

5 Partons and DGLAP equation

During our introduction to QCD and the running coupling we focused on the ultraviolet or high-energy behavior of the theory and noticed how using a running coupling resums potentially large logarithms. In this section we will follow a similar approach to the infrared behavior, where we will find that collinear divergences lead to the DGLAP equations for parton densities, which resum collinear logarithms.

5.1 Incoming partons

To describe scattering processes at hadron colliders, we need to introduce parton distribution functions to describe the probability of finding a collinear parton with momentum fraction x in a proton. Such a pdf is not an observable, but a distribution in the mathematical sense: it has to produce reasonable results when we integrate it with a differential partonic cross section as the test function. Using parton densities we compute a hadronic cross section as

$$\sigma_{\text{tot}} = \int_0^1 dx_1 \int_0^1 dx_2 \sum_{ij} f_i(x_1) f_j(x_2) \hat{\sigma}_{ij}(x_1 x_2 S) , \quad (5.1)$$

where i, j are the incoming partons with the momentum fractions $x_{i,j}$. The partonic energy of the scattering process is $s = x_1 x_2 S$ with the LHC proton energy of $\sqrt{S} = 13.6$ TeV. The partonic cross section $\hat{\sigma}$ includes all θ and δ functions needed for energy–momentum conservation. When we express an n -particle cross section $\hat{\sigma}$ including its phase space integration, the x_i integrations and the phase space integrations can be exchanged, but Jacobians will make life hard.

Different parton densities of the proton have very different behavior — for the valence quarks (uud) they peak below $x < 1/3$, while the gluon pdf is small at $x \sim 1$ and grows rapidly towards small x . For typical LHC kinematics, the gluon density scales roughly like

$$f_g(x) \propto \frac{1}{x^2} \quad \text{for} \quad x = 10^{-3} \dots 10^{-1} . \quad (5.2)$$

We cannot compute parton distribution functions $f_i(x)$ as a function of the longitudinal momentum fraction x , but there are a few predictions we can make based on symmetries and properties of the hadrons, leading to sum rules:

1. The antiproton and proton parton densities are related through the CP-symmetry, which is exact for QCD,

$$f_q^{\bar{p}}(x) = f_{\bar{q}}(x) \quad f_{\bar{q}}^{\bar{p}}(x) = f_q(x) \quad f_g^{\bar{p}}(x) = f_g(x) . \quad (5.3)$$

2. The proton consists of three valence quarks uud , plus quantum fluctuations leading to gluons or quark–antiquark pairs. This sea-quark contribution is symmetric between quarks and antiquarks, so the expectation values for the up and down quark densities in a proton have to fulfill

$$\langle N_u \rangle = \int_0^1 dx (f_u(x) - f_{\bar{u}}(x)) = 2 \quad \langle N_d \rangle = \int_0^1 dx (f_d(x) - f_{\bar{d}}(x)) = 1 . \quad (5.4)$$

3. The expectation value for the proton momentum is split between all parton,

$$\langle \sum x_i \rangle = \int_0^1 dx x \left(\sum_q f_q(x) + \sum_{\bar{q}} f_{\bar{q}}(x) + f_g(x) \right) = 1 . \quad (5.5)$$

If we only take into account the measured quark and antiquark parton densities, we find

$$\int_0^1 dx x \left(\sum_q f_q(x) + \sum_{\bar{q}} f_{\bar{q}}(x) \right) \approx \frac{1}{2} , \quad (5.6)$$

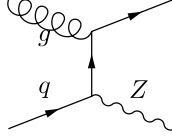
so half of the proton momentum is then carried by quantum-fluctuation gluons.

5.2 Infrared divergences

As the simplest LHC process, let us look at the Drell–Yan process to next-to-leading order or $\sigma_{\text{tot}} \propto G_F \alpha_s$,

$$q\bar{q} \rightarrow Zg. \quad (5.7)$$

Beyond leading order, processes involve renormalization of ultraviolet divergences as well as loop diagrams, which we have to include before we can say anything reasonable, i.e. ultraviolet and infrared finite. To make life easy we look at the crossed NLO contribution



It behaves like a generic other ($2 \rightarrow 2$) process. Because it has a different incoming state than the leading order process, it does not involve virtual corrections or renormalization. Modulo prefactors its amplitude is

$$|\mathcal{M}|^2 \sim -\frac{t}{s} - \frac{s^2 - 2m_Z^2(s + t - m_Z^2)}{st}. \quad (5.8)$$

The Mandelstam variable t for one massless final-state particle can be expressed in terms of the scattering angle

$$t = -s(1 - \tau)y \quad \text{with} \quad y = \frac{1 - \cos \theta}{2} \in [0, 1] \quad \text{and} \quad \tau = \frac{m_Z^2}{s} < 1. \quad (5.9)$$

Similarly, we obtain $u = -s(1 - \tau)(1 - y)$, so we can confirm that $t + u = -s(1 - \tau) = -s + m_Z^2$. The collinear limit, when the gluon splits into a quark-antiquark pair in the beam direction, is given by

$$\begin{aligned} y \rightarrow 0 & \Leftrightarrow t \rightarrow 0 \Leftrightarrow u = -s + m_Z^2 < 0 \\ \Rightarrow |\mathcal{M}|^2 & \rightarrow \frac{s^2 - 2m_Z^2 s + 2m_Z^4}{s(s - m_Z^2)y} + \mathcal{O}(y^0). \end{aligned} \quad (5.10)$$

The cross section diverges for small angles y . We can translate this $1/y$ divergence into the transverse momentum of the gluon or Z using

$$p_T^2 = \frac{tu}{s} = s(1 - \tau)^2 y(1 - y) = \frac{(s - m_Z^2)^2}{s} y + \mathcal{O}(y^2) \quad (5.11)$$

In terms of p_T , the collinear limit of Eq.(5.10) reads

$$|\mathcal{M}|^2 \sim \frac{s^2 - 2sm_Z^2 + 2m_Z^4}{s^2} \frac{s - m_Z^2}{p_T^2} + \mathcal{O}(p_T^0). \quad (5.12)$$

It has a leading divergence proportional to $1/p_T^2$. To compute the total cross section we need to integrate the matrix element over the two-particle phase space. Approximating the matrix element as C'/y or C/p_T^2 this gives us

$$\begin{aligned} \sigma_{\text{tot}} & \sim \int_{y^{\min}}^{y^{\max}} dy \frac{C'}{y} \\ & = \int_{p_T^{\min}}^{p_T^{\max}} dp_T^2 \frac{C}{p_T^2} = 2 \int_{p_T^{\min}}^{p_T^{\max}} dp_T p_T \frac{C}{p_T^2} \simeq 2C \int_{p_T^{\min}}^{p_T^{\max}} dp_T \frac{1}{p_T} = 2C \log \frac{p_T^{\max}}{p_T^{\min}} \end{aligned} \quad (5.13)$$

The approximate form C/p_T^2 is valid in the collinear limit; over the non-collinear phase space C is not constant. For this not integrable divergence we follow the same strategy as for the ultraviolet divergence: (i) we regularize it for example using dimensional regularization, and (ii) we find a well-defined way to get rid of it.

