „Vergröserung" / „Maßstab"

$$R_{11} = (x|x_0) = M = \frac{B}{G} = -\frac{b}{g}$$

$$R_{22} = (x'|x'_0) = M^{-1} = -\frac{g}{b}$$

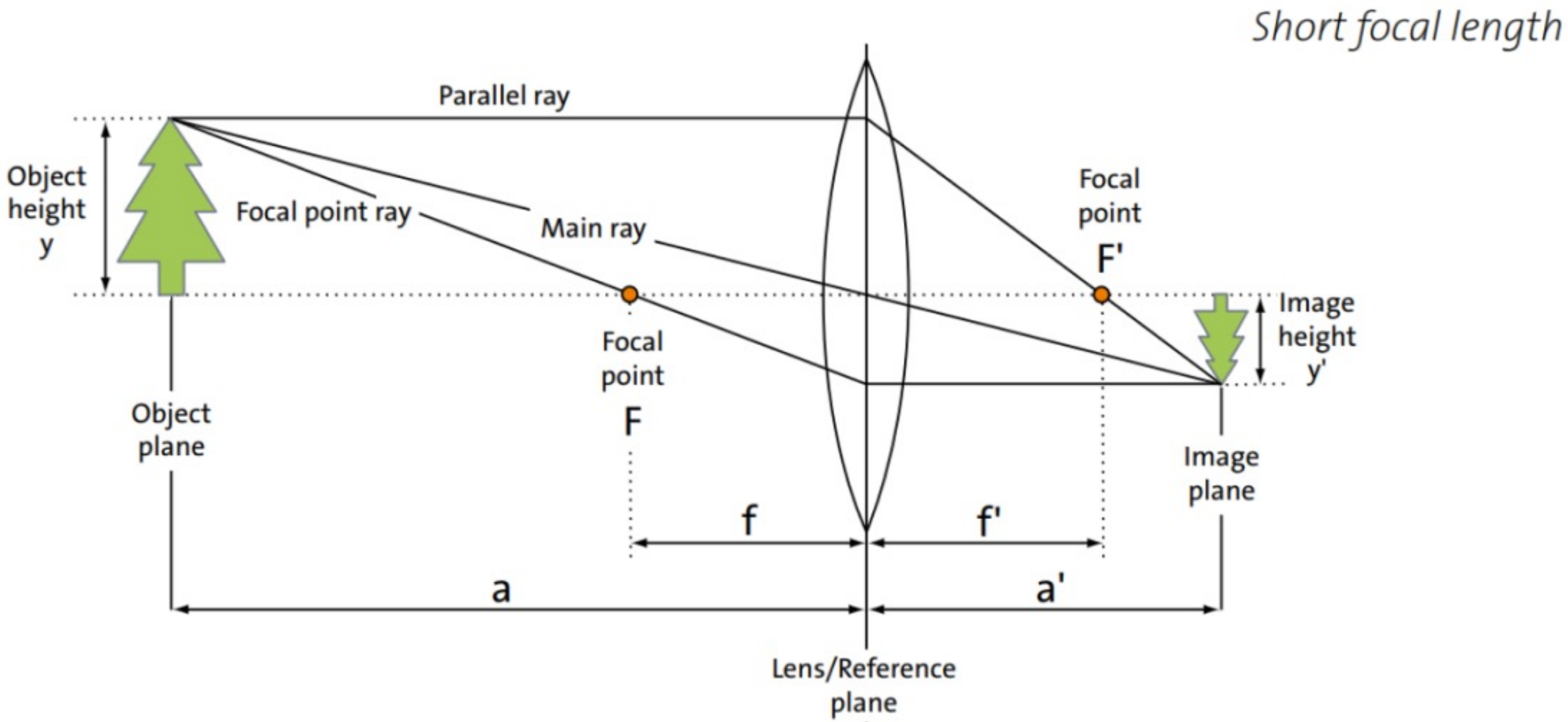
$$R_{21} = (x'|x) = -\frac{1}{f}$$

Focal length

Scaling of transformation (amplification)

7. Geometrical Optics

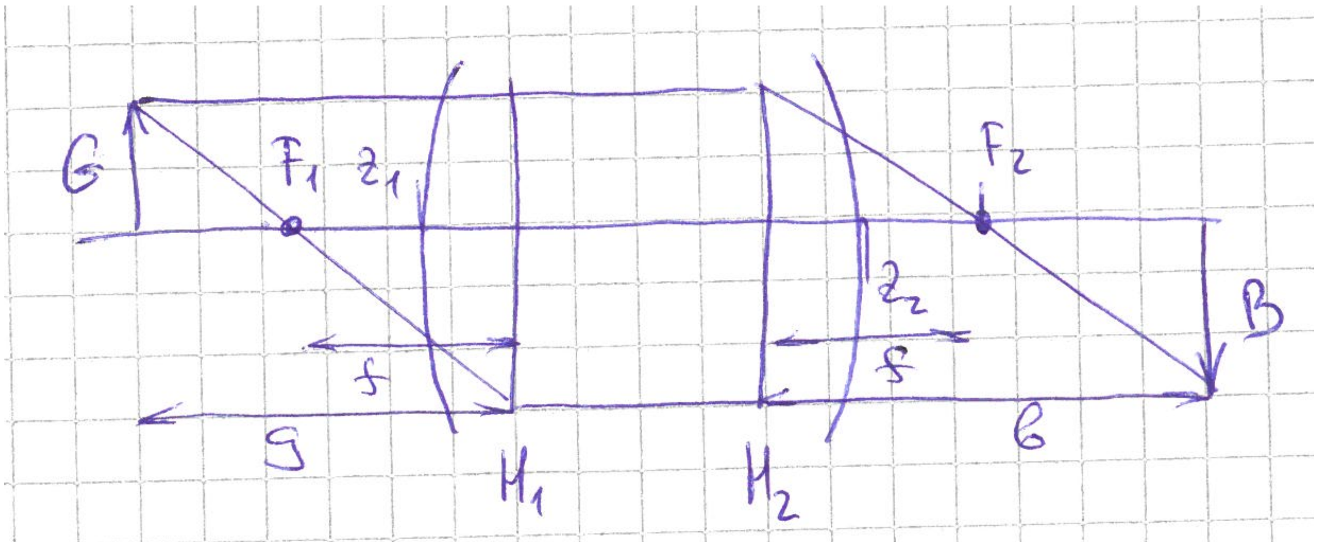
Thin lense approximation



7. Geometrical Optics

Thick lense

Additional drift, principal planes (H), gaps (z)



$$\mathrm{R} = \begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & z_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{z_2}{f} & z1 + z_2 - \frac{z_1 z_2}{f} \\ -\frac{1}{f} & 1 - \frac{z_1}{f} \end{bmatrix}$$

