Partial wave analysis for elastic scattering

Cross section without imposing any limitation on the strength of V(r)

Potential is spherical symmetric

-> angular momentum of the incident particle will be conserved, i.e. a particle scattering from a central potential will have the same angular momentum before and after the collision

Assuming that the incident wave is in the z direction:

$$\boldsymbol{\Phi_{inc}(\vec{r}) = exp(ikr\ cos\theta)}$$

We want to express this in terms of angular momentum eigenstates, each with a definite angular moment $\boldsymbol{l}$:

$$e^{i\vec{k}\vec{r}} = e^{ikrcos\Theta} = \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(cos\Theta)$$

Partial wave expansion for elastic scattering

Starting with the Schrödinger equation in the CM frame:

$$\frac{-\hbar^2}{2\mu}\vec{\nabla}^2\psi(\vec{r}) + \hat{V}(r)\psi(\vec{r}) = E\psi(\vec{r})$$

The most general solution of the Schrödinger equation is

$$\psi(\vec{r}) = \sum_{lm} C_{lm} R_{kl}(r) Y_{lm}(\theta,\phi)$$

Since $V(r)$ is central, the system is symmetrical about the z-axis. Therefore, the scattered wave function must not depend on the azimuthal angle $\phi \rightarrow m = 0$
With $Y_{l0}(\theta,\phi) \sim P_l(cos\theta)$ the scattered wave function becomes

$$\boxed{\psi(r,\theta) = \sum_{l} a_l R_{kl}(r) P_l(cos\theta)}$$

$$here\ k^2 = \frac{2mE}{\hbar^2}$$

where $R_{kl}(r)$ obeys the following radial equation

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2}\right] rR_{kl}(r) = \frac{2m}{\hbar^2} V(r) rR_{kl}(r)$$

each term, known as partial wave, is a joint eigenfunction of L^2 and L_z

Partial wave analysis for elastic scattering

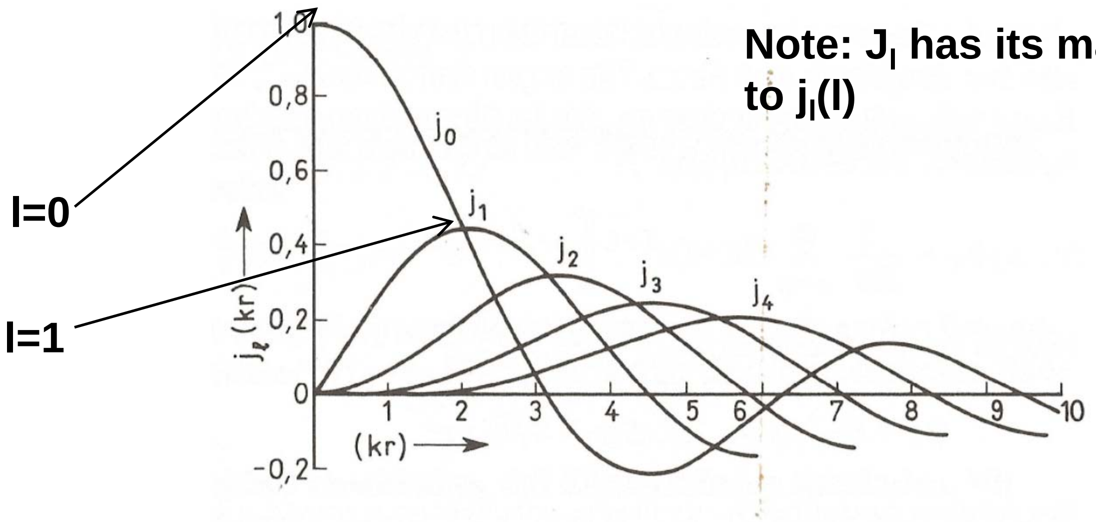
for a free particle $V(r) = 0$

$$rR_l(kr) = rj_l(kr) = r\sqrt{\frac{\pi}{2kr}}\,\underbrace{J_{l+\frac{1}{2}}(kr)}_{\text{Bessel function}}$$

