A few underlying concepts

Units in particle physics

When quoting particle masses, momenta and energies, SI units are not very practical. Instead we use GeV (MeV) as the base unit:

Mass	GeV/c^{2}	$E=mc^{2}$	
Momentum	GeV/c		
Energy	GeV		
Length	$(\mathrm{GeV}/\hbar c)^{-1}$	→	$0.197\,\mathrm{fm}$
Time	$(\mathrm{GeV}/\hbar)^{-1}$	→	$0.658\cdot 10^{-24}\,s$
Cross section	$(\mathrm{GeV}/\hbar c)^{-2}$	→	$0.389\cdot\mathrm{mb}$
			$0.389\cdot 10^{-27}\,\mathrm{cm}^{2}$

$$\hbar c = 0.197\,\mathrm{GeV}\cdot\mathrm{fm}$$

$$\hbar = 0.658\,\mathrm{GeV}\cdot 10^{-24}\,s$$

$$(\hbar c)^{2} = 0.389\,\mathrm{GeV}^{2}\cdot\mathrm{mbarn} \qquad 1\,\mathrm{barn} = 1\,\mathrm{b} = 10^{-28}\mathrm{m}^{2}$$

Natural units $\hbar = c = 1$ in addition: $\varepsilon_0 = \mu_0 = 1$

Simplification of formulae: $E^2 = p^2 + m^2$

All components of a 4-vetor have the same dimension / unit.

All units can be now expressed in GeV. To calculate a result in SI units, need to multiply result with powers of $\hbar c$, $\hbar$, or c

Definition of α: $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$ (In SI units: $\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137}$)

E.g.: Lifetime of a particle τ.

To determine the lifetime of a particle the total decay widths Γ (=sum of partial decay widths Γ_i) is calculated. The dimension of the decay widths is energy. With $\hbar = 1$ the relation between lifetime and Γ simplifies to:

$$\tau = \frac{1}{\Gamma}$$

Lifetime in units 1/GeV can be easily converted into seconds by multiplying with $\hbar = 0.658\,\mathrm{GeV} \cdot 10^{-24}\,s$

Relativistic Kinematics

Special relativity:

Particles in particle physics are often relativistic: Need relativistic description of the kinematics and a relativistic formulation of quantum mechanics.

4-vectors:

contravariant form: $\left(x^{\mu}\right)=(x^{0},x^{1},x^{2},x^{3})=(t,\vec{x})=x$ (time, space)

$\left(p^{\mu}\right)=(p^{0},p^{1},p^{2},p^{3})=(E,\vec{p})=p$ (energy, momentum)

covariant form: $\left(x_{\mu}\right)=(x^{0},-x^{1},-x^{2},-x^{3})=(t,-\vec{x})$

$\left(p_{\mu}\right)=(p^{0},-p^{1},-p^{2},-p^{3})=(E,-\vec{p})$

$x_{\mu}=g_{\mu\nu}x^{\nu}$ and $x^{\mu}=g^{\mu\nu}x_{\nu}$ with $g^{\mu\nu}=g_{\mu\nu}=\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$

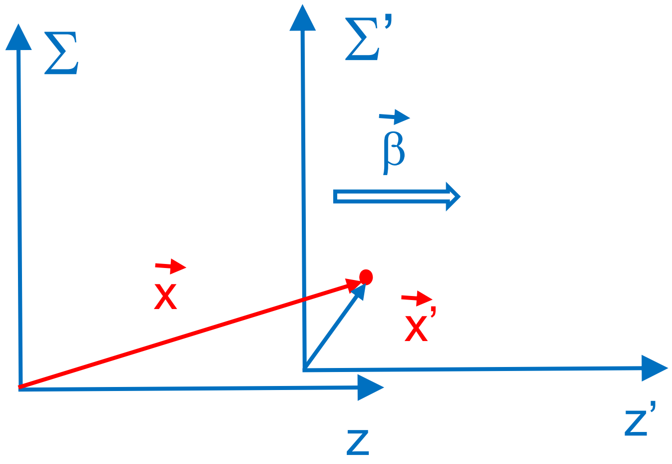
metric tensor

Scalar product: $a\cdot b=a_{\mu}b^{\mu}=g_{\mu\nu}a^{\nu}b^{\mu}$

Lorentz transformations:

System Σ' moves with speed $\vec{\beta}$ relative to Σ:

($\vec{\beta}$ is the velocity in units of c)



$$\begin{aligned} x' &= x \\ y' &= y \\ z' &= \gamma(z-\beta t) \\ t' &= \gamma(t-\beta z) \end{aligned}$$

or using 4-vectors

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & & & -\beta\gamma \\ & 1 & & \\ & & 1 & \\ -\beta\gamma & & & \gamma \end{pmatrix}}\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Lorentz trf. of contravariant 4-vect: $x^{\mu} = (\Lambda)^{\mu}_{\nu} \quad x^{\nu}$

For the covariant form one finds: $\left(x'_{\mu}\right) = \Lambda^{-1}\left(x_{\mu}\right)$

Λ^{-1} same form as
Λ but $-\beta\gamma \rightarrow +\beta\gamma$

One can easily show that scalar products $a_{\mu}b^{\mu}$ are Lorentz invariant ($\Lambda\Lambda^{-1} = 1$)

4-vector derivatives:

$$\frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \xrightarrow[\text{Transformation}]{\text{Lorentz}}$$

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial z'} = \left(\frac{\partial z}{\partial z'} \right) \frac{\partial}{\partial z} + \left(\frac{\partial t}{\partial z'} \right) \frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial t'} = \left(\frac{\partial z}{\partial t'} \right) \frac{\partial}{\partial z} + \left(\frac{\partial t}{\partial t'} \right) \frac{\partial}{\partial t}$$

one finds:

$$\begin{pmatrix} \partial / \partial t' \\ \partial / \partial x' \\ \partial / \partial y' \\ \partial / \partial z' \end{pmatrix} = \begin{pmatrix} \gamma & & & +\beta\gamma \\ & 1 & & \\ & & 1 & \\ +\beta\gamma & & & \gamma \end{pmatrix} \begin{pmatrix} \partial / \partial t \\ \partial / \partial x \\ \partial / \partial y \\ \partial / \partial z \end{pmatrix}$$

i.e. the 4-derivative $\frac{\partial}{\partial x^{\mu}}$ transforms like a covariant vector. Therefore

$$\frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \partial_{\mu}$$

$$\frac{\partial}{\partial x_{\mu}} = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right) = \partial^{\mu}$$

$$\Rightarrow \partial_{\mu}\partial^{\mu} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \Box^2$$

D'Alembert operator

Scalar products of 4-momenta and kinematics.

Lorentz invariance of scalar products is very useful if the kinematics of scattering or decay processes is discussed. One can change to the most appropriate coordinate system (most often CMS) to calculates the kinematic quantities.

Example: 4-momentum p^{μ} of a particle with rest mass m

$$p_{\mu}p^{\mu} = E^2 - \vec{p}^2 = m^2$$

where the last equality can easily be calculated in the particles rest frame where E=m and $\vec{p}$=1.

Example: Lab-system $e^+, E_1 = 2\text{GeV} \xrightarrow[\vec{p}_1]{} \quad \xleftarrow[\vec{p}_2]{} \; e^-, E_2 = 8\text{GeV}$

Kinematics in CMS: $\xrightarrow[\vec{p}_1^*, E_1^*]{} \quad \xleftarrow[\vec{p}_2^*, E_2^*]{}$ with $\vec{p}_1^* = -\vec{p}_2^*$

$$E_1^* = E_2^* = \frac{E_{CMS}}{2}$$

$$(p_1^\mu + p_2^\mu)^2 = \underbrace{(E_1 + E_{2,}, \vec{p}_1 + \vec{p}_2)^2}_{=\left(E_1 + E_2\right)^2 - \left(\vec{p}_1 + \vec{p}_2\right)^2} = (p_1^{*,\mu} + p_2^{*,\mu})^2 = \underbrace{(E_1^* + E_2^*, \vec{p}_1^* + \vec{p}_2^*)^2}_{=\left(E_1^* + E_2^*, 0\right)^2 = E_{CMS}^2}$$