Dimensional regularization means writing the two-particle phase space in $n = 4 - 2\epsilon$ dimensions. Just for reference, the complete formula for the y -distribution reads

$$s \frac{d\sigma}{dy} = \frac{\pi(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)} \left(\frac{\mu_F^2}{m_Z^2} \right)^\epsilon \frac{\tau^\epsilon(1-\tau)^{1-2\epsilon}}{y^\epsilon(1-y)^\epsilon} |\mathcal{M}|^2 \sim \left(\frac{\mu_F^2}{m_Z^2} \right)^\epsilon \frac{|\mathcal{M}|^2}{y^\epsilon(1-y)^\epsilon} . \quad (5.14)$$

The factor $1/y^\epsilon$ regularizes the integral at $y \rightarrow 0$, as long as $\epsilon < 0$ slightly increases the suppression of the integrand in the infrared. Integrating the leading collinear divergence $1/y^{1+\epsilon}$ gives us a pole $1/(-\epsilon)$. This means that for infrared divergences we choose $n = 4 + 2|\epsilon|$ space-time dimensions.

As for the ultraviolet case, a scale $\mu_F^{2\epsilon}$ appears with the n -dimensional integral. When it arises from an infrared regularization we call it factorization scale. The actual removal of the infrared pole — corresponding to the renormalization in the ultraviolet case — is called mass factorization and works exactly the same way as renormalizing a parameter: in a well-defined scheme we subtract the pole from the fixed-order result.

5.3 Parton splitting

Let us understand the patterns behind infrared divergences for massless quarks and gluons in the initial or final state. At the LHC, the most important case is the splitting of one gluon into two,

$$g(p_a) \rightarrow g(p_b) + g(p_c) \quad \text{with} \quad p_a^2 \equiv (-p_b - p_c)^2 \gg p_b^2, p_c^2 . \quad (5.15)$$

The two daughter gluons are close to mass shell while the mother has to have a finite positive invariant mass.

We begin with the phase space, common to all parton splittings. In our case we assume an outgoing gluon as part of an n -particle process, leading to an $(n+1)$ -particle process with the two outgoing gluons. We describe the kinematics of this approximately collinear process in terms of the energy fractions z and $1-z$ defined as

$$z = \frac{|E_b|}{|E_a|} = 1 - \frac{|E_c|}{|E_a|} \quad p_a^2 = (-p_b - p_c)^2 = 2(p_b p_c) = 2z(1-z)(1 - \cos \theta) E_a^2 \\ \Leftrightarrow \quad \theta \equiv \theta_b + \theta_c . \quad (5.16)$$

in the collinear limit and in terms of the opening angle θ between $\vec{p}_b$ and $\vec{p}_c$. Using this parameterization we split the $(n+1)$ -particle phase space into an n -particle phase space and a splitting process of quarks and gluons,

$$d\Phi_{n+1} = \cdots \frac{d^3\vec{p}_b}{2(2\pi)^3|E_b|} \frac{d^3\vec{p}_c}{2(2\pi)^3|E_c|} = \cdots \frac{d^3\vec{p}_a}{2(2\pi)^3|E_a|} \frac{d^3\vec{p}_c}{2(2\pi)^3|E_c|} \frac{|E_a|}{|E_b|} \\ \equiv d\Phi_n \frac{dp_{c,3} dp_T p_T d\phi}{2(2\pi)^3|E_c|} \frac{1}{z} \\ = d\Phi_n \frac{dp_{c,3} dp_T^2 d\phi}{4(2\pi)^3|E_c|} \frac{1}{z} \quad (5.17)$$

This separation does not include any (collinear) approximation. Next, we translate $p_{c,3}$ and p_T^2 into z and $p_a^2 \neq 0$ in the collinear limit,

$$p_a^2 = z(1-z)E_a^2\theta^2 + \mathcal{O}(\theta^4) \\ \theta \simeq \frac{1}{|E_a|} \sqrt{\frac{p_a^2}{z(1-z)}} \\ \Rightarrow \quad \frac{dp_{c,3}}{|E_c|} = \frac{dz}{1-z} (1 + \mathcal{O}(\theta)) \quad \text{and} \quad dp_T^2 = z(1-z)dp_a^2 . \quad (5.18)$$

Assuming azimuthal symmetry in an addition step this gives us the the separated collinear phase space

$$d\Phi_{n+1} = d\Phi_n \frac{dz dp_a^2 d\phi}{4(2\pi)^3} (1 + \mathcal{O}(\theta)) = d\Phi_n \frac{dz dp_a^2}{4(2\pi)^2} (1 + \mathcal{O}(\theta)) . \quad (5.19)$$

Adding the transition matrix elements to the factorized phase space gives us a full factorization in the collinear approximation

$$\begin{aligned}
d\sigma_{n+1} &= \overline{|\mathcal{M}_{n+1}|^2} d\Phi_{n+1} \\
&= \overline{|\mathcal{M}_{n+1}|^2} d\Phi_n \frac{dp_a^2 dz}{4(2\pi)^2} (1 + \mathcal{O}(\theta)) \\
&\simeq \frac{2g_s^2}{p_a^2} \hat{P}(z) \overline{|\mathcal{M}_n|^2} d\Phi_n \frac{dp_a^2 dz}{16\pi^2} \quad \text{assuming} \quad \boxed{\overline{|\mathcal{M}_{n+1}|^2} \simeq \frac{2g_s^2}{p_a^2} \hat{P}(z) \overline{|\mathcal{M}_n|^2}} \\
&= d\sigma_n \frac{dp_a^2}{p_a^2} dz \frac{\alpha_s}{2\pi} \hat{P}(z) .
\end{aligned} \tag{5.20}$$

For the case of incoming parton splitting we replace $p_a^2 \rightarrow t$, the usual Mandelstam variable introduced in Sec. 5.8. We can show this assumed factorization by constructing the appropriate splitting kernels $\hat{P}(z)$ for all quark and gluon configurations:

- For gluon splitting into two gluons we can compute the transition amplitude in terms of the gluon momenta and polarizations. We only quote the result

$$\begin{aligned}
\overline{|\mathcal{M}_{n+1}|^2} &= \frac{2g_s^2}{p_a^2} \frac{N_c}{2} 2 \left[\frac{z}{1-z} + z(1-z) + \frac{1-z}{z} \right] \overline{|\mathcal{M}_n|^2} \\
&\equiv \frac{2g_s^2}{p_a^2} \hat{P}_{g \leftarrow g}(z) \overline{|\mathcal{M}_n|^2} \\
&\Leftrightarrow \quad \hat{P}_{g \leftarrow g}(z) = C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right] ,
\end{aligned} \tag{5.21}$$

using $C_A = N_c$. The splitting kernel is symmetric when we exchange the two gluons z and $(1-z)$. It diverges if either gluon becomes soft. The notation $\hat{P}_{i \leftarrow j} \sim \hat{P}_{ij}$ is inspired by a matrix notation which we can use to multiply the splitting matrix from the right with the incoming parton vector to get the final parton vector.