But a free particle is also described by a plane wave
Hence, rewrite plane wave in terms of eigenfunction of angular momentum

incoming wave $$e^{ikz} = \sum_l A_l(kr)P_l(cos\theta) = \sum_l a_l(kr)\,j_l(kr)P_l(cos\theta)$$

with coefficients $a_l(kr) = i^l(2l+1)$



Note: J_l has its maximum close to $j_l(l)$

Partial wave analysis for elastic scattering

Now we have to write in-coming and out-going waves in terms of the same basis:
We worked out that the total wave function is:

$$\psi_T(\vec{r}\,) = \psi_{in}(\vec{r}) + \psi_{out}(\vec{r})$$

$$= A \cdot \left(e^{i\vec{k_0}\cdot\vec{r}} + f(\theta,\phi)\left(\frac{e^{i\vec{k}\vec{r}}}{r}\right)\right)$$

with $\phi = 0$ and $k = k_0$ for elastic scattering

$$\psi_T(r,\theta) \approx \sum_{l=0} i^l(2l+1)j_l(kr)P_l(cos\theta) + f(\theta)\frac{e^{ikr}}{r}$$

Since we are only interested in the solutions at large distances, we use an approximation of the Bessel functions:

$$j_l(kr) \rightarrow \frac{sin\left(kr - \frac{l\pi}{2}\right)}{kr} \qquad (r \rightarrow \infty)$$

Then the asymptotic form of $\psi(r,\theta)$ is given by

$$\psi(r,\theta) \rightarrow \sum_l i^l(2l+1)P_l(cos\,\theta)\frac{sin\left(kr - \frac{l\pi}{2}\right)}{kr} + f(\theta)\left(\frac{e^{ikr}}{r}\right)$$

Partial wave analysis for elastic scattering

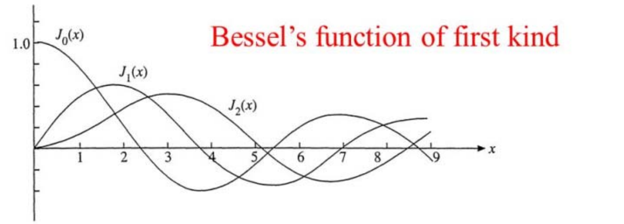
With $sin\left(kr - \frac{l\pi}{2}\right) = \left[(-i)^l e^{ikr} - i^l e^{-ikr}\right]/2i$

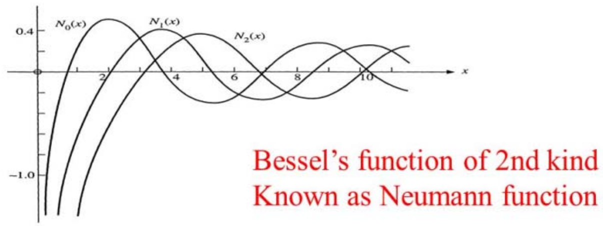
because $e^{\pm il\pi/2} = (\pm i)^l$

One obtains:

$$\psi(r,\theta) \rightarrow -\frac{e^{-ikr}}{2ikr}\sum_l i^{2l}(2l+1)P_l(cos\theta) + \frac{e^{ikr}}{r}[f(\theta + \frac{1}{2ik}\sum_l i^l(-i)^l(2l+1)P_l(cos\theta)]$$

Now we have to find the asymptotic solution of the SE:
At large values of r the scattering potential is effectively 0 -> radial equation becomes





$$\left(\frac{d^2}{dr^2} + k^2\right) rR_{kl}(r) = 0$$

The general solution o this equation is given by linear combination of spherical Bessel and Neumann functions. In order to have a physical solution one has to introduce the phase shift δ_l :

$$R_{kl}(r) = C_l[cos\delta_l\, j_l(kr) - sin\delta_l\, n_l(kr)]$$

Partial wave analysis for elastic scattering

The asymptotic form of the radial function can be written as:

$$\boldsymbol{R_{kl}(r) \rightarrow \frac{C_l\left(cos\delta_l\, sin\left(kr - \frac{l\pi}{2}\right) - sin\delta_l\, cos\left(kr - \frac{l\pi}{2}\right)\right)}{kr} \quad (r \rightarrow \infty)}$$

$$\boxed{R_{kl}(r) \rightarrow C_l\,\frac{sin\left(kr - \frac{l\pi}{2} + \delta_l\right)}{kr} \qquad (r \rightarrow \infty)}$$

