# Experimental tests of QCD

1. Motivation for color as addition quantum number
2. Recap: QCD Feynman-Rules and color factors
3. Discovery of the gluon and its spin
4. $q\bar{q} \rightarrow q\bar{q}$: s- and t-channel processes
5. Measurement of alphas and its running

# 1. Color as an additional quantum number

Historically, an additional internal quantum number "color charge" with 3 possible values (often called r, g, b, or correspondingly $\bar{r}$, $\bar{g}$, $\bar{b}$ for anti-quarks ) was introduced to cure a symmetry-problem of the $\Delta^{++}$ (u↑ u↑u↑ , J=3/2) quark wave function:

$$\Psi(uuu,\uparrow\uparrow\uparrow) = \underbrace{\psi_{space} \cdot \phi_{flavor} \cdot \chi_{spin}}_{\text{symmetric under particle exchange}} \cdot \xi_{color}$$

with $\psi_{space} =$ symmetric (L=0)

$\phi_{flavor} =$ symmetric (uuu)

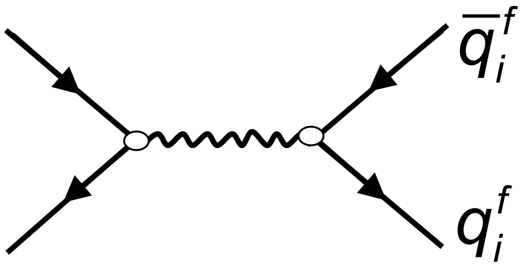
$\chi_{spin} =$ symmetric (↑↑↑)

⇨ Need additional color wave function $\xi_{color}$ fully anti-symmetric

$$\xi_{color} = \frac{1}{\sqrt{6}} \sum_{i,j,k} \varepsilon_{ijk} u_i u_j u_k$$ with color indices i, j, k

# Experimental evidence for 3 different "color states": $N_C = 3$

Hadronic cross section $e^+e^- \rightarrow q\bar{q}$




f flavor

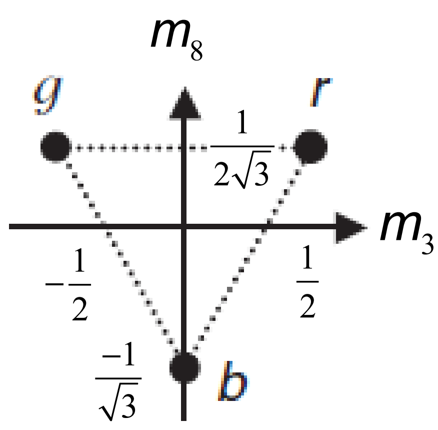
i color

$$R_{had} = \frac{\sigma_{had}}{\sigma_{\mu\mu}} = N_C \cdot \sum_f Q_f^2$$

(we discussed $R_{had}$ already)

Since the color of a quark cannot be observed we postulate a SU(3) color symmetry: r ↔ g ↔ b.

This symmetry was inspired by the u ↔ d ↔ s quark flavor symmetry which was established to describe the hadron spectrum within the static quark model.




Details in lecture by Skyler!

Reminder:

In the fundamental representation the generators $T_a$ of SU(3) are given by the 8 Gell-Mann matrices $\lambda_a$ with a=1…8: $T_a = \frac{1}{2} \lambda_a$ (see QCD).

The 3x3 Gell-Mann matrices $\lambda_a$ are traceless, hermitian and unitary.

The 3 color states in this representation are given by (Skyler's notation)

$$|r\rangle = \left|\frac{1}{2},\frac{1}{2\sqrt{3}}\right\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad |g\rangle = \left|-\frac{1}{2},\frac{1}{2\sqrt{3}}\right\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad |b\rangle = \left|0,-\frac{1}{\sqrt{3}}\right\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

States labelled
by eigenvalues
$m_3$ $m_8$ of $T_3$ and $T_8$

# Reminder Gell-Mann matrices (Recap)

SU(3) Generators: $T_a = \frac{1}{2}\lambda_a$

Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$[\lambda_3, \lambda_8] = 0$ (diagonal)

Lie-Algebra: $[T_a, T_b] = i f_{abc} T_c$

$f_{abc}$ anti-symmetric SU(3) structure constants w/ $f_{acd} f_{bcd} = N_C \delta_{ab}$

$$tr(T_a T_b) = \frac{1}{2}\delta_{ab}$$

# 2. QCD Feynman rules and color factors (recap)

Requiring local gauge invariance introduces 8 vector fields $A_a^\mu$ (gluon fields) and the quark-gluon interaction which depends on the color index i of the quarks. The Lagrangian w/o the gauge-fixing term and w/o ghost term:

$$\mathscr{L}_{\mathrm{QCD}} = \bar{q}_i i\gamma^\mu (D_\mu)_{ij} q_j - \frac{1}{4} F_{a,\mu\nu} F_a^{\mu\nu} \qquad \text{with} \quad i,j = 1,...,3 \qquad a = 1,...,8$$

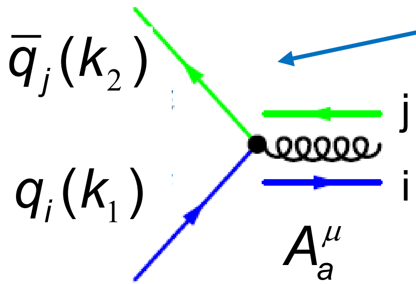
w/ covariant derivative: $$(D_\mu)_{ij} = \partial_\mu \mathbb{1}_{ij} - i g_s A_{\mu,a} (T_a)_{ij}$$

and field non-abelian tensor. $$F_{a,\mu\nu} = \partial_\mu A_{a,\nu} - \partial_\nu A_{a,\mu} - i g_s [A_\mu, A_\nu]_a$$

## Feynman rules:

Colored quarks interact w/ bicolored gluons

Color flow




$$-ig_s(T_a)_{ji}\bar{q}_j(k_2)\gamma^\mu q_i(k_2)$$

(coupling quark current)

Up to additional color factor same as in QED

Gluon propagator


$$-\frac{g_{\mu\nu}}{q^2+i\varepsilon}\delta_{ab}$$

As pointed out by Tilman, the color specific factors and the and Dirac algebra of the $\gamma$ matrices factorize → relevant traces of Gell-Mann matrices separate.