- A second kernel describes the splitting of a gluon into two quarks. Again, we omit the calculation and quote

$$\hat{P}_{q \leftarrow g}(z) = T_R [z^2 + (1-z)^2] . \tag{5.22}$$

It is symmetric under $z \leftrightarrow (1-z)$ because QCD does not distinguish between the outgoing quark and antiquark.

- The third splitting is gluon radiation off a quark line,

$$\hat{P}_{q \leftarrow q}(z) = C_F \frac{1+z^2}{1-z} . \tag{5.23}$$

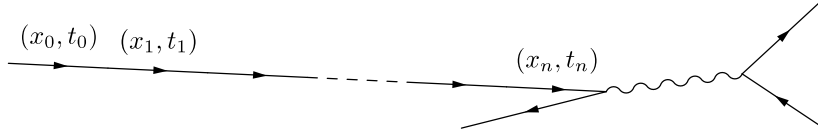
- Just switching $z \leftrightarrow (1-z)$ we can read off the kernel for a quark splitting into the final-state gluon

$$\hat{P}_{g \leftarrow q}(z) = C_F \frac{1+(1-z)^2}{z} . \tag{5.24}$$

A key property of QCD is that, similar to ultraviolet divergences, these collinear splitting kernels are universal. They factorize in the sense that they do not depend on the hard n -particle matrix element as part of the full $(n+1)$ -particle process.

5.4 DGLAP equation

To describe successive splittings we start with a quark inside the proton with an energy fraction x_0 , as it enters the hadronic phase space integral. As this quark is confined inside the proton, it can only have small transverse momentum, which means its four-momentum squared t_0 is negative and its absolute value is small, $|t_0| = p^2 \approx 0$. Let us simplify our argument by assuming that there exists only successive gluon radiation off an incoming quark,



In that case each collinear gluon radiation will decrease the quark energy and increase its virtuality through recoil,

$$x_{j+1} < x_j \quad \text{and} \quad |t_{j+1}| = -t_{j+1} > -t_j = |t_j|. \quad (5.25)$$

We know what the successive splitting means in terms of splitting probabilities and can describe how the parton density $f(x, -t)$ evolves in the (x, t) plane as depicted in Figure 1. We start at (x_0, t_0) and interpret each branching as a step down in $x_j \rightarrow x_{j+1}$ and a step up in $|t_{j+1}|$. The splitting path in the $(x - t)$ plane is marked by discrete points. The probability of a splitting to occur is given by Eq.(5.20),

$$\frac{\alpha_s}{2\pi} \hat{P}(z) \frac{dt}{t} dz \equiv \frac{\alpha_s}{2\pi} \hat{P}_{q \leftarrow q}(z) \frac{dt}{t} dz. \quad (5.26)$$

The evolved parton density at (x_n, t_n) enters the hard scattering, including its energy–momentum conservation.

The link between partonic and hadronic cross section is described by the probability or parton density $f(x, -t)$ over an infinitesimal square,

$$[x_j, x_j + \delta x] \quad \text{and} \quad [|t_j|, |t_j| + \delta t]. \quad (5.27)$$

In our (x, t) plane we can compute the flows into this square and out of this square, which together define the net change in f in the sense of a differential equation,

$$\delta f_{\text{in}} - \delta f_{\text{out}} = \delta f(x, -t). \quad (5.28)$$

We compute the incoming and outgoing flows from the history of the (x, t) evolution. At this stage our picture becomes a little subtle; the way we define the path between two splittings in Figure 1 it can enter and leave the square either vertically or horizontally. Because we want to arrive at a differential equation in t we choose the vertical drop, such that the area the incoming and outgoing flows see is given by δt . If we define a splitting as a vertical drop in x at the target value t_{j+1} , an incoming path hitting the square can come from any x -value above the square. Using this

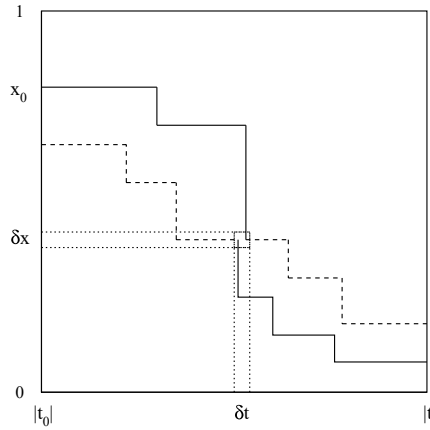


Figure 1: Path of an incoming parton in the $(x - t)$ plane. Because we define t as a negative number its axis is labelled $|t|$.

convention and following the fat solid lines in Figure 1 the vertical flow into the square (x, t) is

$$\begin{aligned}
\delta f_{\text{in}}(-t) &= \delta t \left(\frac{\alpha_s \hat{P}}{2\pi t} \otimes f \right) (x, -t) \\
&= \frac{\delta t}{t} \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}(z) f\left(\frac{x}{z}, -t\right) \\
&\equiv \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}(z) f\left(\frac{x}{z}, -t\right) \quad \text{assuming } f(x', -t) = 0 \text{ for } x' > 1.
\end{aligned} \tag{5.29}$$

We use the definition of a convolution

$$(f \otimes g)(x) = \int_0^1 dx_1 dx_2 f(x_1) g(x_2) \delta(x - x_1 x_2) = \int_0^1 \frac{dx_1}{x_1} f(x_1) g\left(\frac{x}{x_1}\right) = \int_0^1 \frac{dx_2}{x_2} f\left(\frac{x}{x_2}\right) g(x_2). \tag{5.30}$$

The outgoing flow also leaves the infinitesimal square vertically, again following the fat solid line in Figure 1

$$\delta f_{\text{out}}(-t) = \delta t \int_0^1 dy \frac{\alpha_s \hat{P}(y)}{2\pi t} f(x, -t) = \frac{\delta t}{t} f(x, -t) \int_0^1 dy \frac{\alpha_s}{2\pi} \hat{P}(y). \tag{5.31}$$

