5 Partons and DGLAP equation

During our introduction to QCD and the running coupling we focused on the ultraviolet or high-energy behavior of the theory and the running coupling relating subtracted divergences to resummed logarithms. In this section we will follow a similar approach to infrared divergences, where we will find that collinear divergences lead to the DGLAP equations for parton densities, which resum collinear logarithms.

5.1 Incoming partons

To predict for instance the Drell–Yan process at a hadron collider, we need to introduce parton distribution functions, describing the probability of finding a collinear parton with momentum fraction x in a proton. A pdf is not an observable, only a distribution in the mathematical sense: it has to produce reasonable results when we integrate it together with a test function. Different parton densities have very different behavior — for the valence quarks (uud) they peak (quite a bit) below $x < 1/3$, while the gluon pdf is small at $x \sim 1$ and grows very rapidly towards small x. For some typical part of the relevant parameter space ($x = 10^{-3} \cdots 10^{-1}$) the gluon density roughly scales like $f_g(x) \propto x^{-2}$. Towards smaller x values it becomes even steeper.

While we cannot compute parton distribution functions $f_i(x)$ as a function of the momentum fraction x there are a few predictions we can make based on symmetries and properties of the hadrons, leading to sum rules:

1. The parton distributions in an antiproton are linked to those inside a proton through the CP-symmetry, which is exact for QCD. Therefore,
$$f_q^{\bar{p}}(x) = f_{\bar{q}}(x) \qquad\qquad f_{\bar{q}}^{\bar{p}}(x) = f_q(x) \qquad\qquad f_g^{\bar{p}}(x) = f_g(x) \; . \tag{5.1}$$
2. If the proton consists of three valence quarks uud, plus quantum fluctuations from the vacuum which can either involve gluons or quark–antiquark pairs, the contribution from the sea quarks has to be symmetric in quarks and antiquarks. The expectation values for the signed numbers of up and down quarks inside a proton have to fulfill
$$\langle N_u \rangle = \int_0^1 dx \; \left(f_u(x) - f_{\bar{u}}(x) \right) = 2 \qquad\qquad \langle N_d \rangle = \int_0^1 dx \; \left(f_d(x) - f_{\bar{d}}(x) \right) = 1 \; . \tag{5.2}$$
3. The total momentum of the proton has to consist of sum of all parton momenta. We can write this as the expectation value
$$\left\langle \sum x_i \right\rangle = \int_0^1 dx \; x \; \left(\sum_q f_q(x) + \sum_{\bar{q}} f_{\bar{q}}(x) + f_g(x) \right) = 1 \; . \tag{5.3}$$
What makes this prediction interesting is that we can compute the same sum only taking into account the measured quark and antiquark parton densities. We find
$$\int_0^1 dx \; x \; \left(\sum_q f_q(x) + \sum_{\bar{q}} f_{\bar{q}}(x) \right) \approx \frac{1}{2} \; . \tag{5.4}$$
Half of the proton momentum is then carried by gluons.

With this pdf we can compute a hadronic cross section from its partonic counterpart,

$$\boxed{\sigma_{\text{tot}} = \int_0^1 dx_1 \int_0^1 dx_2 \; \sum_{ij} f_i(x_1) \, f_j(x_2) \, \hat{\sigma}_{ij}(x_1 x_2 S)} \; , \tag{5.5}$$

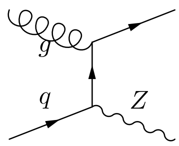
where i, j are the incoming partons with the momentum factions $x_{i,j}$. The partonic energy of the scattering process is $s = x_1 x_2 S$ with the LHC proton energy of $\sqrt{S} = 13.6$ TeV. The partonic cross section $\hat{\sigma}$ includes all the necessary θ and δ functions for energy–momentum conservation. When we express a general n–particle cross section $\hat{\sigma}$ including the phase space integration, the x_i integrations and the phase space integrations can of course be interchanged, but Jacobians will make life hard.