Drift **Lense** **Drift**

7. Geometrical Optics

Thick lense

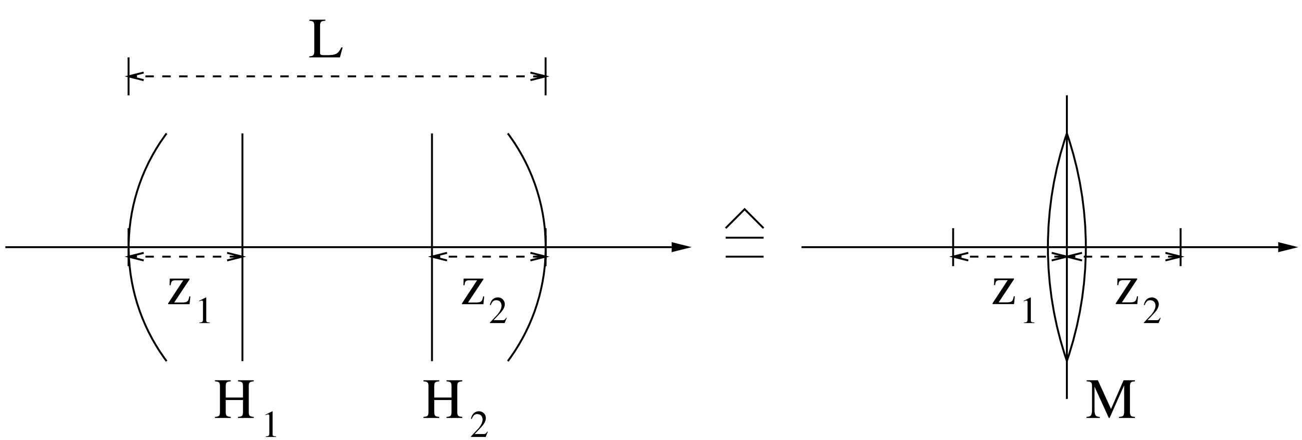
Assuming a focusing (defocusing) system, we can solve the inverse problem

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} 1 & z_1 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$\frac{1}{f} = -R_{21}$$

$$z_1 = \frac{R_{22} - 1}{R_{21}}$$

$$z_2 = \frac{R_{11} - 1}{R_{21}}$$



$$\Delta L = L - (z_1 + z_2)$$

7. Geometrical Optics

Any focusing/defocusing element can be represented as a combination of corresponding thin lense and drifts

1) Radially focusing QP

$$\frac{1}{f} = \sqrt{k_x}\sin(\sqrt{k_x}L)$$

$$z_1 = \frac{\cos(\sqrt{k_x}L) - 1}{-\sqrt{k_x}\sin(\sqrt{k_x}L)}$$

$$z_2 = z_1$$

2) Sector magnet

$$z_1 = z_2 = \rho_0 \tan(\alpha/2)$$

Complete matrix: edge – magnet-edge

$$\mathrm{R_x} = \begin{bmatrix} 1 & \rho_0\tan(\alpha/2) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\sin(\alpha)/\rho_0 & 1 & \sin(\alpha) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \rho_0\tan(\alpha/2) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Account of dispersion

4. Solution of the Equation of Motion

(linear approximation)

Characteristic solutions for transverse motion (drift, quadrupole, dipole)

$k_x(s) > 0$ & $k_y(s) > 0$	$k_x(s) = k_y(s) = 0$	$k_x(s) < 0$ & $k_y(s) < 0$
$C_x(s) = \cos(\sqrt{k_x}s)$	$C_x(s) = 0$	$C_x(s) = \cosh(\sqrt{\lvert k_x\rvert}s)$
$S_x(s) = \frac{\sin(\sqrt{k_x}s)}{\sqrt{k_x}}$	$S_x(s) = s$	$S_x(s) = \frac{\sinh(\sqrt{\lvert k_x\rvert}s)}{\sqrt{\lvert k_x\rvert}}$
$d_x(s) = \frac{h}{k_x}[1 - \cos(\sqrt{k_x}s)]$	$d_x(s) = 0$	$d_x(s) = \frac{h}{\lvert k_x\rvert}[\cosh(\sqrt{\lvert k_x\rvert}s) - 1]$
$C_y(s) = \cos(\sqrt{k_y}s)$	$C_y(s) = 1$	$C_y(s) = \cosh(\sqrt{\lvert k_y\rvert}s)$
$S_y(s) = \frac{\sin(\sqrt{k_y}s)}{\sqrt{k_y}}$	$S_y(s) = s$	$S_y(s) = \frac{\sinh(\sqrt{\lvert k_y\rvert}s)}{\sqrt{\lvert k_y\rvert}}$

periodic

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$$

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4. Transfer matrix

Radially focusing and axially de-focusing quadrupole: $k_x = k \text{ and } k_y = -k$

$$\mathbf{R} = \begin{bmatrix} \cos(\sqrt{k}L) & \frac{\sin(\sqrt{k}L)}{\sqrt{k}} & 0 & 0 & 0 & 0 \\ -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(\sqrt{k}L) & \frac{\sinh(\sqrt{k}L)}{\sqrt{k}} & 0 & 0 \\ 0 & 0 & \sqrt{k}\sinh(\sqrt{k}L) & \cosh(\sqrt{k}L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Radially de-focusing and axially focusing quadrupole: $k_x = -k \text{ and } k_y = k$

$$\mathbf{R} = \begin{bmatrix} \cosh(\sqrt{k}L) & \frac{\sinh(\sqrt{k}L)}{\sqrt{k}} & 0 & 0 & 0 & 0 \\ \sqrt{k}\sinh(\sqrt{k}L) & \cosh(\sqrt{k}L) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\sqrt{k}L) & \frac{\sin(\sqrt{k}L)}{\sqrt{k}} & 0 & 0 \\ 0 & 0 & -\sqrt{k}\sin(\sqrt{k}L) & \cos(\sqrt{k}L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & L/\gamma^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Transfer matrix