With $\boldsymbol{\delta_l = 0}$, the radial function $\boldsymbol{R_{kl}(r)}$ is finite at r=0, because $\boldsymbol{R_{kl}(r)}$ reduces to $\boldsymbol{j_l(kr)}$.

$\boldsymbol{\delta_l}$, phase shift of the $\boldsymbol{l}$th partial wave, vanishes for all values of l in absence of the scattering potential. It measures the distortion of $\boldsymbol{R_{kl}\ (r)}$ from the "free" solution $\boldsymbol{j_l(kr)}$.

Such the scattered wave function in the asymptotic limit runs as follows:

$$\boldsymbol{\psi(r,\theta) \rightarrow \sum_l a_l P_l(cos\theta)\,\frac{sin\left(kr - \frac{l\pi}{2} + \delta_l\right)}{kr} \qquad (r \rightarrow \infty)}$$

This wave function is called the distorted plane wave, which differs from the plane wave by the phase shift $\boldsymbol{\delta_l}$

Partial wave analysis for elastic scattering

with $sin\left(kr - \frac{l\pi}{2} + \delta_l\right) = \left[(-i)^l e^{ikr} e^{i\delta_l} - i^l e^{-ikr} e^{-i\delta_l}\right]/2i$

one rewrites the distorted plane wave:

$$\psi(r,\theta) \rightarrow -\frac{e^{-ikr}}{2ikr}\sum_l a_l i^l e^{-i\delta_l} P_l(cos\theta) + \frac{e^{ikr}}{2ikr}\sum_l a_l (-i)^l\, e^{i\delta_l} P_l(cos\theta)$$

and compares this to

$$\psi(r,\theta) \rightarrow -\frac{e^{-ikr}}{2ikr}\sum_l i^{2l}(2l+1) P_l(cos\theta) + \frac{e^{ikr}}{r}\left[f(\theta + \frac{1}{2ik}\sum_l i^l (-i)^l (2l+1) P_l(cos\theta)\right]$$

one obtains $(2l+1)i^{2l} = a_l i^l e^{-i\delta_l}$

$$a_l = (2l+1) i^l e^{i\delta_l}$$

Partial wave analysis for elastic scattering

By comparing the coefficients for $\frac{e^{ikr}}{r}$ in the two equations one obtains:

$$f(\theta) + \frac{1}{2ik}\sum_l i^l(-i)^l(2l+1)P_l(cos\theta) = \frac{1}{2ik}\sum_l (2l+1)i^l(-i)^l e^{2i\delta_l}P_l(cos\theta)$$

by using $\frac{e^{2i\delta_l-1}}{2i} = e^{i\delta_l}sin\delta_l$ and $i^l(-i)^l = 1$ one gets:

$$f(\theta) = \sum_l f_l(\theta) = \frac{1}{2ik}\sum_l (2l+1)P_l(cos\theta)\,(e^{2i\delta_l} - 1)$$
$$= \frac{1}{k}\sum_l (2l+1)e^{i\delta_l}sin\delta_l\, P_l(cos\theta)$$

where $f_l(cos\theta)$ is the partial wave amplitude.