5.2 Infrared divergences

Let us look at the radiation of additional partons in the Drell–Yan process. We can start for example by computing the cross section for the partonic process

$$q\bar{q} \to Zg \; . \tag{5.6}$$

This partonic process involves renormalization of ultraviolet divergences as well as loop diagrams which we have to include before we can say anything reasonable, *i.e.* ultraviolet and infrared finite. To make life easier we study collinear infrared divergences for the crossed process



It should behave like any other $(2 \to 2)$ jet radiation process, except that it has a different incoming state than the leading order Drell–Yan process and hence does not involve virtual corrections. This means we do not have to deal with ultraviolet divergences and renormalization, and can concentrate on parton or jet radiation from the initial state.

The amplitude for this $(2 \to 2)$ process is — modulo charges and averaging factors, but including all Mandelstam variables

$$|\mathcal{M}|^2 \sim -\frac{t}{s} - \frac{s^2 - 2m_Z^2(s+t-m_Z^2)}{st} \; . \tag{5.7}$$

The Mandelstam variable t for one massless final–state particle can be expressed in terms of the rescaled emission angle

$$t = -s(1-\tau)y \qquad \text{with} \qquad y = \frac{1-\cos\theta}{2} \in [0,1] \qquad \text{and} \qquad \tau = \frac{m_Z^2}{s} < 1 \; . \tag{5.8}$$

Similarly, we obtain $u = -s(1-\tau)(1-y)$, so as a first check we can confirm that $t + u = -s(1-\tau) = -s + m_Z^2$. The collinear limit when the gluon splits in the beam direction is given by

$$\begin{aligned} y \to 0 \qquad &\Leftrightarrow \qquad t \to 0 \qquad \Leftrightarrow \qquad u = -s + m_Z^2 < 0 \\ |\mathcal{M}|^2 &\to \frac{s^2 - 2sm_Z^2 + 2m_Z^4}{s(s-m_Z^2)} \; \frac{1}{y} + \mathcal{O}(y^0) \; . \end{aligned} \tag{5.9}$$

This expression is divergent for collinear gluon radiation or gluon splitting, *i.e.* for small angles y. We can translate this $1/y$ divergence for example into the transverse momentum of the gluon or Z

$$sp_T^2 = tu = s^2(1-\tau)^2 \; y(1-y) = (s-m_Z^2)^2 y + \mathcal{O}(y^2) \tag{5.10}$$

In terms of p_T, the collinear limit our matrix element squared in Eq.(5.9) becomes

$$|\mathcal{M}|^2 \sim \frac{s^2 - 2sm_Z^2 + 2m_Z^4}{s^2} \; \frac{s - m_Z^2}{p_T^2} + \mathcal{O}(p_T^0) \; . \tag{5.11}$$

The matrix element for the tree level process $qg \to Zq$ has a leading divergence proportional to $1/p_T^2$. To compute the total cross section for this process we need to integrate the matrix element over the two-particle phase space. Approximating the matrix element as C'/y or C/p_T^2 this gives us

$$\int_{y^{\text{min}}}^{y^{\text{max}}} dy \frac{C'}{y} = \int_{p_T^{\text{min}}}^{p_T^{\text{max}}} dp_T^2 \frac{C}{p_T^2} = \; 2 \int_{p_T^{\text{min}}}^{p_T^{\text{max}}} dp_T \; p_T \; \frac{C}{p_T^2} \simeq 2C \int_{p_T^{\text{min}}}^{p_T^{\text{max}}} dp_T \frac{1}{p_T} = 2C \; \log \frac{p_T^{\text{max}}}{p_T^{\text{min}}} \tag{5.12}$$

The form C/p_T^2 for the matrix element is of course only valid in the collinear limit; in the non–collinear phase space C is not a constant.