- Homogeneous dipole (n=0), sector magnet

dispersion

drift

$$\mathbf{R} = \begin{bmatrix} \cos(\alpha) & \rho_0 \sin(\alpha) & 0 & 0 & 0 & \rho_0(1-\cos(\alpha)) \\ \frac{\sin(\alpha)}{\rho_0} & \cos(\alpha) & 0 & 0 & 0 & \sin(\alpha) \\ 0 & 0 & 1 & \rho_0\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin(\alpha) & -\rho_0(1-\cos(\alpha)) & 0 & 0 & 1 & \rho_0\frac{\alpha}{\gamma^2} - \rho_0(\alpha - \sin(\alpha)) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Orbit (particle traveling on the inner or outter trajectory relative to the reference orbit) and velocity (velocity spread) effects

8. Beam Properties

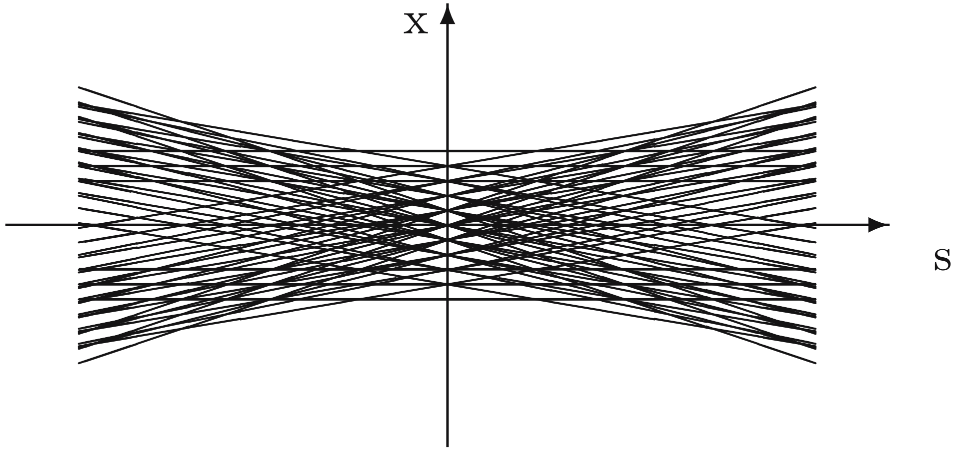
Phase ellipse

Up to now we considered a single particle
Now we shall consider a bunch of particles

The same principle: $$\vec{x}(s) = R(s)\vec{x}(0)$$

Superposition of single particles

Density distribution along *s* $$\rho(\vec{x}) = \rho(x, x', y, y', l, \delta)$$



8. Beam Properties

Phase ellipse

Density distribution in *(x,x')* plane $\rho(x, x')$ **can typically presented with an ellipse**

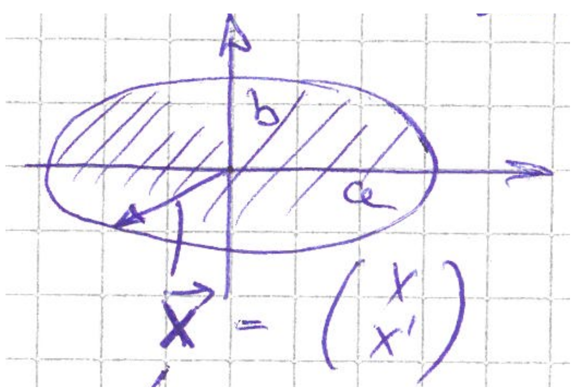
$$\sigma_x = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

$$\sigma_{12} = \sigma_{21}$$

$$\det(\sigma_x) > 0$$

Phase ellipse:

$$\mathbf{X}^{\mathbf{T}} \sigma_x^{-1} \mathbf{X} = 1 \quad (1)$$



Vector from origin to ellipse boundary

$$\Downarrow$$

$$\sigma_x^{-1} = \frac{1}{\det(\sigma_x)} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}$$

$$\frac{x^2}{a^2} + \frac{x'^2}{b^2} = 1$$

8. Beam Properties

Emittance

1 $$\det(\sigma_x) = \sigma_{22}x_1^2 - 2\sigma_{12}x_1x_2 + \sigma_{11}x_2^2 = \epsilon_x^2$$

Area of the ellipse

Emittance:

$$E_x = \pi\epsilon_x = \pi\sqrt{\det(\sigma_x)} = \pi\sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}$$

$$1\ [\mathrm{mm}] \cdot [\mathrm{mrad}] = 1 \cdot 10^{-6}\ [\mathrm{m}] \cdot [\mathrm{rad}]$$

Often this is emittance

Maximal values:

$$x_{\mathrm{max}} = \sqrt{\sigma_{11}} \qquad x'_{\mathrm{max}} = \sqrt{\sigma_{22}}$$