Partial wave analysis for elastic scattering

We obtain for the differential cross section:

$$\frac{d\sigma}{d\Omega} = |\,\mathrm{f}(\theta)|^2 = \frac{1}{k^2}\sum_{l}\sum_{l'}(2l+1)(2l'+1)e^{i(\delta_l-\delta_{l'})}sin\delta_l\; sin\delta_{l'}P_l(cos\theta)P_{l'}(cos\theta)$$

and the total cross section:

$$\sigma = \int \frac{d\sigma}{d\Omega}d\Omega = \int_o^{\pi}|f(\theta)|^2 sin\theta d\theta \int_0^{2\pi} d\phi = 2\pi\int_0^{\pi}|f(\theta)|^2 sin\theta\; d\theta$$

$$= 2\pi/k^2\sum_{l}\sum_{l'}(2l+1)(2l'+1)e^{i(\delta_l-\delta_{l'})}sin\delta_l\; sin\delta_{l'}\int_0^{\pi}P_l(cos\theta)P_{l'}(cos\theta)sin\theta\; d\theta$$

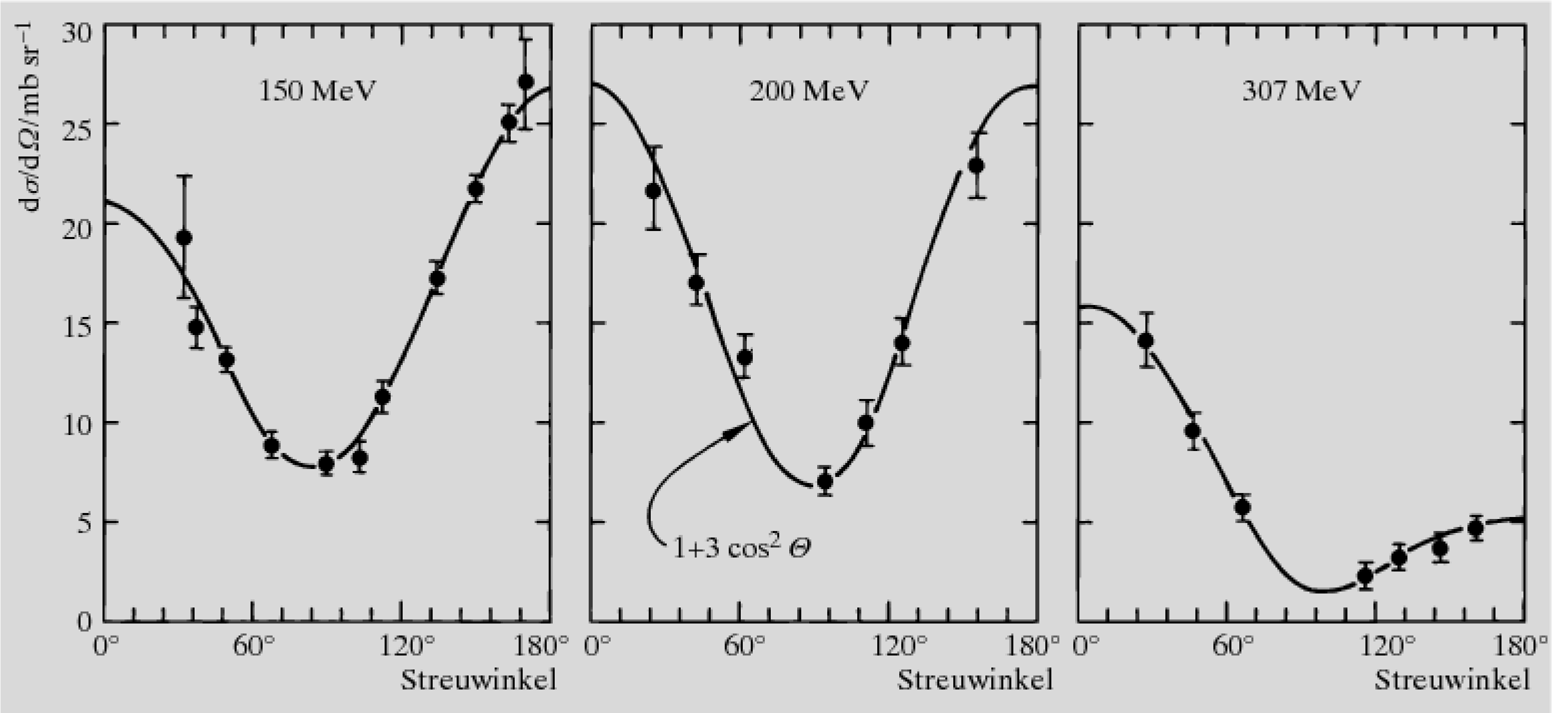
Using the relation $\int_0^{\pi}P_l(cos\theta)P_{l'}(cos\theta)\;\sin\theta\; d\theta = \frac{2}{2l+1}\delta_{ll'}$