The y -integration is not a convolution, because we know the starting condition and integrate over all final configurations. Combining Eq.(5.29) and Eq.(5.31) we can compute the change in the quark pdf as

$$\begin{aligned}
\delta f(x, -t) &= \delta f_{\text{in}} - \delta f_{\text{out}} = \frac{\delta t}{t} \left[\int_0^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}(z) f\left(\frac{x}{z}, -t\right) - \int_0^1 dy \frac{\alpha_s}{2\pi} \hat{P}(y) f(x, -t) \right] \\
&= \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \left[\hat{P}(z) - \delta(1-z) \int_0^1 dy \hat{P}(y) \right] f\left(\frac{x}{z}, -t\right) \\
&\equiv \frac{\delta t}{t} \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}(z)_+ f\left(\frac{x}{z}, -t\right) \\
\Leftrightarrow \quad \frac{\delta f(x, -t)}{\delta(-t)} &= \frac{1}{(-t)} \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}(z)_+ f\left(\frac{x}{z}, -t\right)
\end{aligned} \tag{5.32}$$

Strictly speaking, we require α_s to only depend on t and introduce the plus subtraction

$$F(z)_+ \equiv F(z) - \delta(1-z) \int_0^1 dy F(y) \quad \text{or} \quad \int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \left(\frac{f(z)}{1-z} - \frac{f(1)}{1-z} \right). \tag{5.33}$$

For the second definition we choose $F(z) = 1/(1-z)$, multiply it with an arbitrary test function $f(z)$ and integrate over z . The plus-subtracted integral is by definition finite in the soft limit $z \rightarrow 1$, where some splitting kernels diverge. It is related to dimensional regularization, defined as

$$\int_0^1 dz \frac{1}{(1-z)^{1-\epsilon}} = \int_0^1 dz \frac{1}{z^{1-\epsilon}} = \frac{z^\epsilon}{\epsilon} \Big|_0^1 = \frac{1}{\epsilon} \quad \text{with } \epsilon > 0, \tag{5.34}$$

corresponding to $4 + 2\epsilon$ dimensions. We can relate the dimensionally regularized integral to the plus subtraction as

$$\begin{aligned}
\int_0^1 dz \frac{f(z)}{(1-z)^{1-\epsilon}} &= \int_0^1 dz \frac{f(z) - f(1)}{(1-z)^{1-\epsilon}} + f(1) \int_0^1 dz \frac{1}{(1-z)^{1-\epsilon}} \\
&= \int_0^1 dz \frac{f(z) - f(1)}{1-z} (1 + \mathcal{O}(\epsilon)) + \frac{f(1)}{\epsilon} \\
&= \int_0^1 dz \frac{f(z)}{(1-z)_+} (1 + \mathcal{O}(\epsilon)) + \frac{f(1)}{\epsilon} \quad \text{by definition} \\
\Leftrightarrow \quad \int_0^1 dz \frac{f(z)}{(1-z)^{1-\epsilon}} - \frac{f(1)}{\epsilon} &= \int_0^1 dz \frac{f(z)}{(1-z)_+} (1 + \mathcal{O}(\epsilon)).
\end{aligned} \tag{5.35}$$

The dimensionally regularized integral minus the pole, i.e. the finite part of the dimensionally regularized integral, is the same as the plus-subtracted integral modulo terms of the order ϵ . This means that the plus-subtraction plays a similar role as $\overline{\text{MS}}$ renormalization for UV-divergences.

To regularize our splitting kernel $\hat{P}_{q \leftarrow q}$ in Eq.(5.23) we can, actually, define two subtraction schemes,

$$\begin{aligned} \left(\frac{1+z^2}{1-z} \right)_+ - (1+z^2) \left(\frac{1}{1-z} \right)_+ &= \frac{1+z^2}{1-z} - \delta(1-z) \int_0^1 dy \frac{1+y^2}{1-y} - \frac{1+z^2}{1-z} + \delta(1-z) \int_0^1 dy \frac{1+z^2}{1-y} \\ &= -\delta(1-z) \int_0^1 dy \left(\frac{1+y^2}{1-y} - \frac{2}{1-y} \right) \\ &= \delta(1-z) \int_0^1 dy \frac{y^2-1}{y-1} = \delta(1-z) \int_0^1 dy (y+1) = \frac{3}{2} \delta(1-z). \end{aligned} \quad (5.36)$$

This means we can write the quark splitting kernel in two equivalent ways, where $\delta(1-z)$ implies that the difference is a contribution to a soft-radiation phase space integral. We simplify the notation as $\hat{P}_+ \equiv P$ and write

$$\begin{aligned} P_{q \leftarrow q}(z) &\equiv C_F \left(\frac{1+z^2}{1-z} \right)_+ = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right] \\ \Rightarrow \quad &\boxed{\frac{df_q(x, -t)}{d \log(-t)} = -t \frac{df_q(x, -t)}{d(-t)} = \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P_{q \leftarrow q}(z) f_q\left(\frac{x}{z}, -t\right)}. \end{aligned} \quad (5.37)$$

This is the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi or DGLAP equation for quarks radiating gluons. We have to generalize this DGLAP equation to quarks and gluons. For the quark density on the left hand side we write

$$\frac{df_q(x, -t)}{d \log(-t)} = -t \frac{df_q(x, -t)}{d(-t)} = \sum_{j=q,g} \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P_{q \leftarrow j}(z) f_j\left(\frac{x}{z}, -t\right). \quad (5.38)$$

The splitting from gluon to quark can be included as

$$\delta f_q(x, -t) = \frac{\delta t}{t} \left[\int_0^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}_{q \leftarrow q}(z) f_q\left(\frac{x}{z}, -t\right) + \int_0^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \hat{P}_{q \leftarrow g}(z) f_g\left(\frac{x}{z}, -t\right) \right]. \quad (5.39)$$

The second term is a convolution proportional to the gluon pdf. Quarks can be produced in gluon splitting but cannot vanish into it. Therefore Eq.(5.39) $P_{q \leftarrow g}$ does not include a plus-regulator

$$P_{q \leftarrow g}(z) \equiv \hat{P}_{q \leftarrow g}(z) = T_R [z^2 + (1-z)^2]. \quad (5.40)$$

This kernel is indeed missing a soft-radiation divergence for $z \rightarrow 1$.