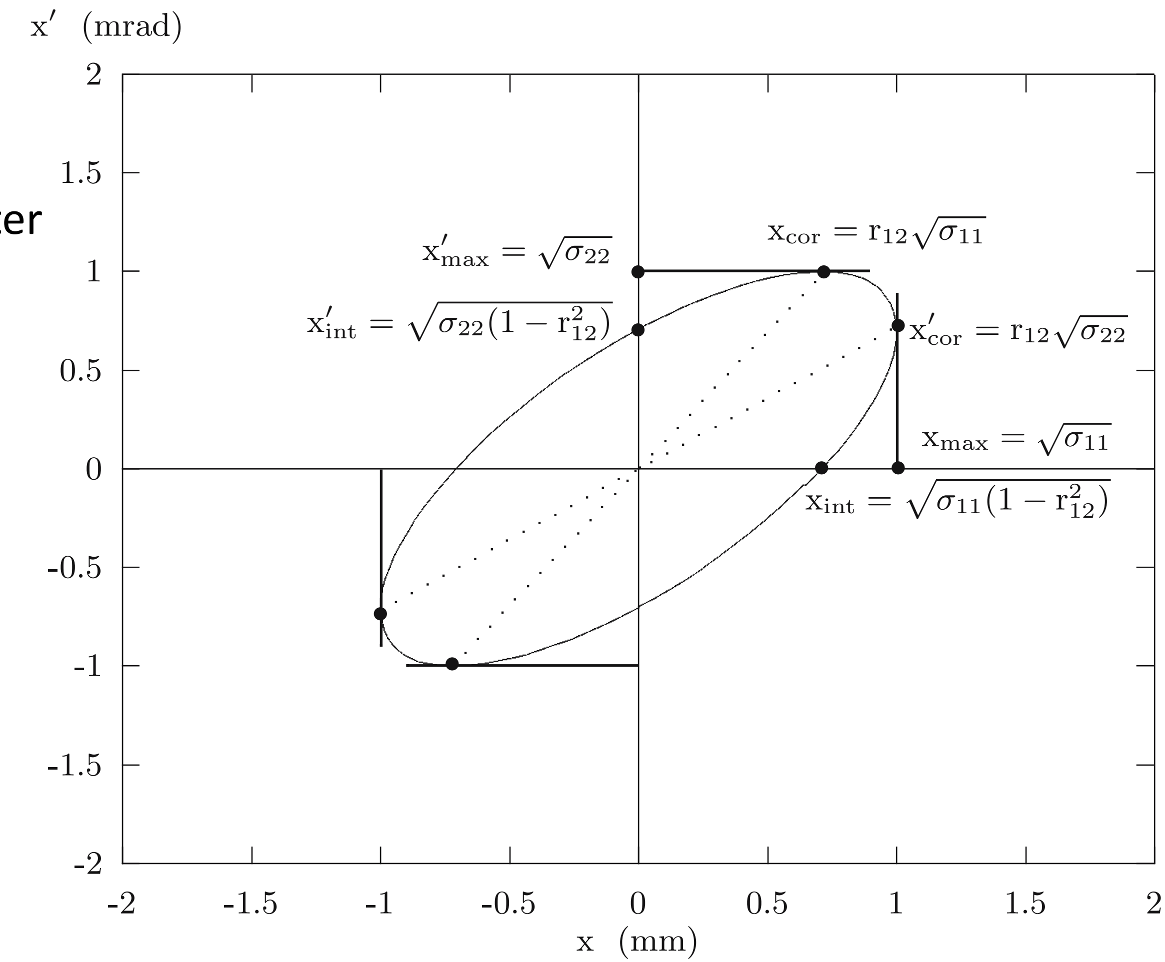
Emittance

σ_{12} - Correlation between *x* and *x'*

$\Downarrow$

Dimensionless correlation parameter

$$r_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}} \in [-1, 1]$$



8. Beam Properties

Density distribution

Most commonly assumed density distribution in phase-space ellipse is a 2D Gaussian

$$\rho(\vec{x}) = \frac{1}{2\pi\epsilon_x} exp\left(- \frac{1}{2}\vec{x}^T \sigma_x^{-1} \vec{x} \right)$$

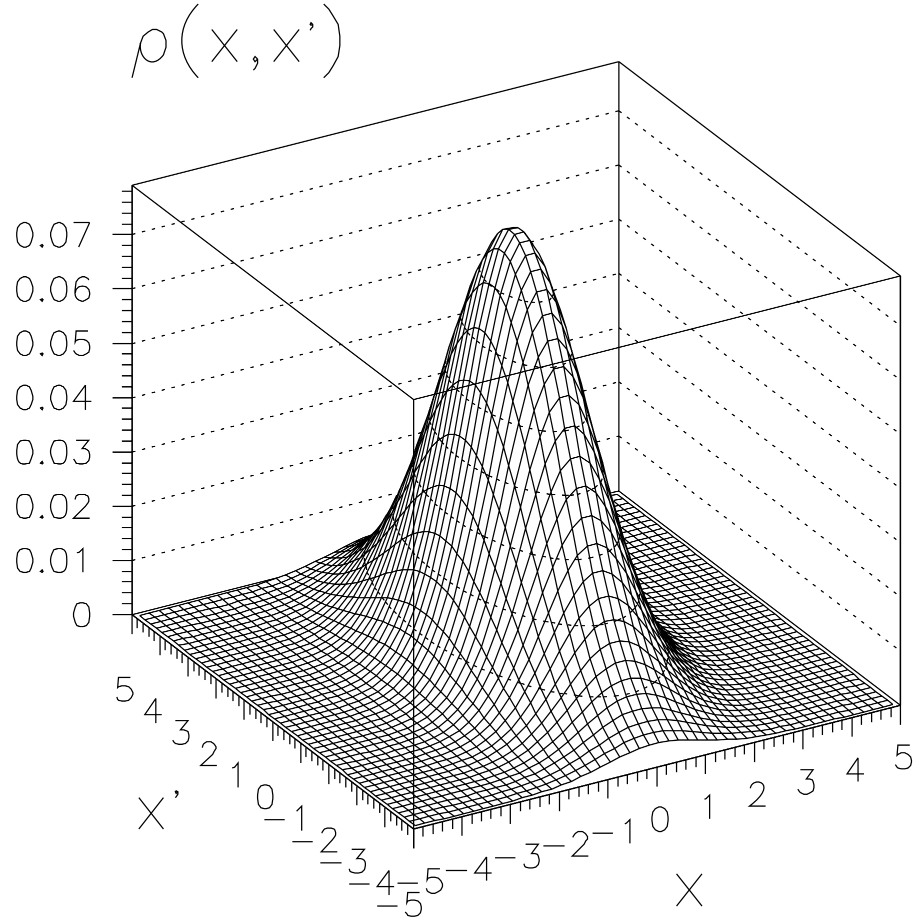
Standard deviation^2 = STD^2

STD^2=1 Encloses 39.3% of particles

$$\epsilon_x^{1\sigma} = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2}$$