one yields

$$\sigma = \sum_{l}\sigma_l = \frac{4\pi}{k^2}\sum_{l}(2l+1)sin^2\delta_l$$

Partial wave analysis for elastic scattering

π+p scattering at different pion energies



At 200 MeV p-wave scattering

https://link.springer.com/chapter/10.1007/978-3-642-41753-5_2

Partial wave analysis for elastic scattering

- The differential cross section is a superposition of different angular momenta and gives rise to interference patterns between different partial waves corresponding to different values of l .
- The interference terms vanish in the total cross section
- If V(r)=0 the phase shifts vanish and the cross section is ZERO

- In the case of low energy scattering between particles, i.e. l=0, the scattering amplitude is.

$$f_0 = \frac{1}{k} e^{i\delta_0} sin\delta_0 \text{ with } P_0(cos\theta) = 1$$

Since f_0 does not depend on θ, the differential and total cross sections are given by:

$$\frac{d\sigma}{d\Omega} = |\mathrm{f}_0|^2 = \frac{1}{\mathrm{k}^2} \sin^2\delta_0, \qquad \sigma = 4\pi|f_0|^2 = \frac{4\pi}{k^2} sin^2\delta_0 \quad (l = 0)$$

The optical theorem

The total cross section can be related to the forward scattering amplitude $f(0)$.

Since $P_l(cos\theta) = P_l(1) = 1$ for $\theta = 0$:

$$f(0) = \frac{1}{k}\sum_l (2l+1)(sin\delta_l\, cos\delta_l + i\, sin^2\delta_l)$$

yields

$$\frac{4\pi}{k}\mathrm{Im}f(0) = \sigma = \frac{4\pi}{k^2}\sum_l (2l+1)sin^2\delta_l$$

This relation is known as the optical theorem.

The physical origin of this theorem is the conversation of particle numbers.

Partial wave analysis for inelastic scattering

The scattering amplitude can be rewritten as:

$$f(\theta) = \sum_l (2l+1) f_l(k) P_l(cos\theta)$$

where

$$f_l(k) = \frac{1}{k}(e^{i\delta_l} - 1) = 1/2ik\,(S_l(k) - 1)$$

with

$$S_l(k) = e^{2i\delta_l}$$

In case of NO beam particle losses, $|S_l(k)| = 1$

If there is absorption of the incident beam $S_l(k)$ is redefined by

$$S_l(k) = \eta_l(k) e^{2i\delta_l}$$

with $0 < \eta_l(k) \leq 1$

$$f_l(k) = \frac{\eta_l e^{2i\delta_l} - 1}{2ik} = \frac{1}{2k}[\eta_l\, sin2\delta_l + i(1 - \eta_l\, cos2\delta_l)]$$

$$f(\theta) = \frac{1}{2k}\sum_l (2l+1)[\eta_l\, sin2\delta_l + i(1 - \eta_l\, cos2\delta_l)] P_l(cos\theta)$$

Total elastic and inelastic cross section

The total elastic scattering cross section is then

$$\sigma_{el} = 4\pi \sum_{l} (2l+1)|f_l|^2 = \frac{\pi}{k^2} \sum_{l} (2l+1)(1+\eta_l^2 - 2\eta_l \cos 2\delta_l)$$

The total inelastic scattering cross section, which describes the loss of flux:

$$\sigma_{inel} = \frac{\pi}{k^2} \sum_{l} (2l+1)(1-\eta_l^2(k))$$

If $\eta_{l(k)} = 1$ there is no inelastic scattering, but if $\eta_{l(k)} = 0$ there is max absorption, but still elastic scattering in this wave.