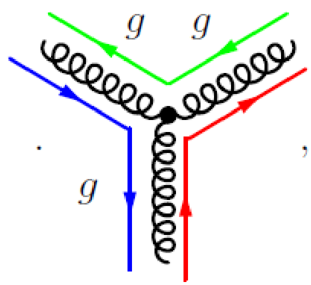
$$|\mathcal{M}|^2 \sim (T_a)_{ji}(T_b)_{ij} = tr(T_a T_b)$$

Remark: consistent w/ Tilman's Langrangian but I use i (j) for in (out) going color index

Convention:

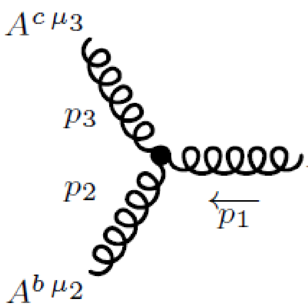
$$\underset{\text{out, in}}{-ig_s(T_a)_{ji}}\;\underset{\text{out}}{\bar{q}_j}(k_2)\gamma^\mu \underset{\text{in}}{q_i}(k_2)$$

Due to the non-abelian structure of the field tensor there are in addition triple and quartic gluon couplings:



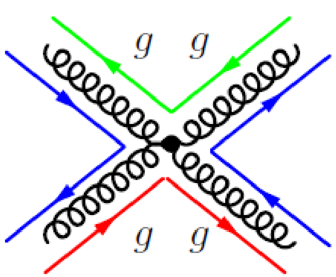

$\sim g_s$

Feynman rule:



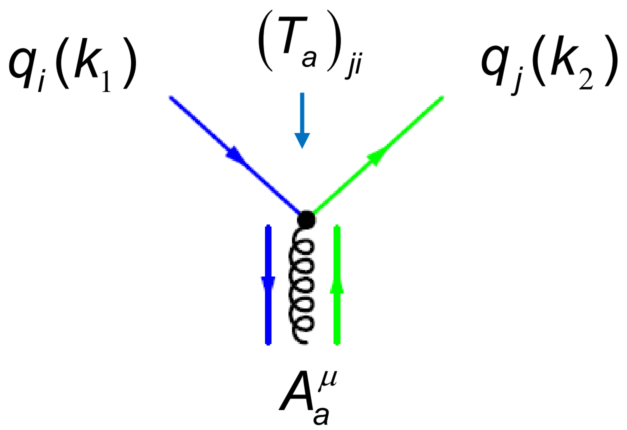

$$: \quad -g_s f^{abc}\left[g^{\mu_1\mu_2}(p_1-p_2)^{\mu_3}+g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1}+g^{\mu_3\mu_1}(p_3-p_1)^{\mu_2}\right]$$

$\times$ 3 gluon fields




$\sim g_s^2$ (even more complicated expression)

# QCD color flow for "pedestrians" I




Gluon carries color and anti-color, represented by $(T_a)_{ji}$ to couple to the corresponding color states $q_i$ and $q_j$

Chosing the color states:

$$|r\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |g\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |b\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and the Gell-Man matrices one has combinations such as:

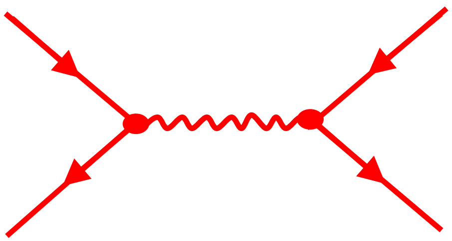
$$\langle g|T_a|b\rangle = \frac{1}{2}(\lambda_a)_{23} \qquad \langle b|T_a|b\rangle = \frac{1}{2}(\lambda_a)_{33}$$

$$\langle r|T_a|b\rangle = \frac{1}{2}(\lambda_a)_{13}$$

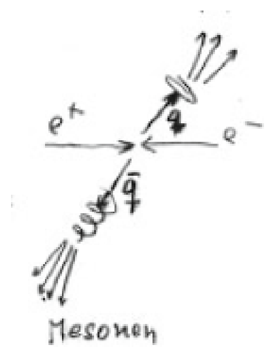
Depending on the indices only a selection of the Gell-Mann matrices contribute. Need to sum over all possible combinations