STD^2=4 86.5% $\epsilon_x^{2\sigma} = 4\epsilon_x^{1\sigma}$

STD^2=9 98.5% $\epsilon_x^{3\sigma} = 9\epsilon_x^{1\sigma}$



In reality, the distribution is not Gaussian (scattering, collimators, walls …)

8. Beam Properties

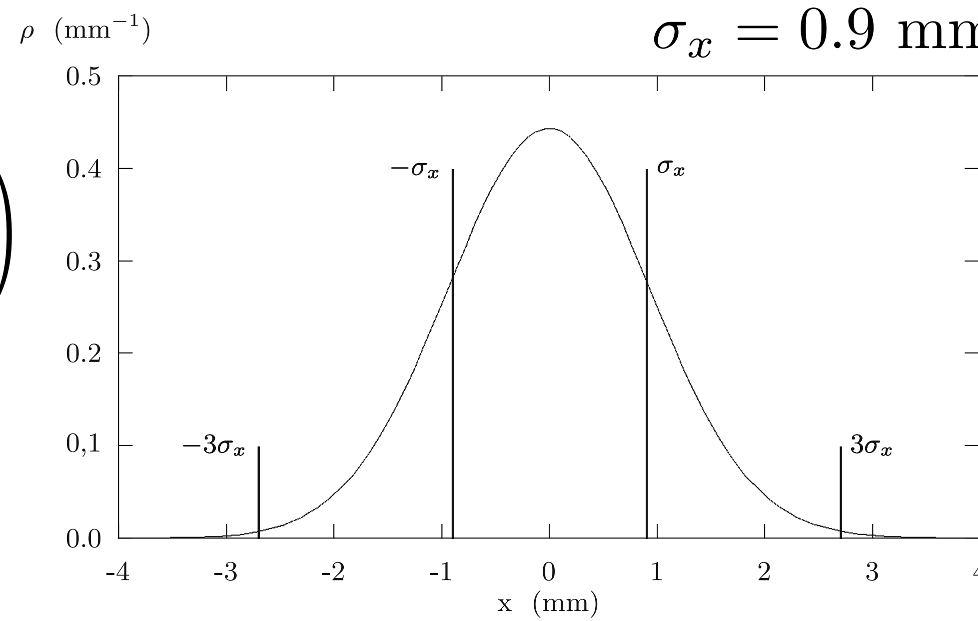
Beam Envelope

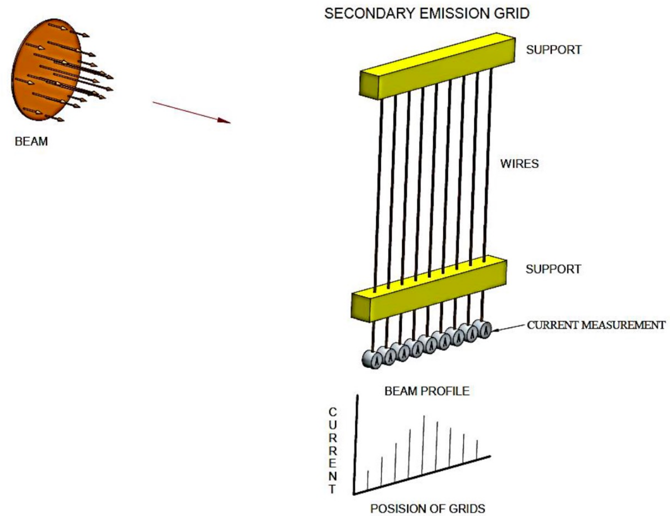
Beam profile – 1D projection of the density distribution

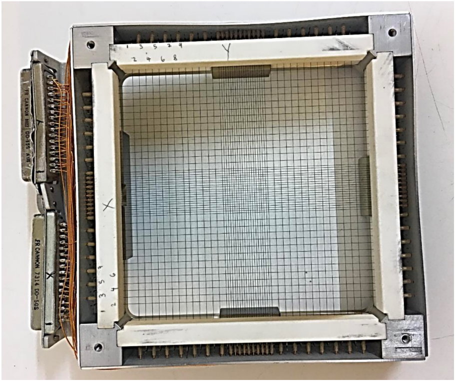
$$\sigma_x = \sqrt{\sigma_{11}}$$

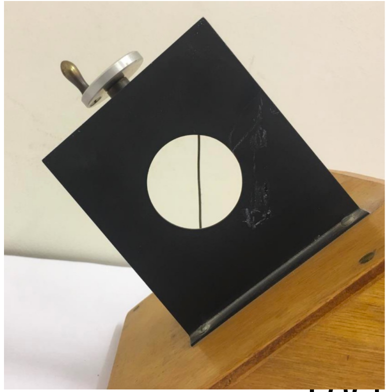
$$\rho(x) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_{11}}} \exp\left(-\frac{1}{2}\frac{x^2}{\sigma_{11}^2} \right)$$

$\sigma_x = 0.9$ mm







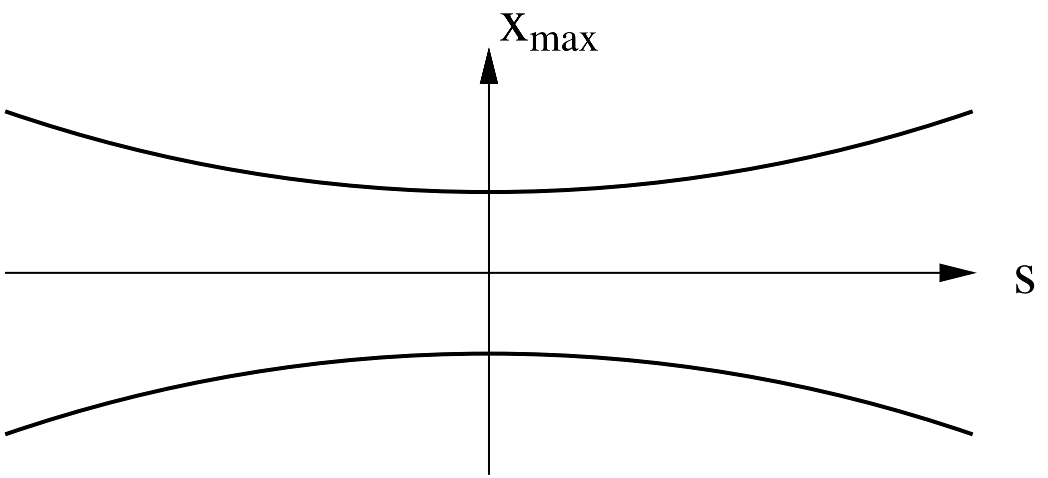


8. Beam Properties

Beam Envelope

Beam envelope (RMS envelope)