The second parton density we have to study is the gluon density. The incoming contribution to the infinitesimal square is given by the sum of four splitting scenarios each leading to a gluon with virtuality $-t_{j+1}$

$$\begin{aligned} \delta f_{\text{in}}(-t) &= \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \left[\hat{P}_{g \leftarrow g}(z) \left(f_g\left(\frac{x}{z}, -t\right) + f_g\left(\frac{x}{1-z}, -t\right) \right) + \hat{P}_{g \leftarrow q}(z) \left(f_q\left(\frac{x}{z}, -t\right) + f_{\bar{q}}\left(\frac{x}{z}, -t\right) \right) \right] \\ &= \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \left[2\hat{P}_{g \leftarrow g}(z) f_g\left(\frac{x}{z}, -t\right) + \hat{P}_{g \leftarrow q}(z) \left(f_q\left(\frac{x}{z}, -t\right) + f_{\bar{q}}\left(\frac{x}{z}, -t\right) \right) \right], \end{aligned} \quad (5.41)$$

using $P_{g \leftarrow \bar{q}} = P_{g \leftarrow q}$ in the first line and $P_{g \leftarrow g}(1-z) = P_{g \leftarrow g}(z)$ in the second. To leave the volume element in (x, t) -space a gluon can either split into two gluons or radiate one of n_f light-quark flavors. Combining the incoming and outgoing flows we find

$$\begin{aligned} \delta f_g(x, -t) &= \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} \left[2\hat{P}_{g \leftarrow g}(z) f_g\left(\frac{x}{z}, -t\right) + \hat{P}_{g \leftarrow q}(z) \left(f_q\left(\frac{x}{z}, -t\right) + f_{\bar{q}}\left(\frac{x}{z}, -t\right) \right) \right] \\ &\quad - \frac{\delta t}{t} \int_0^1 dy \frac{\alpha_s}{2\pi} \left[\hat{P}_{g \leftarrow g}(y) + n_f \hat{P}_{q \leftarrow g}(y) \right] f_g(x, -t) \end{aligned} \quad (5.42)$$

Unlike in the quark case these terms do not immediately correspond to regularizing the diagonal splitting kernel using the plus prescription.

First, the contribution to δf_{in} proportional to f_q or $f_{\bar{q}}$ which is not matched by the outgoing flow. From the quark case we already know how to deal with it. The corresponding splitting kernel does not need any regularization, so we define

$$P_{g \leftarrow q}(z) \equiv \hat{P}_{g \leftarrow q}(z) = C_F \frac{1 + (1 - z)^2}{z} . \quad (5.43)$$

We see that the structure of the DGLAP equation implies that the two off-diagonal splitting kernels do not include any plus prescription $\hat{P}_{i \leftarrow j} = P_{i \leftarrow j}$. This is expected because the kernels are finite in the soft limit, $z \rightarrow 1$.

Next, we can compute the y -integral describing the gluon splitting into a quark pair directly,

$$\begin{aligned} - \int_0^1 dy \frac{\alpha_s}{2\pi} n_f \hat{P}_{q \leftarrow g}(y) &= - \frac{\alpha_s}{2\pi} n_f T_R \int_0^1 dy [1 - 2y + 2y^2] \quad \text{using Eq.(5.40)} \\ &= - \frac{\alpha_s}{2\pi} n_f T_R \left[y - y^2 + \frac{2y^3}{3} \right]_0^1 \\ &= - \frac{2}{3} \frac{\alpha_s}{2\pi} n_f T_R . \end{aligned} \quad (5.44)$$

Finally, the two terms proportional to the pure gluon splitting $P_{g \leftarrow g}$ in Eq.(5.42) require some work. The y -integral from the outgoing flow has to consist of a finite term and a term we can use to define the plus prescription for $\hat{P}_{g \leftarrow g}$. The integral gives

$$\begin{aligned} - \int_0^1 dy \frac{\alpha_s}{2\pi} \hat{P}_{g \leftarrow g}(y) &= - \frac{\alpha_s}{2\pi} C_A \int_0^1 dy \left[\frac{y}{1-y} + \frac{1-y}{y} + y(1-y) \right] \quad \text{using Eq.(5.21)} \\ &= - \frac{\alpha_s}{2\pi} C_A \int_0^1 dy \left[\frac{2y}{1-y} + y(1-y) \right] \\ &= - \frac{\alpha_s}{2\pi} C_A \int_0^1 dy \left[\frac{2(y-1)}{1-y} + y(1-y) \right] - \frac{\alpha_s}{2\pi} C_A \int_0^1 dy \frac{2}{1-y} \\ &= - \frac{\alpha_s}{2\pi} C_A \int_0^1 dy [-2 + y - y^2] - \frac{\alpha_s}{2\pi} 2C_A \int_0^1 dz \frac{1}{1-z} \\ &= - \frac{\alpha_s}{2\pi} C_A \left[-2 + \frac{1}{2} - \frac{1}{3} \right] - \frac{\alpha_s}{2\pi} 2C_A \int_0^1 dz \frac{1}{1-z} \\ &= \frac{\alpha_s}{2\pi} \frac{11}{6} C_A - \frac{\alpha_s}{2\pi} 2C_A \int_0^1 dz \frac{1}{1-z} . \end{aligned} \quad (5.45)$$

The second term in this result is what we need to replace the first term in the splitting kernel of Eq.(5.21) proportional to $1/(1-z)$ by $1/(1-z)_+$. We can see this using $f(z) = z$ and correspondingly $f(1) = 1$ in Eq.(5.33). The two finite terms in Eq.(5.44) and Eq.(5.45) are included in the definition of $\hat{P}_{g \leftarrow g}$ ad hoc. Because the regularized splitting kernel appears in a convolution, the two finite terms require an explicit factor $\delta(1-z)$. Collecting all of them we arrive at

$$P_{g \leftarrow g}(z) = 2C_A \left(\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right) + \frac{11}{6} C_A \delta(1-z) - \frac{2}{3} n_f T_R \delta(1-z) . \quad (5.46)$$

In its standard form and in terms of the factorization scale $\mu_F^2 \equiv -t$ the complete DGLAP equation reads

$$\boxed{\frac{df_i(x, \mu_F^2)}{d \log \mu_F^2} = \sum_j \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P_{i \leftarrow j}(z) f_j\left(\frac{x}{z}, \mu_F\right) = \frac{\alpha_s}{2\pi} \sum_j (P_{i \leftarrow j} \otimes f_j)(x, \mu_F)} . \quad (5.47)$$

Let us briefly recapitulate: for the full quark and gluon particle content of QCD we have derived the DGLAP equation which describes a factorization scale dependence of the quark and gluon parton densities. The universality of the splitting kernels is obvious from the way we derive them — no information on the n -particle process ever enters the derivation.

The DGLAP equation is formulated in terms of four splitting kernels of gluons and quarks which are linked to the splitting probabilities, but which for the DGLAP equation have to be regularized. With the help of a plus-subtraction all kernels $P_{i \leftarrow j}(z)$ become finite, including in the soft limit $z \rightarrow 1$. However, splitting kernels are only regularized when needed, so the finite off-diagonal quark–gluon and gluon–quark splittings are unchanged. This means the plus prescription really acts as an infrared renormalization, moving universal infrared divergences into the definition of the parton densities. The original collinear divergence has vanished as well.