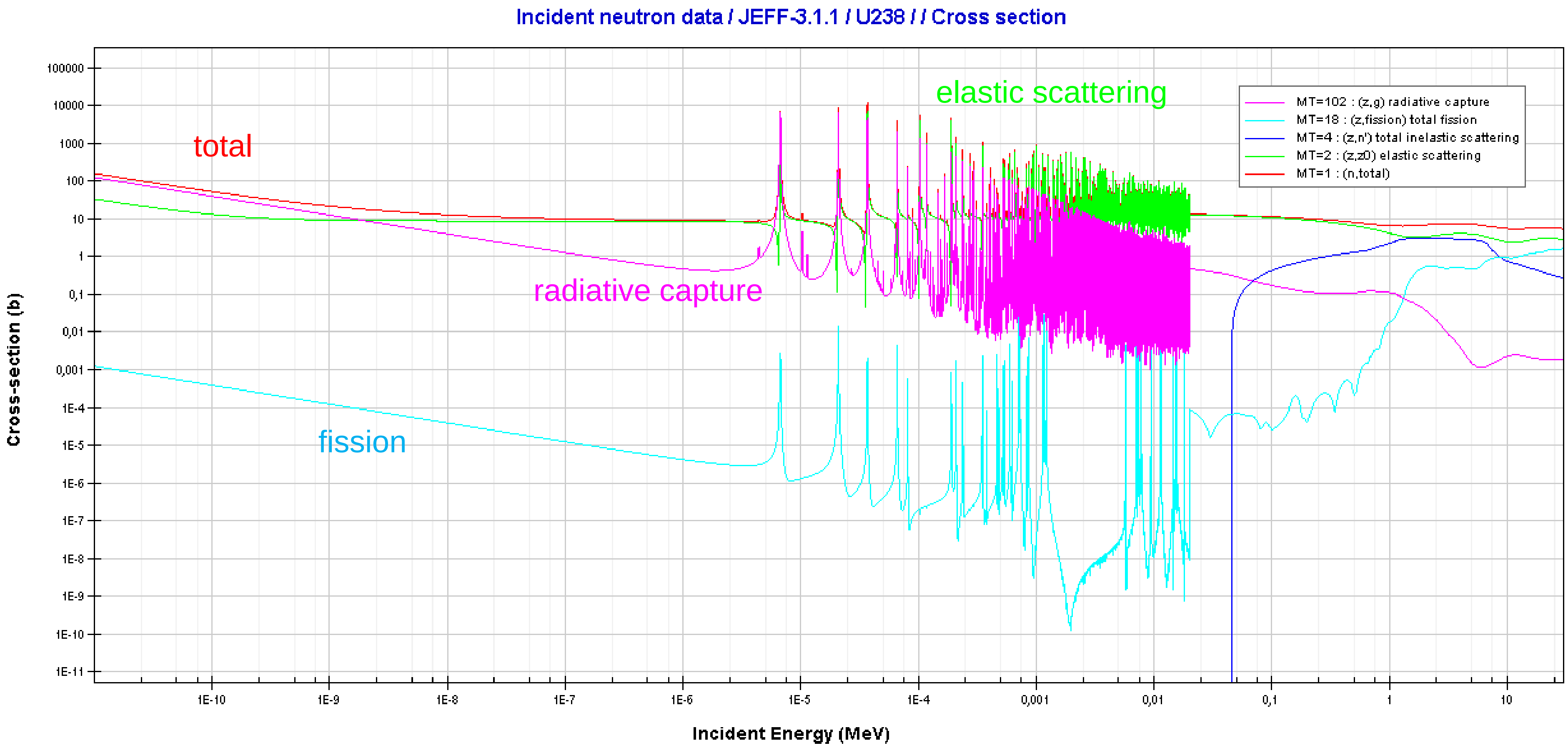
Total cross section:

$$\sigma_{tot} = \sigma_{el} + \sigma_{inel} = \frac{2\pi}{k^2} \sum_{l} (2l+1)(1-\eta_l \cos 2\delta_l)$$

The optical theorem is also valid:

$$\mathrm{Im} f(0) = k\sigma_{tot}/4\pi$$

Resonances: n scattered on U238



Resonance scattering

Resonance characterized by spin and parity => contribution from few partial waves only (conservation of orbital angular momentum)