$$x_{\max}(s) = \sqrt{\sigma_{11}(s)}$$



Beam waist („Strahltaille“) / focus

$r_{12} < 0$ $r_{12} > 0$

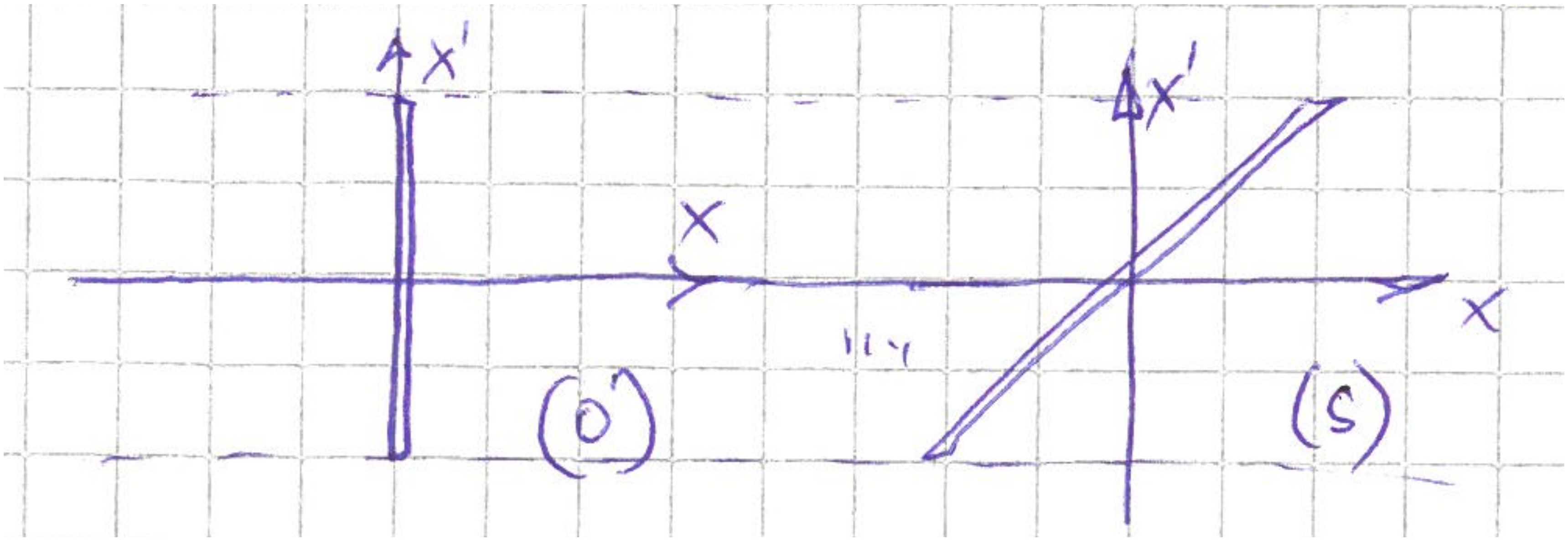
8. Beam Properties

Transformation of phase ellipses

$$\sigma_x(s) = R_x(s)\sigma_x(0)R_x^T(s)$$

Derivation Hinterberger

Drift



8. Beam Properties

Transformation of phase ellipses

Drift

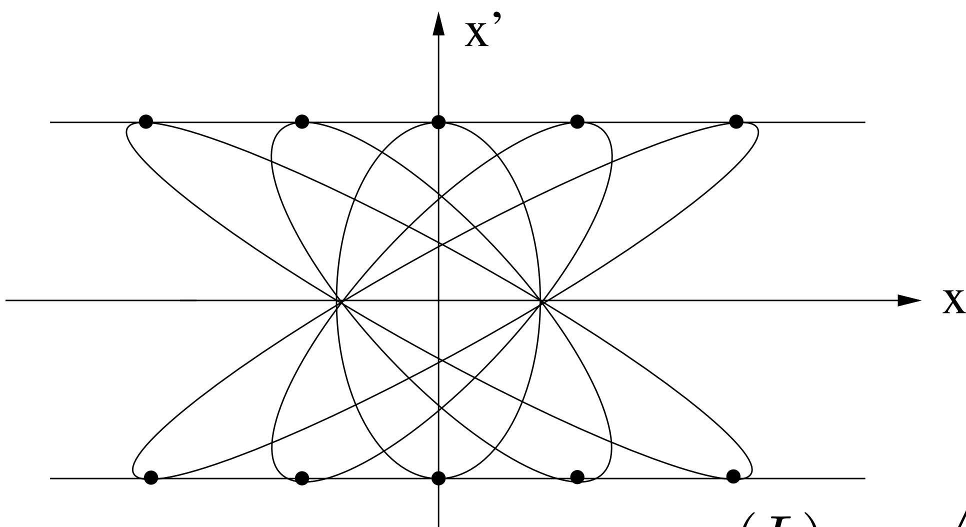
!!!