5.5 Resumming collinear logarithms

In our discussion of the DGLAP equation and its solution we encounter the splitting probability in the exponent. To make sense of such a structure we remind ourselves that ratios of α_s values to some power can appear as a result of a resummed series. Such a series would need to include powers of $(\mathcal{M}[\hat{P}])^n$ summed over n which corresponds to a sum over splittings with a varying number of partons in the final state. This suggests that parton densities cannot be formulated in terms of a fixed final state because they include effects from any number of collinear partons summed over the number of such partons, like

$$\boxed{pp \rightarrow \mu^+ \mu^- + X} \quad \text{where } X \text{ includes any number of collinear jets.} \quad (5.48)$$

The same argument leads us towards the logarithms the running parton densities re-sum. To identify them we build a physical model based on collinear splitting, but without using the DGLAP equation. We then solve it to see the resulting structure of the solutions and compare it to the structure of the DGLAP solutions in Eq.(5.67).

We start from the basic equation defining the physical picture of parton splitting in Eq.(5.20). To treat initial state splittings, we need a definition of the virtuality t . We introduce a positive transverse momentum variable

$$-t = \frac{\vec{p}_T^2}{1-z} > 0 \quad \Rightarrow \quad \frac{dt}{t} = \frac{d\vec{p}_T^2}{\vec{p}_T^2}, \quad (5.49)$$

where $\vec{p}_T^2$ is the transverse three-momentum of the parton pair after splitting. Equation(5.20) with a running strong coupling and a generic splitting kernel $P(z)$ then becomes

$$\sigma_{n+1}(x, \mu_F) \approx \int_{x_0}^1 \frac{dx_n}{x_n} P\left(\frac{x}{x_n}\right) \sigma_n(x_n, \mu_0) \int_{\mu_0}^{\mu_F} \frac{d\vec{p}_{T,n}^2}{\vec{p}_{T,n}^2} \frac{\alpha_s(\mu_R^2)}{2\pi}. \quad (5.50)$$

Because the splitting kernel is infrared divergent we cut off the convolution integral at x_0 . Similarly, the transverse momentum integral is bounded by an infrared cutoff μ_0 and the physical external scale μ_F . They give the range in which an additional collinear radiation is included in σ_{n+1} . For splitting the two integrals in Eq.(5.50) it is crucial that μ_0 is the only scale the matrix element σ_n depends on. The other integration variable, the transverse momentum, does not feature in σ_n because collinear factorization is defined in the limit $\vec{p}_T^2 \rightarrow 0$.

If μ_F is the global upper boundary of the transverse momentum integration for collinear splitting, we can apply the recursion formula in Eq.(5.50) iteratively

$$\begin{aligned} \sigma_{n+1}(x, \mu_F) &\approx \int_{x_0}^1 \frac{dx_n}{x_n} P\left(\frac{x}{x_n}\right) \cdots \int_{x_0}^1 \frac{dx_1}{x_1} P\left(\frac{x_2}{x_1}\right) \sigma_1(x_1, \mu_0) \\ &\times \int_{\mu_0}^{\mu_F} \frac{d\vec{p}_{T,n}^2}{\vec{p}_{T,n}^2} \frac{\alpha_s(\mu_R^2)}{2\pi} \cdots \int_{\mu_0}^{\mu_F} \frac{d\vec{p}_{T,1}^2}{\vec{p}_{T,1}^2} \frac{\alpha_s(\mu_R^2)}{2\pi}. \end{aligned} \quad (5.51)$$

Next, we will look for assumptions which allow us to solve Eq.(5.50) and compare the result to the solution of the DGLAP equation. To develop this physics picture of the DGLAP equation we make two assumptions:

1. We identify the scale of the strong coupling α_s with the transverse momentum scale of the splitting,

$$\mu_R^2 = \bar{p}_T^2. \quad (5.52)$$

2. Finally, we assume strongly ordered splittings in the transverse momentum. If the ordering of the splitting is fixed externally by the chain of momentum fractions x_j , this means

$$\mu_0^2 < \bar{p}_{T,1}^2 < \bar{p}_{T,2}^2 < \dots < \mu_F^2 \quad (5.53)$$

Under these two assumptions the transverse momentum integrals in Eq.(5.51) become

$$\begin{aligned} & \int_{\mu_0}^{\mu_F} \frac{d\bar{p}_{T,n}^2}{\bar{p}_{T,n}^2} \frac{\alpha_s(\bar{p}_{T,n}^2)}{2\pi} \dots \int_{\mu_0}^{p_{T,3}} \frac{d\bar{p}_{T,2}^2}{\bar{p}_{T,2}^2} \frac{\alpha_s(\bar{p}_{T,2}^2)}{2\pi} \int_{\mu_0}^{p_{T,2}} \frac{d\bar{p}_{T,1}^2}{\bar{p}_{T,1}^2} \frac{\alpha_s(\bar{p}_{T,1}^2)}{2\pi} \\ &= \int_{\mu_0}^{\mu_F} \frac{d\bar{p}_{T,n}^2}{\bar{p}_{T,n}^2} \frac{1}{2\pi b_0 \log \frac{\bar{p}_{T,n}^2}{\Lambda_{\text{QCD}}^2}} \dots \int_{\mu_0}^{p_{T,3}} \frac{d\bar{p}_{T,2}^2}{\bar{p}_{T,2}^2} \frac{1}{2\pi b_0 \log \frac{\bar{p}_{T,2}^2}{\Lambda_{\text{QCD}}^2}} \int_{\mu_0}^{p_{T,2}} \frac{d\bar{p}_{T,1}^2}{\bar{p}_{T,1}^2} \frac{1}{2\pi b_0 \log \frac{\bar{p}_{T,1}^2}{\Lambda_{\text{QCD}}^2}} \\ &= \frac{1}{(2\pi b_0)^n} \int_{\mu_0}^{\mu_F} \frac{d\bar{p}_{T,n}^2}{\bar{p}_{T,n}^2} \frac{1}{\log \frac{\bar{p}_{T,n}^2}{\Lambda_{\text{QCD}}^2}} \dots \int_{\mu_0}^{p_{T,3}} \frac{d\bar{p}_{T,2}^2}{\bar{p}_{T,2}^2} \frac{1}{\log \frac{\bar{p}_{T,2}^2}{\Lambda_{\text{QCD}}^2}} \int_{\mu_0}^{p_{T,2}} \frac{d\bar{p}_{T,1}^2}{\bar{p}_{T,1}^2} \frac{1}{\log \frac{\bar{p}_{T,1}^2}{\Lambda_{\text{QCD}}^2}}. \end{aligned} \quad (5.54)$$

We can solve the individual integrals by switching variables, for example in the last integral

$$\begin{aligned} \int_{\mu_0}^{p_{T,2}} \frac{d\bar{p}_{T,1}^2}{\bar{p}_{T,1}^2} \frac{1}{\log \frac{\bar{p}_{T,1}^2}{\Lambda_{\text{QCD}}^2}} &= \int_{\log \log \mu_0^2 / \Lambda^2}^{\log \log p_{T,2}^2 / \Lambda^2} d \log \log \frac{\bar{p}_{T,1}^2}{\Lambda_{\text{QCD}}^2} \quad \text{with} \quad \frac{d(ax)}{(ax) \log x} = d \log \log x \\ &= \log \frac{\log p_{T,2}^2 / \Lambda_{\text{QCD}}^2}{\log \mu_0^2 / \Lambda_{\text{QCD}}^2}. \end{aligned} \quad (5.55)$$