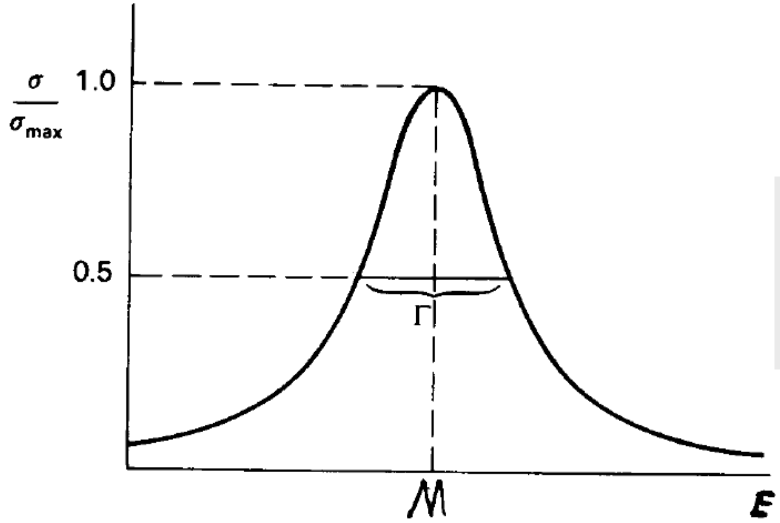
Differential cross section:

$$\frac{d\sigma^{\mathrm{Res}}}{d\Omega} = \frac{1}{|\vec{p}|^2}(2J+1)^2 P_J^2(\cos\theta)\frac{\Gamma^2/4}{(\sqrt{s}-M)^2+\Gamma^2/4}$$

Total cross section:

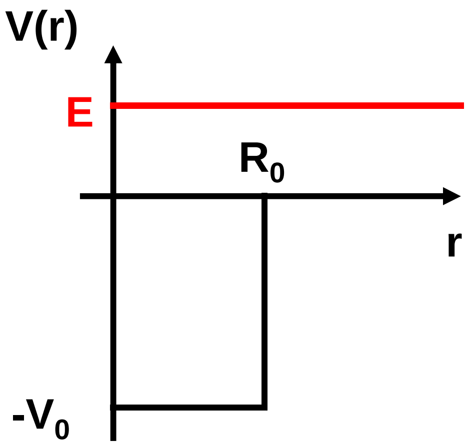
$$\sigma^{\mathrm{Res}} = \frac{4\pi}{k^2}(2J+1)\frac{\Gamma^2/4}{(\sqrt{s}-M)^2+\Gamma^2/4}$$

Breit-Wigner distribution (NR):



$$\sigma(E) = \sigma_0 \frac{\frac{\Gamma^2}{4}}{(E-M)^2+\frac{\Gamma^2}{4}}$$

S – wave scattering



Low relative energy dominated by s-wave scattering.

Condition (non relativistic):

$$mvR_0 << 1$$

$$\Downarrow$$

$$E_{kin} = \frac{p^2}{2m} << \frac{1}{2mR_0^2}$$

QM – solution of scattering problem:

$$\psi(\vec{r}) = R(r)Y_{lm}(\theta,\varphi)$$

$$u(r) = rR(r)$$

Radial SE:

$$u''(r) + k^2u(r) = 0$$

$$u(r) = \alpha e^{ikr} + \beta e^{-ikr}$$

$$r < R_0 : k_1 = \sqrt{2m(E+V_0)} \longrightarrow u_1(r) = A\sin k_1 r$$

$$r > R_0 : k_2 = \sqrt{2mE} \longrightarrow u_{tot}(r) = \frac{1}{k_2} e^{i\delta_0} \sin(k_2 r + \delta_0)$$

Continuity for u and u':

$$\Rightarrow \delta_0 = -k_2R_0 + \arctan\left(\frac{k_2}{k_1}\tan k_1R_0\right)$$