Start with upright ellipse $\sigma_{12} = \sigma_{21} = 0$

$$\sigma_x(L) = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11}(0) & 0 \\ 0 & \sigma_{22}(0) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \sigma_{11}(0) + L^2\sigma_{22}(0) & L\sigma_{22}(0) \\ L\sigma_{22}(0) & \sigma_{22}(0) \end{bmatrix}$$

Dependent on the sign of L – rotating ellipse in the clockwise/anticlockwise direction



$r_{12} > 0$ Divergent beam

$r_{12} < 0$ Convergent beam

$$x'_{\max} = const$$

$$x_{\text{int}} = const$$

$$x_{\max}(L) = \sqrt{\sigma_{11}(L)} = \sqrt{\sigma_{11}(0) + L^2\sigma_{22}(0)}$$

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8. Beam Properties

Transformation of phase ellipses

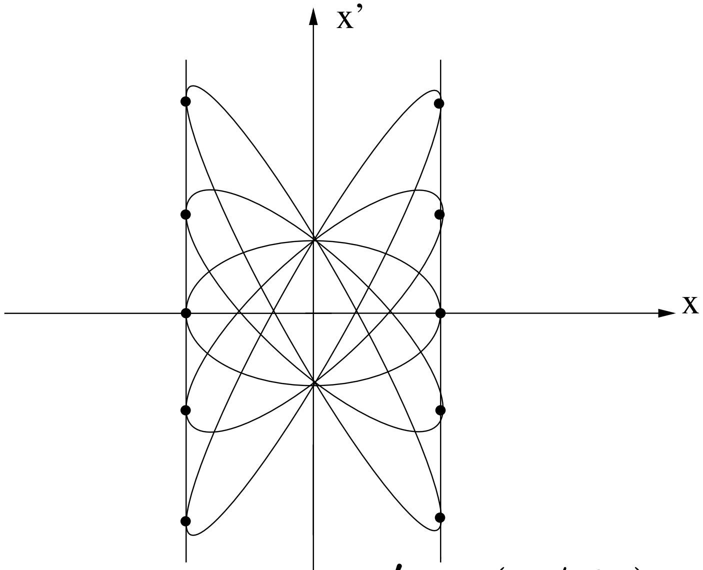
Thin lense **!!!**

Start with upright ellipse $\sigma_{12} = \sigma_{21} = 0$

$$\sigma_x(1/f_x) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f_x} & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11}(0) & 0 \\ 0 & \sigma_{22}(0) \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{f_x} \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \sigma_{11}(0) & -\sigma_{11}(0)/f_x \\ -\sigma_{11}(0)/f_x & \sigma_{22}(0) + \sigma_{11}(0)/f_x^2 \end{bmatrix}$$

Dependent on the sign of L – rotating ellipse in the clockwise/anticlockwise direction



$r_{12} > 0$ Divergent beam

$r_{12} < 0$ Convergent beam

$$x_{\max} = const$$

$$x'_{\text{int}} = const$$

$$x'_{\max}(1/f_x) = \sqrt{\sigma_{11}(1/f_x)} = \sqrt{\sigma_{22}(0) + \sigma_{11}(0)/f_x^2}$$

HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

8. Beam Properties

Phase space in multi dimensions / Phase space ellipsoid

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} & \sigma_{26} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} & \sigma_{34} & \sigma_{35} & \sigma_{36} \\ \sigma_{14} & \sigma_{24} & \sigma_{34} & \sigma_{44} & \sigma_{45} & \sigma_{46} \\ \sigma_{15} & \sigma_{25} & \sigma_{35} & \sigma_{45} & \sigma_{55} & \sigma_{56} \\ \sigma_{16} & \sigma_{26} & \sigma_{36} & \sigma_{46} & \sigma_{56} & \sigma_{66} \end{pmatrix}$$

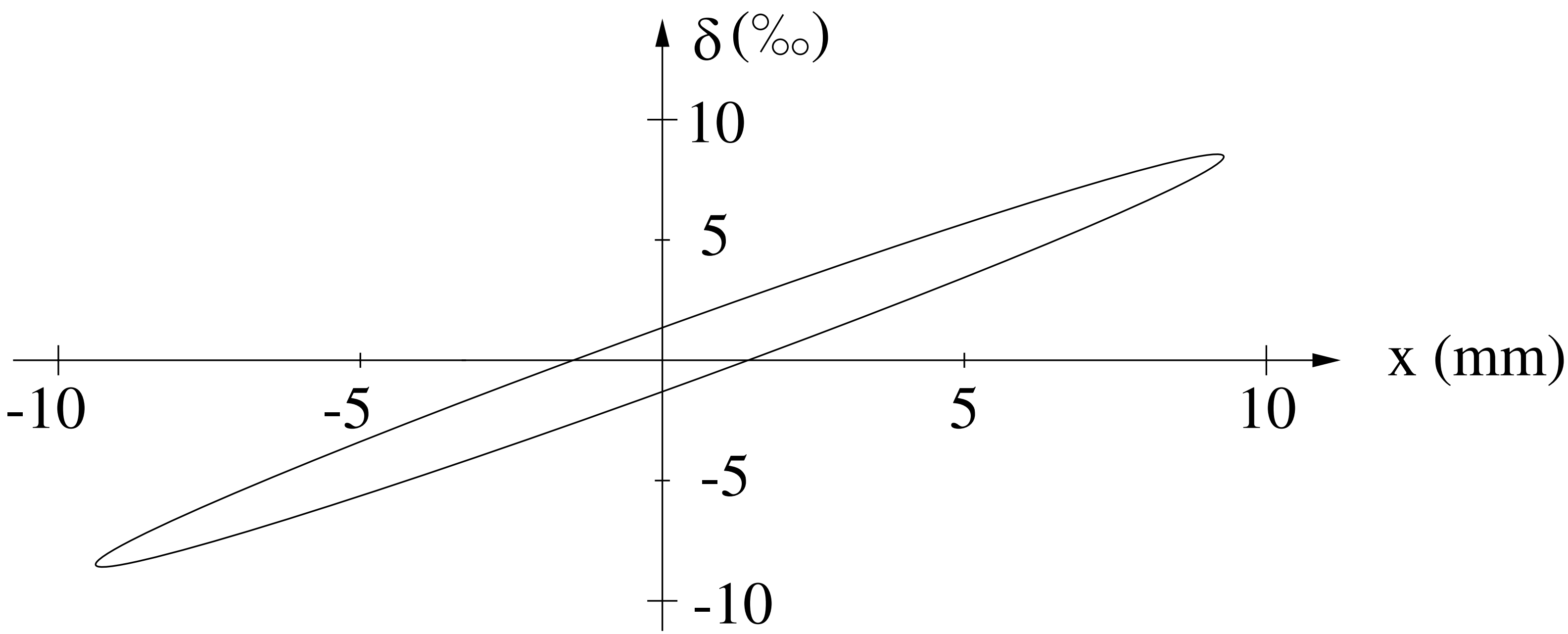
$$V = \frac{16}{3}\pi\sqrt{\det(\sigma)}$$

$$\begin{aligned} \sigma_{ii} &= \overline{(x_i - \overline{x_i})^2} \\ &= \int\int\int\int\int\int (x_i - \overline{x_i})^2 \rho(\boldsymbol{x}) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}x_5 \mathrm{d}x_6 \,, \\ \sigma_{ij} &= \overline{(x_i - \overline{x_i})(x_j - \overline{x_j})} \\ &= \int\int\int\int\int\int (x_i - \overline{x_i})(x_j - \overline{x_j}) \rho(\boldsymbol{x}) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}x_5 \mathrm{d}x_6 \,. \end{aligned}$$