This gives us for the chain of transverse momentum integrals, shifted to get rid of the lower boundaries,

$$\begin{aligned} & \int^{p_{T,n} \equiv \mu_F} d \log \frac{\log \bar{p}_{T,n}^2 / \Lambda_{\text{QCD}}^2}{\log \mu_0^2 / \Lambda_{\text{QCD}}^2} \dots \int^{p_{T,2} \equiv p_{T,3}} d \log \frac{\log \bar{p}_{T,2}^2 / \Lambda_{\text{QCD}}^2}{\log \mu_0^2 / \Lambda_{\text{QCD}}^2} \int^{p_{T,1} \equiv p_{T,2}} d \log \frac{\log \bar{p}_{T,1}^2 / \Lambda_{\text{QCD}}^2}{\log \mu_0^2 / \Lambda_{\text{QCD}}^2} \\ &= \int^{p_{T,n} \equiv \mu_F} d \log \frac{\log \bar{p}_{T,n}^2 / \Lambda_{\text{QCD}}^2}{\log \mu_0^2 / \Lambda_{\text{QCD}}^2} \dots \int^{p_{T,2} \equiv p_{T,3}} d \log \frac{\log \bar{p}_{T,2}^2 / \Lambda_{\text{QCD}}^2}{\log \mu_0^2 / \Lambda_{\text{QCD}}^2} \left(\log \frac{\log \bar{p}_{T,2}^2 / \Lambda_{\text{QCD}}^2}{\log \mu_0^2 / \Lambda_{\text{QCD}}^2} \right) \\ &= \int^{p_{T,n} \equiv \mu_F} d \log \frac{\log \bar{p}_{T,n}^2 / \Lambda_{\text{QCD}}^2}{\log \mu_0^2 / \Lambda_{\text{QCD}}^2} \dots \frac{1}{2} \left(\log \frac{\log \bar{p}_{T,3}^2 / \Lambda_{\text{QCD}}^2}{\log \mu_0^2 / \Lambda_{\text{QCD}}^2} \right)^2 \\ &= \int^{p_{T,n} \equiv \mu_F} d \log \frac{\log \bar{p}_{T,n}^2 / \Lambda_{\text{QCD}}^2}{\log \mu_0^2 / \Lambda_{\text{QCD}}^2} \left(\frac{1}{2} \dots \frac{1}{n-1} \right) \left(\log \frac{\log \bar{p}_{T,n}^2 / \Lambda_{\text{QCD}}^2}{\log \mu_0^2 / \Lambda_{\text{QCD}}^2} \right)^{n-1} \\ &= \frac{1}{n!} \left(\log \frac{\log \mu_F^2 / \Lambda_{\text{QCD}}^2}{\log \mu_0^2 / \Lambda_{\text{QCD}}^2} \right)^n = \frac{1}{n!} \left(\log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu_F^2)} \right)^n. \end{aligned} \quad (5.56)$$

This is the final result for the chain of transverse momentum integrals in Eq.(5.51). After integrating over the transverse momenta, the strong coupling is evaluated at $\mu_R \equiv \mu_F$.

Next, we look at the convolution integrals in Eq.(5.50),

$$\sigma_{n+1}(x, \mu) \approx \frac{1}{n!} \left(\frac{1}{2\pi b_0} \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right)^n \int_{x_0}^1 \frac{dx_n}{x_n} P\left(\frac{x}{x_n}\right) \dots \int_{x_0}^1 \frac{dx_1}{x_1} P\left(\frac{x_2}{x_1}\right) \sigma_1(x_1, \mu_0). \quad (5.57)$$

	renormalization scale μ_R	factorization scale μ_F
source	ultraviolet divergence	collinear (infrared) divergence
poles cancelled	counter terms (renormalization)	parton densities (mass factorization)
summation	resum self energy bubbles	resum parton splittings
parameter evolution	running coupling $\alpha_s(\mu_R^2)$ RGE for α_s	running parton density $f_j(x, \mu_F)$ DGLAP equation
large scales	decrease of σ_{tot}	increase of σ_{tot} for gluons/sea quarks
theory background	renormalizability proven for gauge theories	factorization proven all orders for DIS proven order-by-order DY...

Table 2: Comparison of renormalization and factorization scales appearing in LHC cross sections.

As before, we Mellin-transform the equation into moment space

$$\begin{aligned}
\mathcal{M}[\sigma_{n+1}](m, \mu) &\sim \frac{1}{n!} \left(\frac{1}{2\pi b_0} \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right)^n \mathcal{M} \left[\int_{x_0}^1 \frac{dx_n}{x_n} P \left(\frac{x}{x_n} \right) \cdots \int_{x_0}^1 \frac{dx_1}{x_1} P \left(\frac{x_2}{x_1} \right) \sigma_1(x_1, \mu_0) \right] (m) \\
&= \frac{1}{n!} \left(\frac{1}{2\pi b_0} \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \gamma(m) \right)^n \mathcal{M}[\sigma_1](m, \mu_0) \quad \text{using } \gamma(m) \equiv \mathcal{M}[P](m) \quad (5.58)
\end{aligned}$$

This is still not the structure of the DGLAP solution in Eq.(5.67). However, we get it by summing the production cross sections for up to n collinear jets,

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{M}[\sigma_{n+1}](m, \mu) &= \mathcal{M}[\sigma_1](m, \mu_0) \sum_n \frac{1}{n!} \left(\frac{1}{2\pi b_0} \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \gamma(m) \right)^n \\
&= \mathcal{M}[\sigma_1](m, \mu_0) \exp \left(\frac{\gamma(m)}{2\pi b_0} \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right) \\
&= \mathcal{M}[\sigma_1](m, \mu_0) \left(\frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right)^{\frac{\gamma(m)}{2\pi b_0}}. \quad (5.59)
\end{aligned}$$

This is now the same structure as the solution of the DGLAP equation. This means that we can understand the physics of the DGLAP equation using our model calculation of a successive gluon emission, including the generically variable number of collinear jets in the form of $pp \rightarrow \mu^+ \mu^- + X$, as shown in Eq.(5.48). On the left hand side of Eq.(5.59) we have the sum over any number of additional collinear partons; on the right hand side we see fixed order Drell–Yan production without any additional partons, but with an exponentiated correction factor. Comparing this to the running parton densities we can draw the analogy that any process computed with a scale dependent parton density where the scale dependence is governed by the DGLAP equation includes any number of collinear partons.