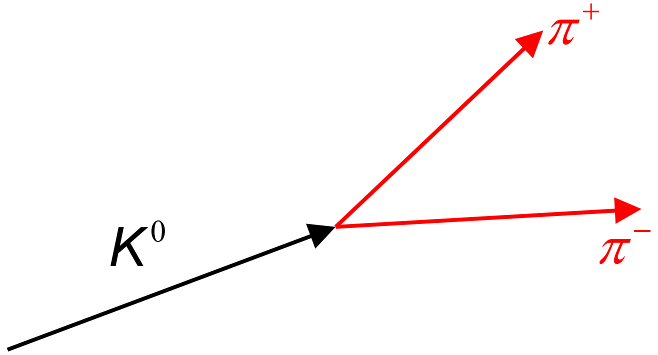
if electron mass is neglected $\left|\vec{p}_i\right| = E_i$

$$= \left[(2+8)^2 - \left(8-2\right)^2\right]\text{GeV}^2$$

$$= 64\ \text{GeV}^2$$

$$\boxed{\begin{array}{l} E_{CMS} = 8\,\text{GeV} \\ \left|\vec{p}_1^*\right| = \left|\vec{p}_2^*\right| = 4\,\text{GeV} \end{array}}$$

Example: Particle decay and invariant mass

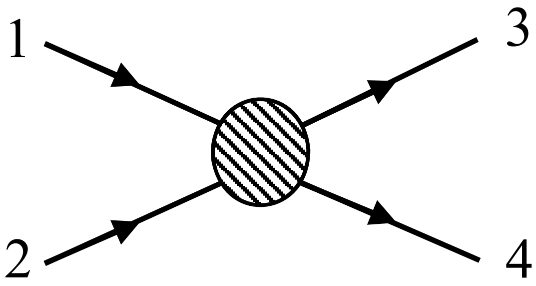


$$\underbrace{m_K^2 = p_K^2 = \left(p_K^\mu\right)^2}_{\text{Rest frame of mother particle K}} = \underbrace{\left(p_{\pi1}^\mu + p_{\pi2}^\mu\right)^2 = \left(p_{\pi1} + p_{\pi2}\right)^2}_{\text{Laboratory frame}}$$

Using the 4-momenta of the two pions one can thus calculate the mass of the kaon!

Mandelstam variables

Cross sections and other Lorentz invariant observables are often expressed in an Lorentz invariant form using scalar products of 4-momenta: **Mandelstam variables** (square of 4-momenta)



$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

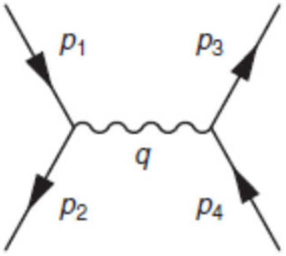
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$

$$\overline{}$$

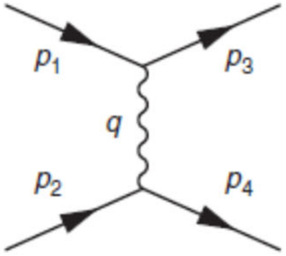
$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

Meaning of s, t, u becomes clear if you look at the q^2 of different scattering processes:



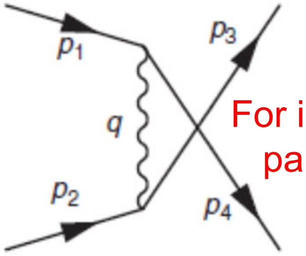
$$q^2 = s = E_{CMS}^2$$

s-channel



$$q^2 = t$$

t-channel



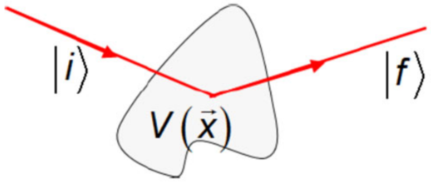
For identical particles

$$q^2 = u$$

u-channel

Non-relativistic perturbation theory

Transition amplitude in quantum mechanics: $V(\vec{x})$ small perturbation



Hamilton-Operator:

$$\hat{H} = \hat{H}_0 + \hat{H}_{WW} = \frac{\hat{p}^2}{2m} + V(\vec{x})$$

$|i\rangle, |f\rangle$ are solutions of the unperturbed free particle Hamiltonian $\hat{H}_0$

$$\psi_j = N \exp(i\vec{k}_j\vec{x} - i\omega_j t) \text{ with } \vec{k}_j = \frac{\vec{p}_j}{\hbar} \text{ and } \omega_j = \frac{E_j}{\hbar}$$

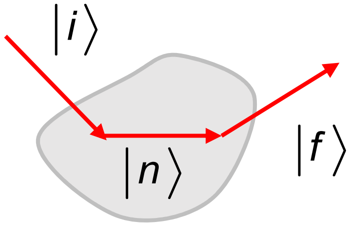
$N =$ normalization

Transition amplitude (describing the transition probability):

$$\mathcal{A}(i \to f) = \mathcal{A}_{fi} = \langle f|\hat{H}_{WW}|i\rangle = \int \psi_f^* V(\vec{x})\psi_i d\vec{x}$$

Considering higher orders:

$$\mathcal{A}_{fi} \to \mathcal{A}_{fi} + \sum_{n\neq i,f} \mathcal{A}_{fn}\frac{1}{E_i - E_n + i\epsilon}\mathcal{A}_{ni} + \cdots$$



Fermi's Golden Rule

Transition rate from initial state $|i\rangle$ to final state $|f\rangle$

$$\omega_{fi} = 2\pi \left|\mathcal{A}_{fi}\right|^2 \rho(E_f)$$

With $\rho(E_f)$ density of final states at energy E_f: $\rho(E_f) = \left.\frac{dn}{dE}\right|_{E_f = E_i}$

Where $\left.\frac{dn}{dE}\right|_{E_f = E_i} = \int \frac{dn}{dE} \delta(E_i - E) dE$

Density of states which fulfill energy conservation: $E_f = E_i$

We therefore can rewrite the "golden rule":

$$\boxed{\omega_{fi} = 2\pi \int \left|\mathcal{A}_{fi}\right|^2 \delta(E_i - E) dn}$$

What is dn or alternatively ρ ?

Density ρ of final-states with energy E_f

State density for a free particle in a cubic box with $V=L^3$

Assuming periodic boundary conditions one finds for one dimension (x):

In 1-D:
Wave number:

$$k_x = \frac{2\pi}{L}\cdot n, \quad n = 1,2,3\ldots$$

$$dk_x = dp_x = \frac{2\pi}{L}dn \qquad \frac{dn}{dp_x} = \frac{L}{2\pi} \qquad \left(= \frac{L}{2\pi\hbar}\right)$$

In 3 dimension:

$$dn = \frac{L^3}{(2\pi)^3}d^3p = \frac{V}{(2\pi)^3}d^3p \longrightarrow \frac{dn}{dp} = 4\pi p^2\frac{V}{(2\pi)^3}$$

thus

$$\rho(E_f) = \left.\frac{dn}{dE}\right|_{E_f} = \frac{dn}{dp}\frac{dp}{dE}$$

N-particle state density (phase space factor):

Only N-1 momenta / particles independent

$$\rho_N = \frac{dn_N}{dE} \quad \text{with} \quad dn_N = \frac{V^{N-1}}{(2\pi)^{3(N-1)}} \cdot d^3p_1 d^3p_2 \cdots d^3p_{N-1}$$

$$\rho_N(E_f) = \frac{d}{dE}$$

$$= \frac{V^{N-1}}{(2\pi)^{3(N-1)}} \delta\left(\vec{P} - \sum_N \vec{p_i}\right) d^3p_1 d^3p_2 \cdots d^3p_N$$

$$= (2\pi)^3 \delta\left(\vec{P} - \sum_N \vec{p_i}\right) \prod_{i=1}^{N} \frac{d^3p_i}{(2\pi)^3}$$

Normalization volume
V often set to 1.