For this divergence we can follow the same strategy as for the ultraviolet divergence. First, we regularize it for example using dimensional regularization. Then, we find a well–defined way to get rid of it. Dimensional regularization means writing the two-particle phase space in $n = 4 - 2\epsilon$ dimensions. Just for reference, the complete formula for the y-distribution reads

$$s\,\frac{d\sigma}{dy} = \frac{\pi(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)} \left(\frac{\mu_F^2}{m_Z^2}\right)^{\epsilon} \frac{\tau^\epsilon(1-\tau)^{1-2\epsilon}}{y^\epsilon(1-y)^\epsilon} |\mathcal{M}|^2 \sim \left(\frac{\mu_F^2}{m_Z^2}\right)^{\epsilon} \frac{|\mathcal{M}|^2}{y^\epsilon(1-y)^\epsilon} \,. \tag{5.13}$$

In the second step we only keep the factors we are interested in. The additional factor $1/y^\epsilon$ regularizes the integral at $y \to 0$, as long as $\epsilon < 0$ by slightly increasing the suppression of the integrand in the infrared regime. This means that for infrared divergences we choose $n = 4 + 2|\epsilon|$ space–time dimensions. After integrating the leading collinear divergence $1/y^{1+\epsilon}$ we are left with a pole $1/(-\epsilon)$.

What is important to notice is again the appearance of a scale $\mu_F^{2\epsilon}$ with the n-dimensional integral. Now it arises from an infrared regularization and is referred to as factorization scale. The actual removal of the infrared pole — corresponding to the renormalization in the ultraviolet case — is called mass factorization and works exactly the same way as renormalizing a parameter: in a well–defined scheme we subtract the pole from the fixed-order matrix element squared.

5.3 Parton splitting

Infrared divergences occur for massless particles in the initial or final state, so we need to go through all ways incoming or outgoing gluons and quark can split into each other. The factorized phase space is common to all different channels. The first and at the LHC most important case is the splitting of one gluon into two,

$$g(p_a) \to g(p_b) + g(p_c) \qquad \text{with} \qquad p_a^2 \gg p_b^2, p_c^2 \,. \tag{5.14}$$

The two daughter gluons are close to mass shell while the mother has to have a finite positive invariant mass. We assign the direction of the momenta as $p_a = -p_b - p_c$ and describe the kinematics of this approximately collinear process in terms of the energy fractions z and $1-z$ defined as

$$z = \frac{|E_b|}{|E_a|} = 1 - \frac{|E_c|}{|E_a|} \qquad p_a^2 = (-p_b - p_c)^2 = 2(p_b p_c) = 2z(1-z)(1-\cos\theta)E_a^2 = z(1-z)E_a^2\theta^2 + \mathcal{O}(\theta^4)$$
$$\Leftrightarrow \qquad \theta \equiv \theta_b + \theta_c \simeq \frac{1}{|E_a|}\sqrt{\frac{p_a^2}{z(1-z)}} \,, \tag{5.15}$$

in the collinear limit and in terms of the opening angle θ between $\vec{p}_b$ and $\vec{p}_c$. Using this phase space parameterization we divide an $(n+1)$-particle process into an n-particle process and a splitting process of quarks and gluons. First, this requires us to split the $(n+1)$-particle phase space alone into an n-particle phase space and the (collinear) splitting,

$$\begin{aligned} d\Phi_{n+1} &= \cdots \frac{d^3\vec{p}_b}{2(2\pi)^3|E_b|} \frac{d^3\vec{p}_c}{2(2\pi)^3|E_c|} = \cdots \frac{d^3\vec{p}_a}{2(2\pi)^3|E_a|} \frac{d^3\vec{p}_c}{2(2\pi)^3|E_c|} \frac{|E_a|}{|E_b|} \\ &\equiv d\Phi_n \, \frac{dp_{c,3}dp_T p_T d\phi}{2(2\pi)^3|E_c|} \, \frac{1}{z} \\ &= d\Phi_n \, \frac{dp_{c,3}dp_T^2 d\phi}{4(2\pi)^3|E_c|} \, \frac{1}{z} \end{aligned} \tag{5.16}$$

We can separate the $(n+1)$-particle space into an n-particle phase space and a $(1 \to 2)$ splitting phase space without any approximation.