# 3. Discovery of the gluon & determination of gluon spin

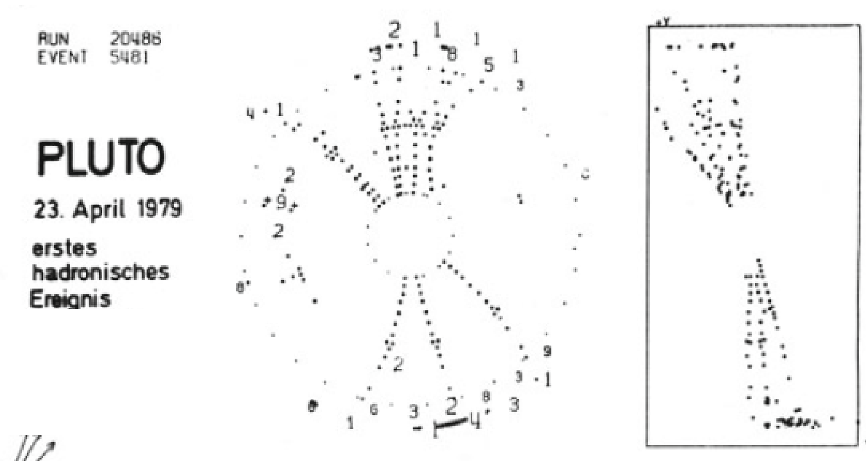
$$e^+e^- \rightarrow q\bar{q}$$

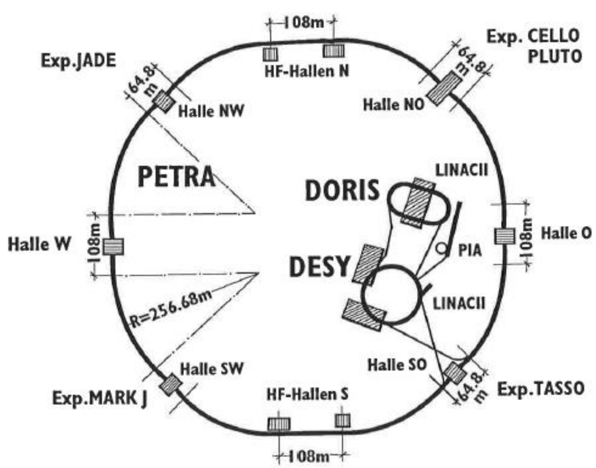


local parton
hadron duality:
partons $\rightarrow$ jets




One of the first 2-jet events at PETRA



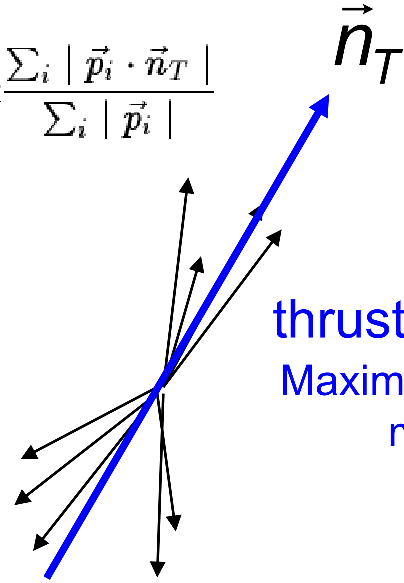

Remark:

PETRA (1978 -) was $e^+e^-$ circular accelerator at DESY: operated at $\sqrt{s}$ between 13 and 46 GeV.

Earlier $e^+e^-$ machines (e.g. SPEAR) with $\sqrt{s}_{max} \approx$ 10 GeV: $ee \rightarrow qq$ events have been observed, however events much less jet-like (more spherical) due to the smaller boost.