We can also identify the logarithms which are resummed by scale dependent parton densities. Going back to Eq.(5.13) reminds us that we start from the divergent collinear logarithms $\log p_T^{\text{max}}/p_T^{\text{min}}$ arising from the collinear phase space integration. In our model for successive splitting we replace the upper boundary by μ_F . The collinear logarithm of successive initial–state parton splitting diverges for $\mu_0 \rightarrow 0$, but it gets absorbed into the parton densities and determines the structure of the DGLAP equation and its solutions. The upper boundary μ_F tells us to what extent we assume incoming quarks and gluons to be a coupled system of splitting partons and what the maximum momentum scale of these splittings is. Transverse momenta $p_T > \mu_F$ generated by hard parton splitting are not covered by the DGLAP equation and hence not a feature of the incoming partons anymore. They belong to the hard process and have to be consistently simulated. While this scale can be chosen freely we have to make sure that it does not become too large, because at some point the collinear approximation $C \simeq \text{constant}$ in Eq.(5.13) ceases to hold.

5.6 Extra: solving the DGLAP equation

While it is hard to solve the DGLAP equation in Eq.(5.47) for vectors of pdfs, we can simplify our life by solving it for eigenvalues in parton space. One such eigenvalue is the non-singlet parton density,

$$f_q^{\text{NS}} = (f_q - f_{\bar{q}}) . \quad (5.60)$$

Since gluons cannot distinguish between quarks and antiquarks, the gluon contribution to their evolution cancels, at least in the massless limit, at arbitrary loop order. The corresponding DGLAP equation is

$$\frac{df_q^{\text{NS}}(x, \mu_F^2)}{d \log \mu_F^2} = \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P_{q \leftarrow q}(z) f_q^{\text{NS}}\left(\frac{x}{z}, \mu_F^2\right) . \quad (5.61)$$

To solve it we need a transformation which simplifies a convolution, leading us to the Mellin transform. Starting from a function $f(x)$ of a real variable x we define the Mellin transform into moment space m as

$$\mathcal{M}[f](m) \equiv \int_0^1 dx x^{m-1} f(x) \quad f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dm \frac{\mathcal{M}[f](m)}{x^m} , \quad (5.62)$$

where for the back transformation we choose an arbitrary appropriate constant $c > 0$, such that the integration contour for the inverse transformation lies to the right of all singularities of the analytic continuation of $\mathcal{M}[f](m)$. The key property is that the Mellin transform of a convolution is the product of the two Mellin transforms, which gives us the transformed DGLAP equation

$$\begin{aligned} \frac{d\mathcal{M}[f_q^{\text{NS}}](m, \mu_F^2)}{d \log \mu_F^2} &= \frac{\alpha_s}{2\pi} \mathcal{M} \left[\int_x^1 \frac{dz}{z} P_{q \leftarrow q}(z) f_q^{\text{NS}}\left(\frac{x}{z}, \mu_F^2\right) \right] (m) \\ &= \frac{\alpha_s}{2\pi} \mathcal{M}[P_{q \leftarrow q}](m) \mathcal{M}[f_q^{\text{NS}}](m, \mu_F^2) . \end{aligned} \quad (5.63)$$

In Mellin space it has the simple solution

$$\begin{aligned} \mathcal{M}[f_q^{\text{NS}}](m, \mu_F^2) &= \mathcal{M}[f_q^{\text{NS}}](m, \mu_{F,0}^2) \exp \left(\frac{\alpha_s}{2\pi} \mathcal{M}[P_{q \leftarrow q}](m) \log \frac{\mu_F^2}{\mu_{F,0}^2} \right) \\ &= \mathcal{M}[f_q^{\text{NS}}](m, \mu_{F,0}^2) \left(\frac{\mu_F^2}{\mu_{F,0}^2} \right)^{\frac{\alpha_s}{2\pi} \mathcal{M}[P_{q \leftarrow q}](m)} \\ &\equiv \mathcal{M}[f_q^{\text{NS}}](m, \mu_{F,0}^2) \left(\frac{\mu_F^2}{\mu_{F,0}^2} \right)^{\frac{\alpha_s}{2\pi} \gamma(m)} , \end{aligned} \quad (5.64)$$

defining $\gamma(m) = \mathcal{M}[P](m)$.

This solution still includes μ_F and α_s as two free parameters. Instead, we can include the running of $\alpha_s(\mu_R^2)$ in the DGLAP equation by identifying the renormalization and factorization scales

$$\mu_F \equiv \mu_R \equiv \mu . \quad (5.65)$$

Physically, this identification is motivated by one-scale problems where we have no freedom to choose either of the two scales. In the DGLAP equation it allows us to replace $\log \mu^2$ by α_s using Eq.(4.54)

$$\frac{d}{d \log \mu^2} = \frac{d \log \alpha_s}{d \log \mu^2} \frac{d}{d \log \alpha_s} = \frac{1}{\alpha_s} \frac{d \alpha_s}{d \log \mu^2} \frac{d}{d \log \alpha_s} = -\alpha_s b_0 \frac{d}{d \log \alpha_s} . \quad (5.66)$$

The pre-factor α_s cancels the factor α_s on the right hand side of the DGLAP equation Eq.(5.63)

$$\begin{aligned}
\frac{d\mathcal{M}[f_q^{\text{NS}}](m, \mu^2)}{d \log \alpha_s} &= -\frac{1}{2\pi b_0} \gamma(m) \mathcal{M}[f_q^{\text{NS}}](m, \mu^2) \\
\mathcal{M}[f_q^{\text{NS}}](m, \mu^2) &= \mathcal{M}[f_q^{\text{NS}}](m, \mu_0^2) \exp \left(-\frac{1}{2\pi b_0} \gamma(m) \log \frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right) \\
&= \mathcal{M}[f_q^{\text{NS}}](m, \mu_0^2) \left(\frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right)^{\frac{\gamma(m)}{2\pi b_0}}.
\end{aligned} \tag{5.67}$$

Among other things, in this derivation we neglect that some splitting functions have singularities and therefore the Mellin transform is not obviously well defined. Our convolution is not really a convolution either, because we cut it off at Q_0^2 etc; but the final structure in Eq.(5.67) really holds.

We can find the same solution in pure Yang–Mills theory, *i.e.* in QCD without quarks. Looking at the different color factors in QCD this limit can also be derived as the leading terms in N_c . In that case there also exists only one splitting kernel defining an anomalous dimension γ , and we find the same solution as in Eq.(5.67).

Remembering how we arrive at the DGLAP equation we notice an analogy to the case of ultraviolet divergences and the running coupling. We start from universal infrared divergences. We describe them in terms of splitting functions which we regularize using the plus prescription. The DGLAP equation plays the role of a renormalization group equation for example for the running coupling. It links parton densities evaluated at different scales μ_F . In analogy to the scaling logarithms considered in Section 4.7 we should test if we can point to a type of logarithm the DGLAP equation resums by reorganizing our perturbative series of parton splitting.