Our next task is to translate $p_{c,3}$ and p_T^2 into z and $p_a^2 \neq 0$. This can be done if we assume approximately collinear collinear splittings, where we find that

$$\frac{dp_{c,3}}{|E_c|} = \frac{dz}{1-z}\left(1+\mathcal{O}(\theta)\right) \qquad \text{and} \qquad dp_T^2 = z(1-z)dp_a^2 \; . \tag{5.17}$$

This gives us the final result for the separated collinear phase space, assuming azimuthal symmetry in an addition step,

$$d\Phi_{n+1} = d\Phi_n \, \frac{dz\, dp_a^2\, d\phi}{4(2\pi)^3}\left(1+\mathcal{O}(\theta)\right) = d\Phi_n \, \frac{dz\, dp_a^2}{4(2\pi)^2}\left(1+\mathcal{O}(\theta)\right) \; . \tag{5.18}$$

Adding the transition matrix elements to this factorization of the phase space we can write a full factorization in the collinear approximation as

$$\begin{aligned} d\sigma_{n+1} &= \overline{|\mathcal{M}_{n+1}|^2}\; d\Phi_{n+1} \\ &= \overline{|\mathcal{M}_{n+1}|^2}\; d\Phi_n \frac{dp_a^2\, dz}{4(2\pi)^2}\;\left(1+\mathcal{O}(\theta)\right) \\ &\simeq \frac{2g_s^2}{p_a^2}\;\hat{P}(z)\;\overline{|\mathcal{M}_n|^2}\; d\Phi_n \frac{dp_a^2\, dz}{16\pi^2} \qquad \text{assuming} \quad \boxed{\overline{|\mathcal{M}_{n+1}|^2} \simeq \frac{2g_s^2}{p_a^2}\;\hat{P}(z)\;\overline{|\mathcal{M}_n|^2}} \\ &= \sigma_n\;\frac{dp_a^2}{p_a^2}\; dz\;\frac{\alpha_s}{2\pi}\hat{P}(z) \; . \end{aligned} \tag{5.19}$$

For splitting incoming partons we replace $p_a^2 \to t$, the usual Mandelstam variable. We can show this assumed factorization by constructing the appropriate splitting kernels $\hat{P}(z)$ for all quark and gluon configurations:

- First comes gluon splitting into two gluons. To compute its transition amplitude we need to write down all gluon momenta and polarizations in a specific frame. We skip the derivation and just quote the result

$$\begin{aligned} \overline{|\mathcal{M}_{n+1}|^2} &= \frac{2g_s^2}{p_a^2}\;\frac{N_c}{2}\;2\left[\frac{z}{1-z} + z(1-z) + \frac{1-z}{z}\right]\;\overline{|\mathcal{M}_n|^2} \\ &\equiv \frac{2g_s^2}{p_a^2}\;\hat{P}_{g\leftarrow g}(z)\;\overline{|\mathcal{M}_n|^2} \\ &\Leftrightarrow \qquad \hat{P}_{g\leftarrow g}(z) = C_A\left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right] \; , \end{aligned} \tag{5.20}$$

  using $C_A = N_c$. The splitting kernel is symmetric when we exchange the two gluons z and $(1-z)$. It diverges if either gluon becomes soft. The notation $\hat{P}_{i\leftarrow j} \sim \hat{P}_{ij}$ is inspired by a matrix notation which we can use to multiply the splitting matrix from the right with the incoming parton vector to get the final parton vector.

- A second kernel describes the splitting of a gluon into two quarks. Again, we omit the calculation and quote

$$\hat{P}_{q\leftarrow g}(z) = T_R\left[z^2 + (1-z)^2\right] \; . \tag{5.21}$$

  It is symmetric under $z \leftrightarrow (1-z)$ because QCD does not distinguish between the outgoing quark and antiquark.