## Quantify the 2-jet-likeness:  thrust T

$$T = \max \frac{\sum_i |\vec{p}_i \cdot \vec{n}_T|}{\sum_i |\vec{p}_i|}$$

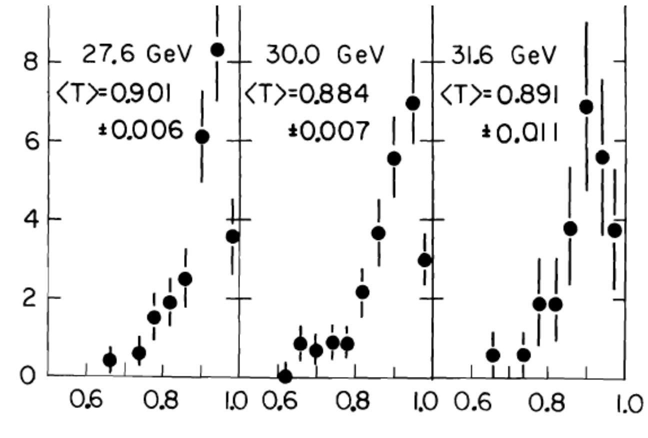



thrust axis = jet axis

Maximizes longitudinal momentum

Expect T close to 1 ⇨

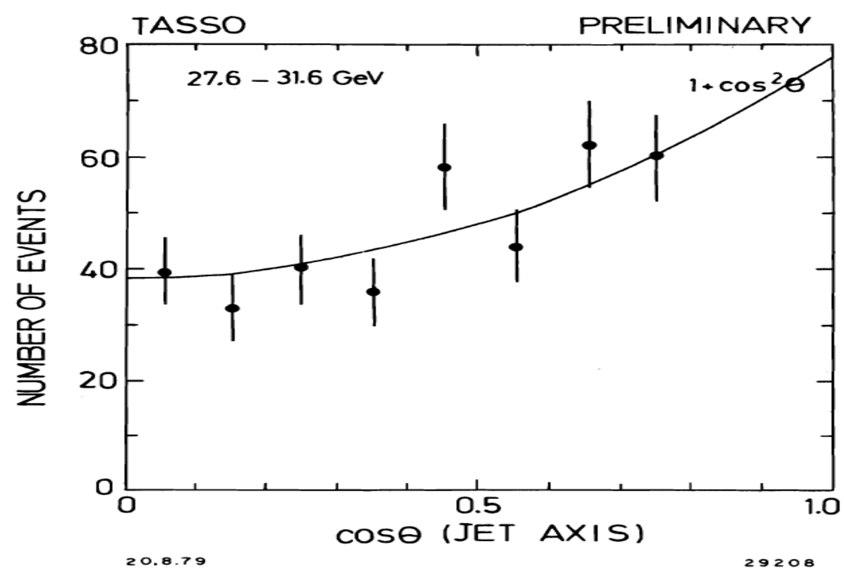
Thrust distribution



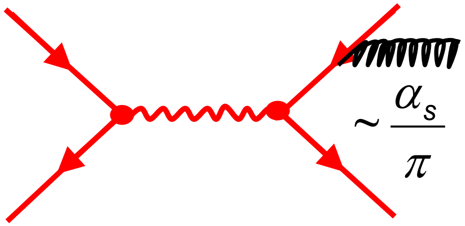

Thrust axis also defines the jet-axis

Jet axis follows $(1+\cos^2\theta)$

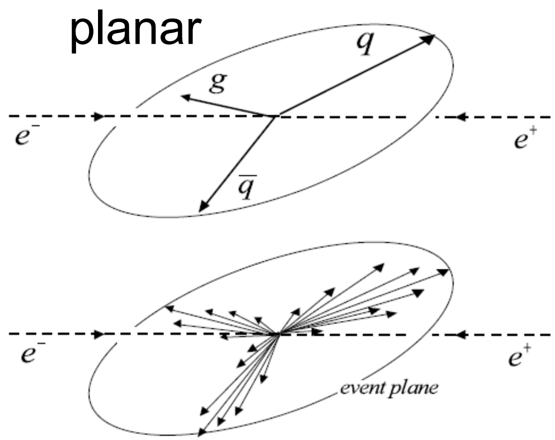
⇨ Quark spin ½



# Gluon radiation: 3-jet events



$\sim \frac{\alpha_s}{\pi}$



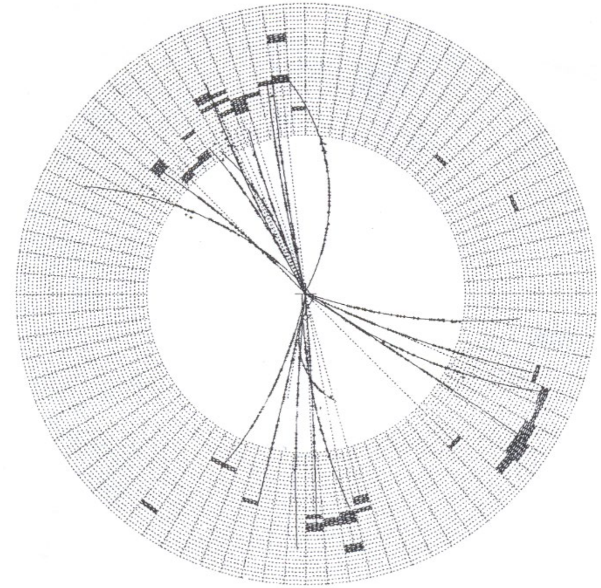
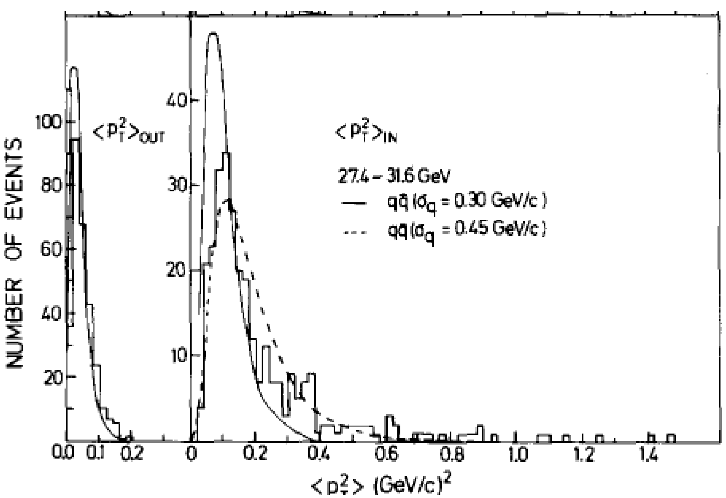



Fig. 11.12 A three-jet event observed by the JADE detector at PETRA.

But: How to exclude that the observed 3-jet signatures are fluctuations?




Check transverse $<p_T^2>$ outside and inside event plane: fluctuations should be the same: Outside $<p_T^2>$ well described by 2-jet model. Inside: "broadening" cannot be described, even not by higher string-tension.

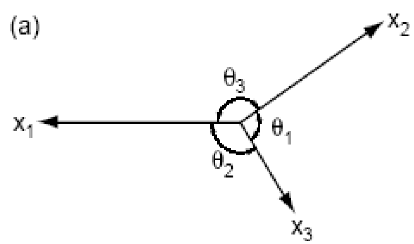
Exp: $$\frac{\#3-\text{jet events}}{\#2-\text{jet events}} \approx 0.15 \sim \frac{\alpha_s}{\pi}$$

# Spin of the Gluon:

## Angular distribution of jets depend on gluon spin:

Ordering of 3 jets: $E_1>E_2>E_3$

Likely to be gluon




Ellis-Karlinger angle

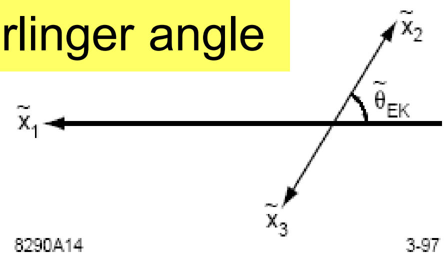



Figure 8: (a) Representation of the momentum vectors in a three-jet event, an (b) definition of the Ellis-Karliner angle.

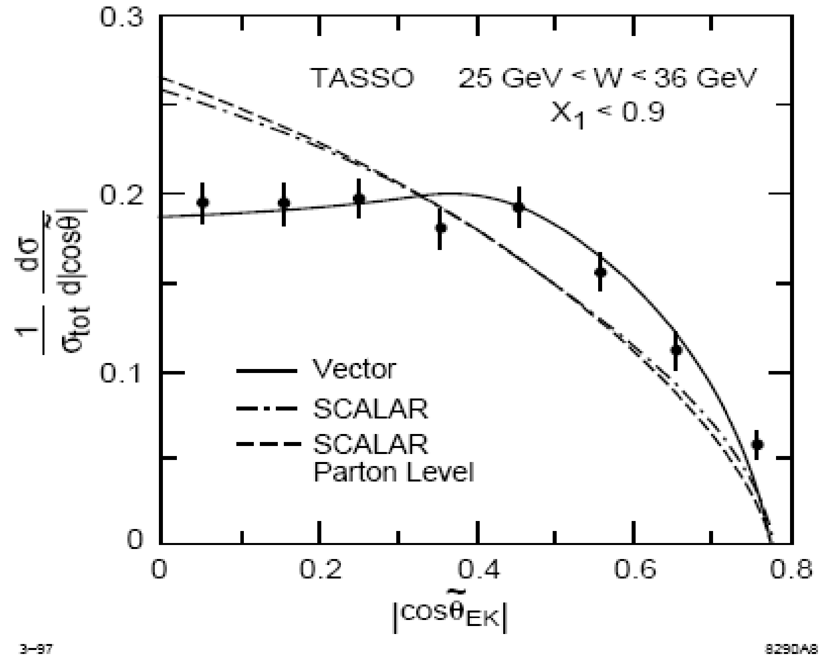



Figure 9: The Ellis-Karliner angle distribution of three-jet events recorded by TASSO at $Q \sim 30$ GeV [18]; the data favour spin-1 (vector) gluons.