Sign of phase shift

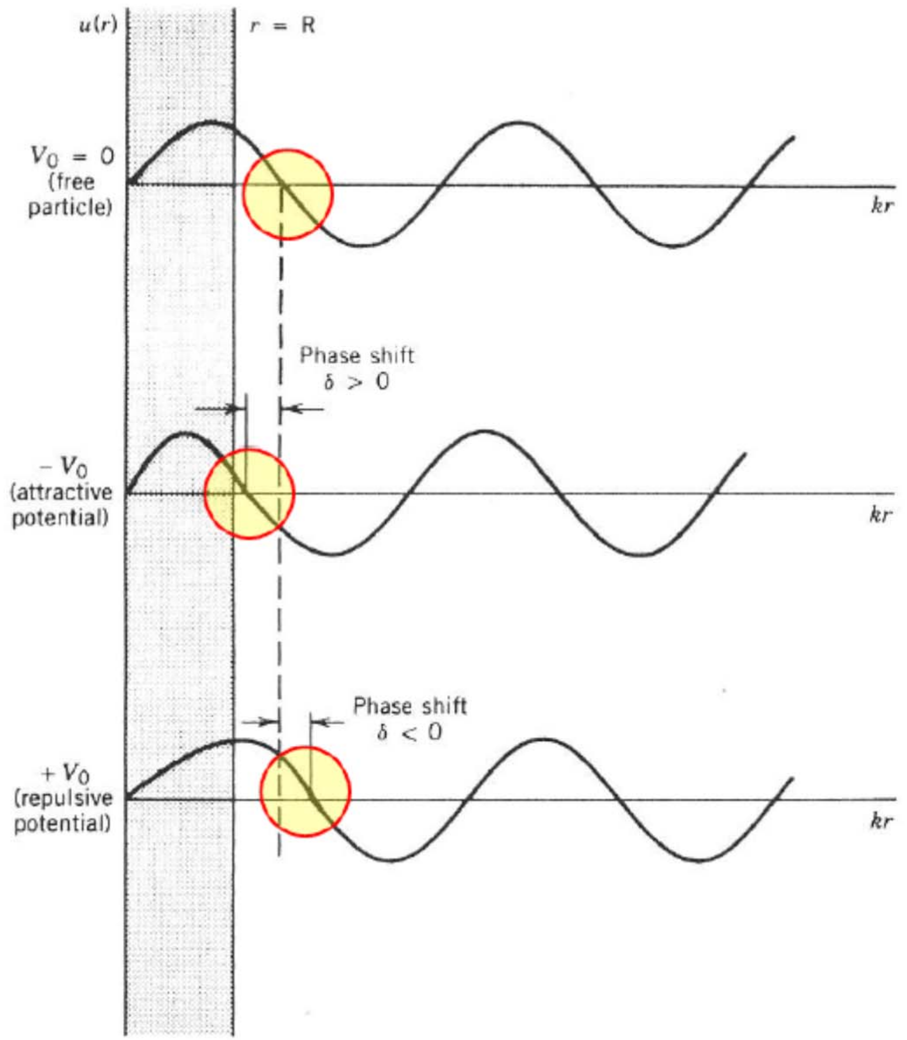
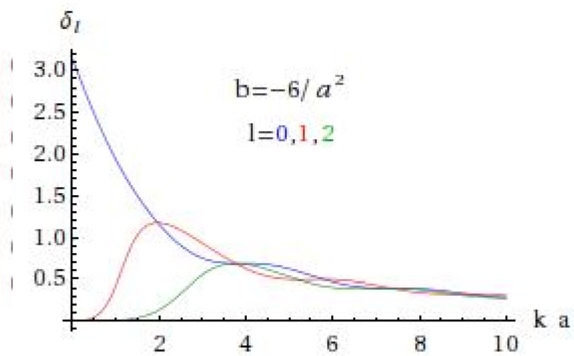


Figure 4.4 The effect of a scattering potential is to shift the phase of the scattered wave at points beyond the scattering regions, where the wave function is that of a free particle.

$$\delta_l \approx \frac{2Mk}{\hbar^2}\int_0^\infty V(r) J_l^2(kr) r^2 dr$$



attractive potential
R=a