Lorentz invariance:

Above, neither phase space factor nor transition amplitude are Lorentz invariant.

For the amplitude the wave-functions are usually normalized to 1 particle in the normalization volume. Instead, to make the amplitude Lorentz invariant, one normalizes to 2E particles. For a process a+b→1+2 one finds the Lorentz invariant transition amplitude (matrix element):

$$\mathcal{M}_{fi} = \left\langle \psi_a' \psi_b' \middle| \hat{H}_{WW} \middle| \psi_1' \psi_2' \right\rangle = (2E_a \cdot 2E_b \cdot 2E_1 \cdot 2E_2)^{1/2}\, \mathcal{A}_{fi}$$

Lorentz invariant Phase space:
In the same way one needs to replace d^3p by $d^3p/2E$ to make the phase space factor Lorentz invariant.

Particle decay

In the following we discuss the two-body decay **a → 1 +2** in a Lorentz invariant formulation (in lab frame, particle a is in rest).

The partial decay width (decay rate) Γ(a→1+2) for the given process is given by the transition rate ω_{fi}. Using the non-invariant expressions one finds:

$$\omega_{fi} = 2\pi \int \left|\mathcal{A}_{fi}\right|^2 \delta(E_i - E)dn$$

$$\Gamma(a \to 1+2) = \omega_{fi} = (2\pi)^4 \int |\mathcal{A}_{fi}|^2 \, \delta(E_a - E_1 - E_2)\delta^3(\vec{p_a} - \vec{p_1} - \vec{p_2}) \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3}$$

Use Lorentz invariant transition amplitude:

$$\Gamma = \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}_{fi}|^2 \, \delta(E_a - E_1 - E_2)\delta^3(\vec{p_a} - \vec{p_1} - \vec{p_2}) \underbrace{\frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2}}_{\text{Lorentz invariant phase space dLISP}}$$

For N particles: $dLISP = \prod_{i=1}^{N} \frac{d^3p_i}{(2\pi)^3 2E_i}$

If there is more than one decay channel of particle a: $\Gamma = \sum_i \Gamma_i$

Branching ratio: $BR_i = \frac{\Gamma_i}{\Gamma}$ lifetime: $\tau = \frac{1}{\Gamma}$ Γ_i can be calculated.

Coming back to the 2-body decay of particle a in rest:

$$\Gamma = \frac{(2\pi)^4}{\underset{\underset{\text{Mass } m_a}{\uparrow}}{2E_a}} \int |\mathcal{M}_{fi}|^2\, \delta(\underset{\underset{\text{Mass } m_a}{\uparrow}}{E_a} - E_1 - E_2)\delta^3(\vec{p_a} - \vec{p_1} - \vec{p_2}) \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2}$$

$\vec{p}_a = 0,\ \vec{p}_1 = -\vec{p}_2$

$$\Gamma = \frac{1}{8\pi^2 m_a} \int |\mathcal{M}_{fi}|^2\, \delta(m_a - E_1 - E_2)\delta^3(\vec{p_1} + \vec{p_2}) \frac{d^3p_1}{2E_1} \frac{d^3p_2}{2E_2}$$

$$= \frac{1}{8\pi^2 m_a} \int |\mathcal{M}_{fi}|^2 \frac{1}{4E_1E_2} \delta(m_a - E_1 - E_2) d^3p_1 \Big|_{\vec{p}_1 = -\vec{p}_2}$$

$$= \frac{1}{8\pi^2 m_a} \int |\mathcal{M}_{fi}|^2\, \delta\left(m_a - \sqrt{m_1^2 - p_1^2} - \sqrt{m_2^2 - p_1^2}\right) \frac{p_1^2}{4E_1E_2} dp_1 d\Omega_1$$