Measure direction of jet-1 in the rest frame of jet-2 and jet-3: $\theta_{EK}$

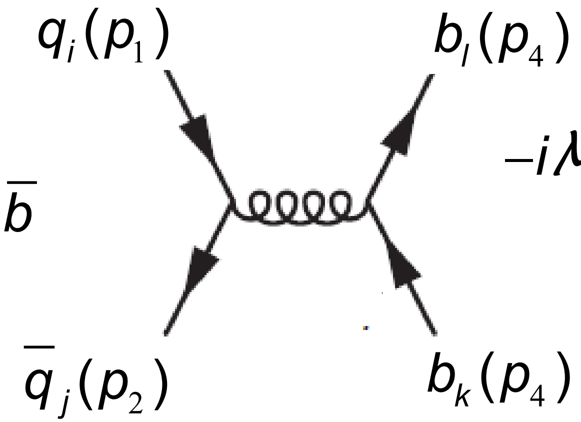
$\boxed{\text{Gluon spin J=1}}$

# 4. $q\bar{q} \rightarrow q\bar{q}$: s- and t-channel processes

The easiest process to calculate is, e.g. $q\bar{q} \rightarrow b\bar{b}$ at tree-level.

As in QED: in-going colored anti-quark (anti-color index j)
corresponds to outgoing colored quark (color index j)

$$q\,\bar{q} \rightarrow b\,\bar{b}$$




$$-i\mathcal{M} = -ig_s(T_a)_{ji}\,\bar{q}_j(p_2)\gamma^{\mu}q_i(p_1)\left(-\frac{g_{\mu\nu}}{q^2}\delta_{ab}\right)$$

$$-\,ig_s(T_b)_{lk}\,\bar{b}_l(p_3)\gamma^{\nu}b_k(p_4)$$

$$= g_s^2 \cdot \underbrace{(T_a)_{ji}(T_a)_{lk}} \cdot (\ldots QED\ldots)$$

$$= g_s^2 \cdot C(i\,\bar{j} \rightarrow l\,\bar{k}) \cdot (\ldots QED\ldots)$$

To calculate $\langle |\mathcal{M}|^2 \rangle$ we need $\langle |C|^2 \rangle$

$$\langle |C|^2 \rangle = \frac{1}{9} \sum_a \sum_{i,j,k,l} |C(i\,j \to k\,l)|^2$$

$$C(i\,j \to k\,l) C^*(i\,j \to k\,l) = (T_a)_{ji} (T_a)_{lk} (T_b)^*_{ji} (T_b)^*_{lk}$$
$$= (T_a)_{ji} (T_a)_{lk} (T_b)_{ij} (T_b)_{kl} = (Tr(T_a T_b))^2$$

$$= \frac{1}{9} \sum_{a,b} (tr(T_a T_b))^2 = \frac{1}{9} \sum_{a,b} \left( \frac{1}{2} \delta_{ab} \right)^2 = \frac{1}{9} \frac{8}{4} = \frac{2}{9}$$

See QCD lecture

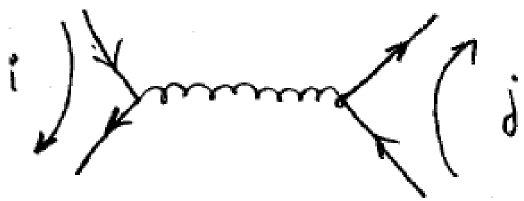
w/ QED matrix element: $\langle |\mathcal{M}|^2 \rangle = 2e^2 \left( \frac{t^2 + u^2}{s^2} \right)$

Here replace e $\leftrightarrow$ $g_s$

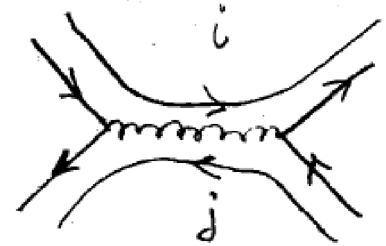
$$\langle |\mathcal{M}(q\bar{q} \to b\bar{b})|^2 \rangle = 2g_s^2 \langle |C|^2 \rangle \left( \frac{t^2 + u^2}{s^2} \right) = 2g_s^2 \frac{2}{9} \left( \frac{t^2 + u^2}{s^2} \right)$$

# Color factors for “pedestrians” II

Color flow for $q\bar{q} \to b\bar{b}$:



$r\bar{r} \to r\bar{r}$



$r\bar{b} \to r\bar{b}$

$$C\begin{pmatrix} r\bar{r} \to r\bar{r} \\ b\bar{b} \to b\bar{b} \\ g\bar{g} \to g\bar{g} \end{pmatrix} = \frac{1}{4}\sum_a (\lambda_a)_{11}\,(\lambda_a)_{11}$$

$$= \frac{1}{4}\left[ (\lambda_3)^2_{11} + (\lambda_8)^2_{11} \right] = \frac{1}{4}\left[ 1 + \left(\frac{1}{\sqrt{3}}\right)^2 \right] = \frac{1}{3}$$