8. Beam Properties

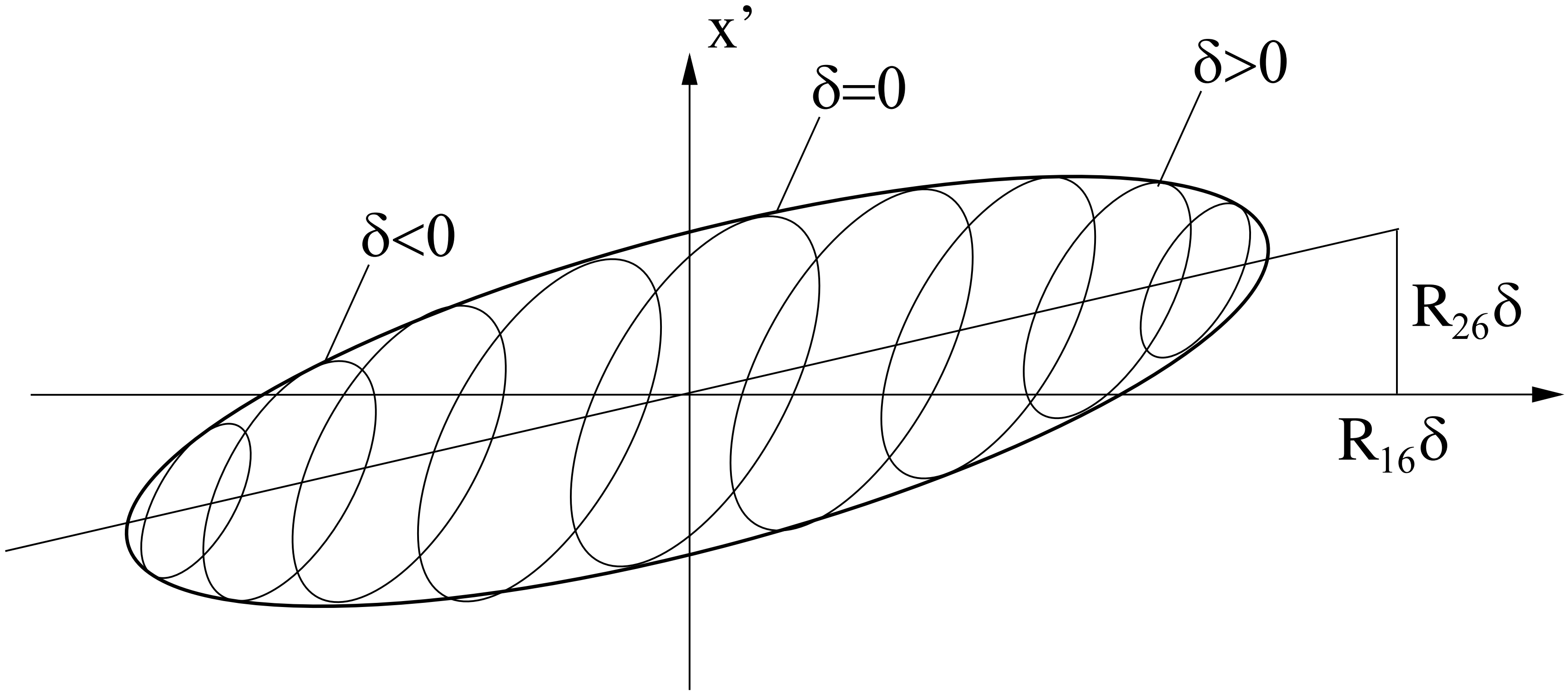
Momentum deviation

$$x_{\max} = \sqrt{\sigma_{11}} \qquad \delta_{\max} = \sqrt{\sigma_{66}} \qquad r_{16} = \frac{\sigma_{16}}{\sqrt{\sigma_{11}\sigma_{66}}}$$



8. Beam Properties

Ellipsoid



$$x_{\max} = \sqrt{\sigma_{11}^{(0)} + R_{16}^2 \sigma_{66}} = \sqrt{\sigma_{11}^{(0)} + \sigma_{16}^2 / \sigma_{66}} \,,$$

$$x'_{\max} = \sqrt{\sigma_{22}^{(0)} + R_{26}^2 \sigma_{66}} = \sqrt{\sigma_{22}^{(0)} + \sigma_{26}^2 / \sigma_{66}} \,.$$

8. Second Order Ion Optics

Multipole expansion:

$$B_x(x,y,s) = \frac{\partial \Phi}{\partial x} = A_{11}y + A_{12}xy + \cdots ,$$

$$B_y(x,y,s) = \frac{\partial \Phi}{\partial y} = A_{10} + A_{11}x + \frac{1}{2!}\left(A_{12}x^2 + A_{30}y^2\right) + \cdots$$

$$B_s(x,y,s) = \frac{1}{1+hx}\frac{\partial \Phi}{\partial s} = \frac{1}{1+hx}\left(A'_{10}y + A'_{11}xy + \cdots\right) .$$

Much more complicated equations of motion
Transfer matrix elements … etc

For details see Berz, Hinterberger

8. Higher-Order Ion Optics

The MAD-X Program

(Methodical Accelerator Design)

Version 5.02.05

User's Reference Manual

Hans Grote
Frank Schmidt
Laurent Deniau
Ghislain Roy (editor)

`MAX_MULT_ORD` (optional parameter, default = 11)

9. Ion Optical Systems

FODO Cell