With some calculation….

$$\boxed{\Gamma = \frac{p_1}{32\pi^2 m_a^2} \int |\mathcal{M}_{fi}|^2 \, d\Omega_1}$$

with $$p_1 = \frac{1}{2m_a}\sqrt{\left[m_a^2 - (m_1 + m_2)^2\right]\left[m_a^2 - (m_1 - m_2)^2\right]}$$

For equal particles 1=2: $$p_1 = \frac{1}{2}\sqrt{m_a^2 - 4m_1^2}$$

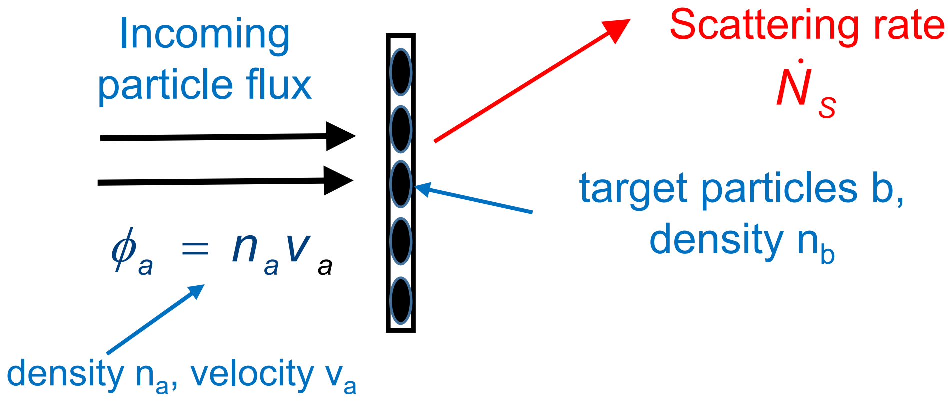
$$p_1 = \left|\vec{p}_1\right|$$

Decay width (rate) is proportional to $|M_{fi}|^2$ (as expected) and the phase space dependence results into a proportionality to p_1

Cross section

Cross section = $\dfrac{\text{Scattering rate per target particle}}{\text{incident flux}}$

$$\boxed{\sigma = \frac{\dot{N}_S}{\phi_a N_b}}$$



Incoming particle flux

$\phi_a = n_a v_a$

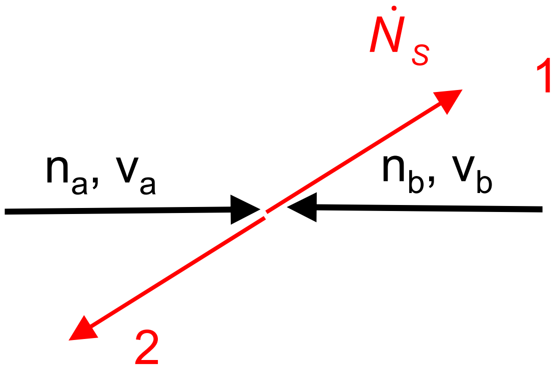
density n_a, velocity v_a

Scattering rate $\dot{N}_S$

target particles b, density n_b

Scattering rate for scattering volume V:

$$\boxed{\begin{aligned} \dot{N}_S &= n_a(v_a + v_b)\cdot n_b V \cdot \sigma \\ \sigma &= \frac{\dot{N}_S}{\phi_a \cdot n_b V} \quad \text{with} \quad \phi_a = n_a(v_a + v_b) \end{aligned}}$$



With appropriate normalization of wave function $n_a = n_b = 1/V$

One can express the cross section through the transition rate (here use V=1):

$$\boxed{\sigma = \frac{\omega_{fi} \cdot V}{(v_a + v_b)} = \frac{\omega_{fi}}{(v_a + v_b)}}$$