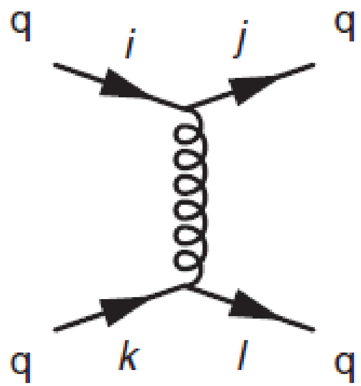
$$C\begin{pmatrix} r\bar{b} \to r\bar{b} \\ r\bar{g} \to \\ b\bar{g} \to \end{pmatrix} = \frac{1}{4}\sum_a (\lambda_a)_{13}\,(\lambda_a)_{31} = \frac{1}{4}\left[ (\lambda_4)_{13}(\lambda_4)_{31} + (\lambda_5)_{13}(\lambda_5)_{31} \right] = \frac{1}{2}$$

$$C\begin{pmatrix} r\bar{r} \to b\bar{b} \\ r\bar{r} \to g\bar{g} \\ b\bar{b} \to \end{pmatrix} = \dots\dots\dots\dots = -\frac{1}{6}$$

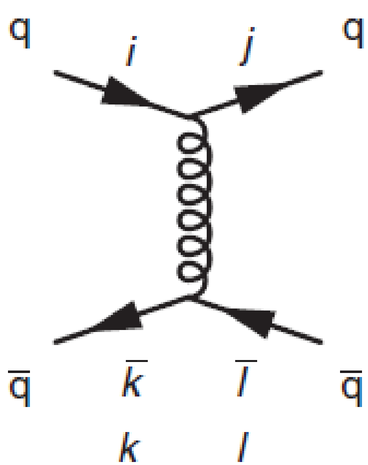
$$\langle |c|^2 \rangle = \frac{1}{9} \cdot \sum_{i,j,k,l} \left| C(ij \to kl) \right|^2$$

$$= \frac{1}{9} \cdot \left[ 3 \cdot \left(\frac{1}{3}\right)^2 + 6 \cdot \left(\frac{1}{2}\right)^2 + 6 \cdot \left(-\frac{1}{6}\right)^2 \right] = \underline{\underline{\frac{2}{9}}}$$

# t-channel quark-(anti)quark scattering:

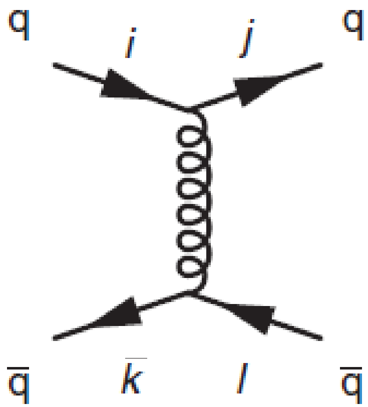



$$(T_a)_{ji}(T_a)_{lk}$$




$$(T_a)_{ji}(T_a)_{kl}$$

# Quark-Antiquark potential ("t-channel"):




In QED: attractive potential.

$$V(\vec{q}) = -\frac{e^2}{|\vec{q}|^2} \quad \text{(in momentum space)}$$

$$\Downarrow$$

In QCD:

$$V(\vec{q}) = -(T_a)_{ji}(T_a)_{kl} \cdot \frac{g_s^2}{|\vec{q}|^2}$$

For the quark-antiquark pair there are two different configurations possible:

- Color singlet: $|q\bar{q}\rangle_S = \delta_{ik}|q_i\bar{q}_k\rangle$

$$C = \delta_{ik} C(i\,k \to j\,l) = C_F \delta_{jl} = \frac{4}{3}\delta_{jl}$$

- Color octet: $|q\bar{q}\rangle_8 \sim (T_a)_{ki}|q_i\bar{q}_k\rangle \longleftarrow$ To describe the color octet qq state assume that they result from gluon splitting

$$C = (T_a)_{ki} C(i\,\bar{k} \to j\,\bar{l}) \sim (T_b T_a T_b)_{jl} = -\frac{1}{2N_C}(T_a)_{jl}$$

# Summary $q\bar{q}$-potential

$$V(\vec{q}) = -\frac{g_s^2}{|\vec{q}|^2} \cdot C \quad \text{with} \quad C = \begin{cases} C_F & \text{for color singlet: attractive} \\ -\dfrac{1}{2N_C} & \text{for color octet: repulsive} \end{cases}$$

This is consistent w/ the fact that only color singlet $q\bar{q}$ pairs are observed as bound states (mesons) in nature.

# 5. Running of strong coupling constant $\alpha_s$

Recap:

Running of $\alpha_s$ as a function of the physical measurement scale $p^2$ :

$$\frac{1}{\alpha_s(p^2)} = \frac{1}{\alpha_s(M^2)}\left(1 + \alpha_s(M^2) b_0 \log\frac{p^2}{M^2}\right)$$

(QCD lecture)

or

$$\alpha_s(p^2) = \frac{\alpha_s(M^2)}{1 + \alpha_s(M^2) b_0 \log\frac{q^2}{M^2}}$$

where

$$b_0 = \frac{1}{4\pi}\left(\frac{11}{3}N_c - \frac{2}{3}n_f\right)$$

$n_f$ = active quark flavors

In QED similar function but w/o first term ~$N_C$ → different sign!!

Using the scale $\Lambda_{QCD}$ (derived from Landau pole – term in brackets vanishes! )

$$\alpha_s(p^2) = \frac{1}{b_0 \log(p^2/\Lambda^2_{QCD})}$$

with $\Lambda_{QCD} \approx 210\,\mathrm{MeV}$

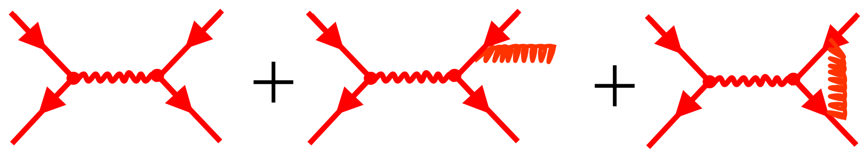
scale at which perturbation theory diverges