Using Fermi's Golden Rule and the Lorentz invariant form of transition amplitude

$$\sigma = \frac{(2\pi)^4}{v_a + v_b} \int |\mathcal{A}_{fi}|^2 \, \delta(E_a + E_b - E_1 - E_2)\delta(\vec{p}_a + \vec{p}_b - \vec{p}_1\vec{p}_2)\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}$$

$\downarrow$ use Lorentz invariant transition amplitude and phase space

$$= \underbrace{\frac{(2\pi)^{-2}}{4E_aE_b(v_a + v_b)}} \int |\mathcal{M}_{fi}|^2 \, \delta(E_a + E_b - E_1 - E_2)\delta(\vec{p}_a + \vec{p}_b - \vec{p}_1\vec{p}_2)\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}$$

$$\begin{aligned} F &= 4E_aE_b(v_a + v_b) \\ &= 4\left(E_a|\vec{p}_b| + E_b|\vec{p}_a|\right) \end{aligned} \quad \downarrow$$

Using $v = \beta = \frac{|\vec{p}|}{E}$

Calculating F^2 one can show that F can be written in a Lorentz invariant form:

$$F = 4\left[(p_a \cdot p_b)^2 - m_a^2 m_b^2\right]^{\frac{1}{2}}$$

In the CMS frame:

$$\vec{p}_a = -\vec{p}_b = \vec{p}_i \quad \text{und} \quad E_a + E_b = \sqrt{s} \quad \longrightarrow \quad F = 4\vec{p}_i\sqrt{s}$$

Cross section im CMS:

$$\sigma = \frac{1}{(2\pi)^2} \frac{1}{4p_i\sqrt{s}} \int |\mathcal{M}_{fi}|^2 \, \delta(\sqrt{s} - E_1 - E_2)\delta(\vec{p}_1 + \vec{p}_2) \frac{d^3p_1}{2E_1} \frac{d^3p_2}{2E_2}$$

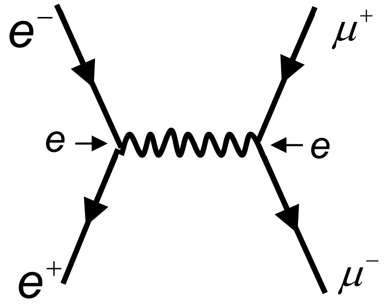
$$\vec{p}_1 = -\vec{p}_2 = \vec{p}_f$$

$$\boxed{\sigma = \frac{1}{64\pi^2 \, s} \frac{p_f}{p_i} \int |\mathcal{M}_{fi}|^2 \, d\Omega_f}$$

Differential cross section:

$$\boxed{\frac{d\sigma}{d\Omega_f} = \frac{1}{64\pi^2 \, s} \frac{p_f}{p_i} |\mathcal{M}_{fi}|^2}$$

Example: $e^+e^- \to \mu^+\mu^-$



The derivation of the Feynman rules and the calculation of the above process will be the topic of the first theory block. Here we just use the result.

If one uses unpolarized electron/positron beams one should average over the helicity states of the incoming electrons. Moreover, the muon polarization is in general not observed. For the cross section one needs to add all possible muon polarization states. The leads to an "average matrix element"

$$\overline{|\mathcal{M}_{fi}|^2} = \underbrace{2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^4}{(p_1 \cdot p_2)^2} = 2e^4 \frac{t^2+u^2}{s^2}}_{\text{if one ignores the fermion masses}} = e^4 (1 + \cos\theta_f)$$

$$\boxed{\frac{d\sigma}{d\Omega_f} = \frac{1}{64\pi^2 s} \frac{p_f}{p_i} e^4 (1+\cos\theta_f) = \frac{\alpha^2}{4s}(1+\cos\theta_f) \quad \Rightarrow \quad \sigma = \frac{4\pi\alpha^2}{3s}}$$