# Measurement of $q^2$ dependence of $\alpha_s$

➡ $\alpha_s$ measurements are done at given scale $q^2$: $\alpha_s(q^2)$

## a) $\alpha_s$ from total hadronic cross section in QED

$e^+e^- \to q\,\bar{q}(g)$



$+$ higher orders

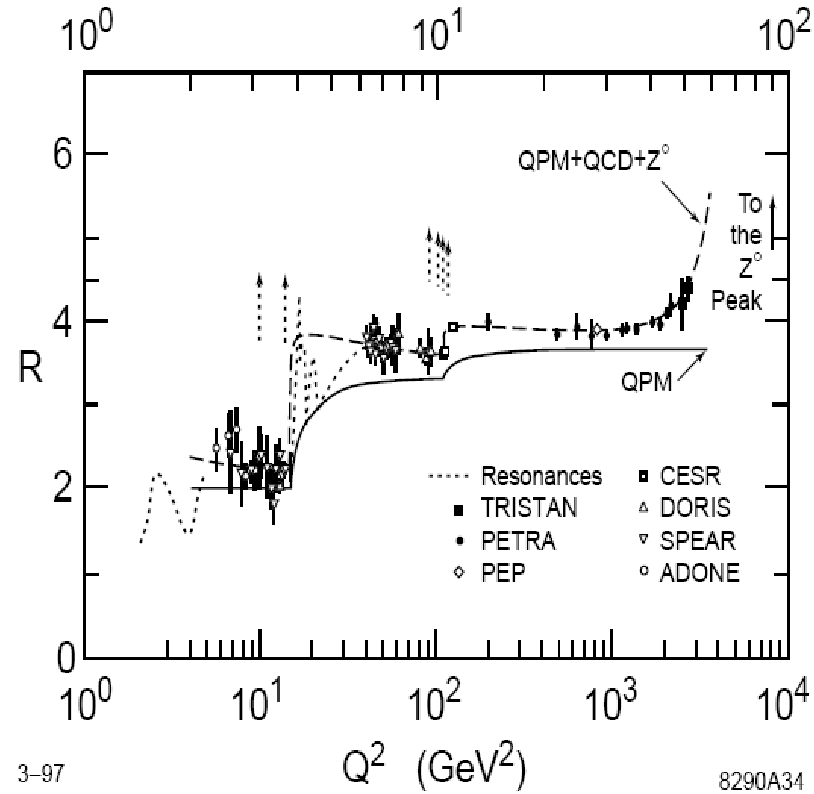
$$\sigma_{had}(s) = \sigma_{had}^{QED}(s)\underbrace{\left[1 + \frac{\alpha_s(s)}{\pi} + 1.4\cdot\frac{\alpha_s(s)^2}{\pi^2} + 12\cdot\frac{\alpha_s(s)^3}{\pi^3} + \ldots\right]}_{1+\delta_{QCD}}$$

$$R_{had} = \frac{\sigma(ee \to hadrons)}{\sigma(ee \to \mu\mu)} = 3\sum Q_q^2\left[1 + \frac{\alpha_s(s)}{\pi} + 1.4\cdot\frac{\alpha_s(s)^2}{\pi^2} + 12\cdot\frac{\alpha_s(s)^3}{\pi^3} + \ldots\right] \quad \text{(QED)}$$

$$\frac{11N_c}{9}$$

# Reminder: $R_{had}$ in QED

$$R_{had} = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_{quarks} Q_q^2$$



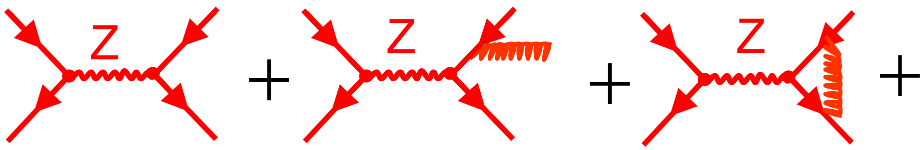

Data lies systematically higher than the prediction from Quark Parton Model (QPM)
$\to$ QCD corrections

$$\sigma(s) = \sigma_{QED}(s)\left[1 + \frac{\alpha_s(s)}{\pi} + 1.4 \cdot \frac{\alpha_s(s)^2}{\pi^2} + \ldots\right]$$

# Z pole: √s = $m_Z$

At the Z-pole instead of the electric charge the relevant couplings are $c_L$ and $c_R$ of the quarks and the muon to the Z.

However QCD corrections stay the same:




$$\sigma_{had}(s) = \sigma_{had}^{QED}(s)\left[1 + \underbrace{1.05 \cdot \frac{\alpha_s(s)}{\pi} + 0.9 \cdot \frac{\alpha_s(s)^2}{\pi^2} + \ldots}_{1+\delta_{QCD}}\right]$$

for $\sigma_{had}^{Z}(s)$ look at EW lecture

The coefficients $C_1, C_2, C_3$ are slightly different from the QED prediction.

Early Z-pole measurement:

$$R_{had}^{Z} = 20.89 \pm 0.13$$

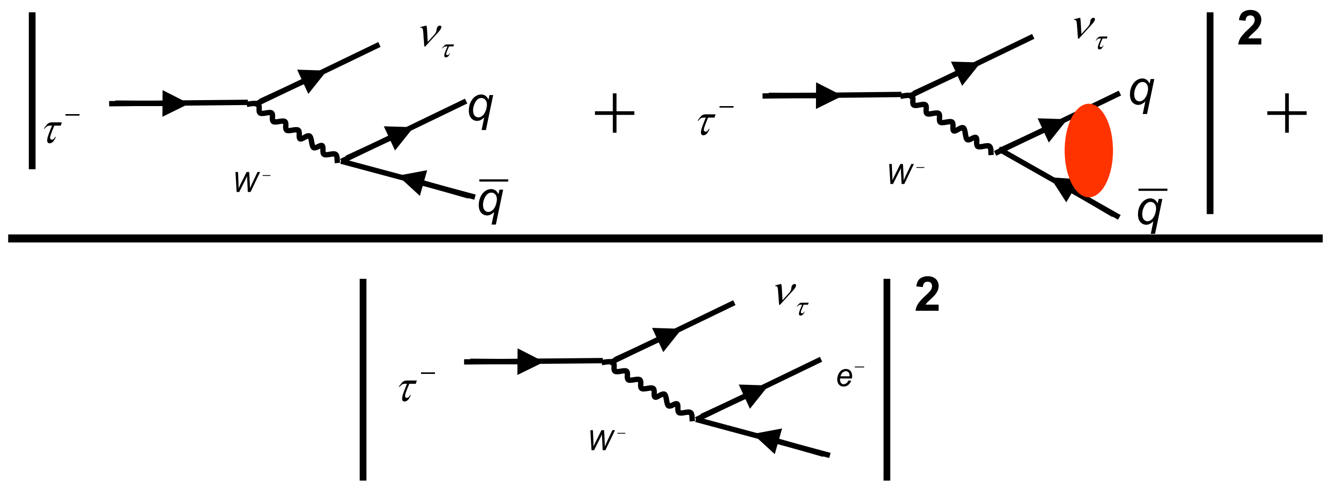
$$\delta_{QCD} = 0.0461 \pm 0.0065$$

$$\alpha_s(m_z) = 0.136 \pm 0.019$$

# c) $\alpha_s$ from hadronic $\tau$ decays: $q^2=m_\tau^2$

Same principle $$R^\tau_{had} = \frac{\Gamma(\tau \to \nu_\tau + Hadrons)}{\Gamma(\tau \to \nu_\tau + e\bar{\upsilon}_e)} \sim f(\alpha_s)$$

$R^\tau_{had} =$




$$R^\tau_{had} = R^{\tau,0}_{had}\left(1 + \frac{\alpha_s(m_\tau^2)}{\pi} + \ldots\right)$$

b) $\alpha_s$ from hadronic event shape variables

d) $\alpha_s$ from DIS (deep inelastic scattering): DGLAP fits to PDFs

(next semester)

# Running of $\alpha_s$ and asymptotic freedom

Experimental determination.

$$\alpha_s(M_Z^2) = 0.1175 \pm 0.0010$$

Alphas from the lattice:

$$\alpha_s(M_Z^2) = 0.1182 \pm 0.0008$$

Unweighted average w/ average uncertainty of the two:

$$\alpha_s(M_Z^2) = 0.1179 \pm 0.0009$$

*Particle Data Group 2021*

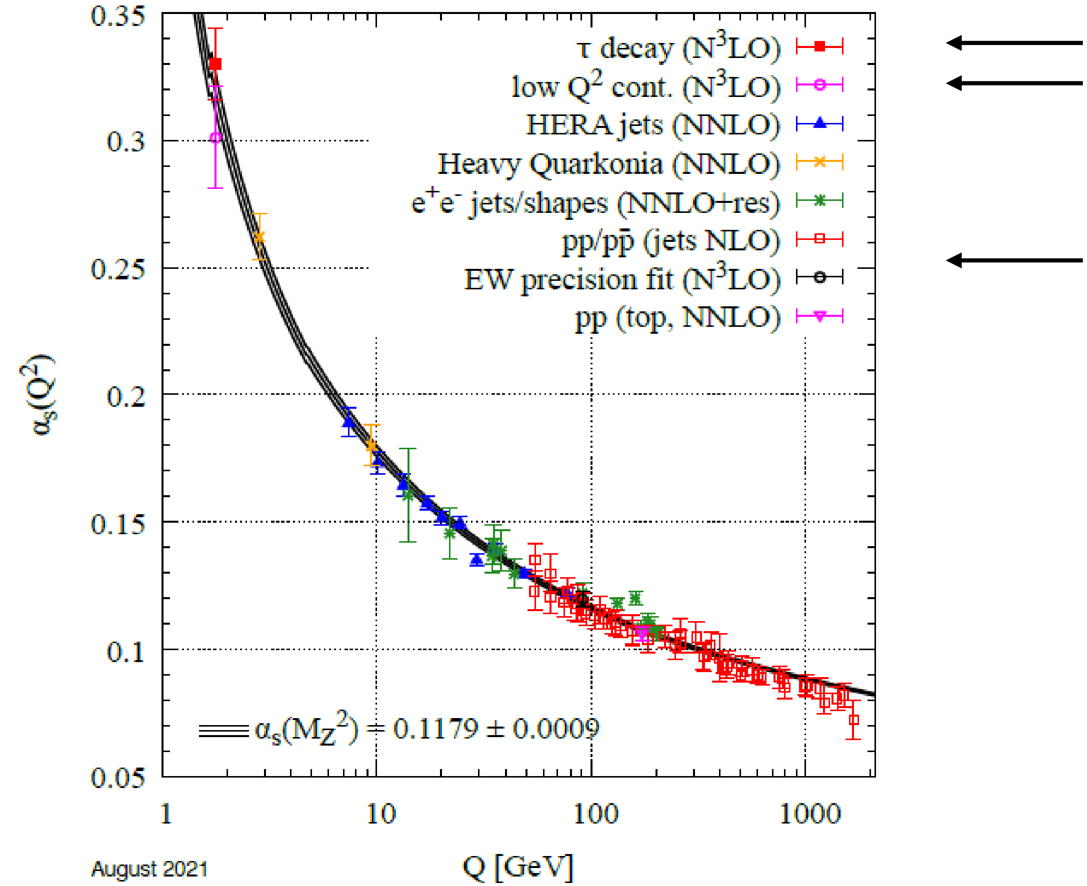



**Figure 9.3:** Summary of measurements of $\alpha_s$ as a function of the energy scale $Q$. The respective degree of QCD perturbation theory used in the extraction of $\alpha_s$ is indicated in brackets (NLO: next-to-leading order; NNLO: next-to-next-to-leading order; NNLO+res.: NNLO matched to a resummed calculation; N$^3$LO: next-to-NNLO).