- The third splitting is gluon radiation off a quark line,

$$\hat{P}_{q\leftarrow q}(z) = C_F \frac{1+z^2}{1-z} \; . \tag{5.22}$$

- Just switching $z \leftrightarrow (1-z)$ we can read off the kernel for a quark splitting into the final–state gluon

$$\hat{P}_{g\leftarrow q}(z) = C_F \frac{1+(1-z)^2}{z} \; . \tag{5.23}$$

Similar to ultraviolet divergences these splitting kernels are universal. They do not depend on the hard n-particle matrix element as part of the full $(n+1)$-particle process. **tp: done to here**

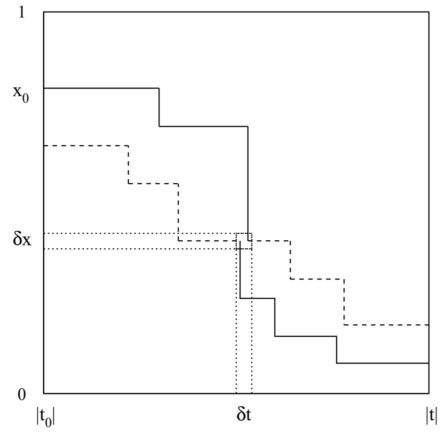
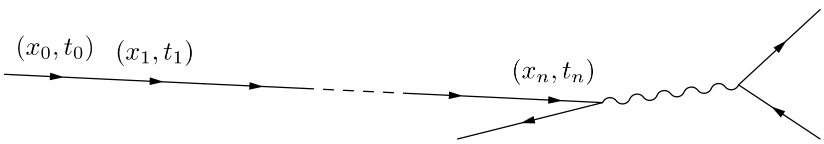


Figure 1: Path of an incoming parton in the $(x - t)$ plane. Because we define t as a negative number its axis is labelled $|t|$.

5.4 DGLAP equation

To describe successive splittings we start with a quark inside the proton with an energy fraction x_0, as it enters the hadronic phase space integral. As this quark is confined inside the proton, it can only have small transverse momentum, which means its four-momentum squared t_0 is negative and its absolute value $|t_0|$ is small. For the incoming partons which if on–shell have $p^2 = 0$ it gives the distance to the mass shell. Let us simplify our kinematic argument by assuming that there exists only one splitting, namely successive gluon radiation off an incoming quark, where the outgoing gluons are not relevant



In that case each collinear gluon radiation will decrease the quark energy and increase its virtuality through recoil,

$$x_{j+1} < x_j \qquad \text{and} \qquad |t_{j+1}| = -t_{j+1} > -t_j = |t_j| \; . \tag{5.24}$$

We know what the successive splitting means in terms of splitting probabilities and can describe how the parton density $f(x, -t)$ evolves in the $(x - t)$ plane as depicted in Figure 1. The starting point (x_0, t_0) is, probabilistically, given by the energy and kinds of parton and hadron. We then interpret each branching as a step downward in $x_j \to x_{j+1}$ and assign to a increased virtuality $|t_{j+1}|$ after the branching. The actual splitting path in the $(x - t)$ plane is made of discrete points. The probability of a splitting to occur is given by Eq.(5.19),

$$\frac{\alpha_s}{2\pi}\,\hat{P}(z)\,\frac{dt}{t}\,dz \equiv \frac{\alpha_s}{2\pi}\,\hat{P}_{q\leftarrow q}(z)\,\frac{dt}{t}\,dz \; . \tag{5.25}$$

At the end of the path we will probe the evolved parton density at (x_n, t_n), entering the hard scattering process and its energy–momentum conservation.

To convert a partonic into a hadronic cross section, we probe the probability or the parton density $f(x, -t)$ over an infinitesimal square,

$$[x_j, x_j + \delta x] \qquad \text{and} \qquad [|t_j|, |t_j| + \delta t] \; . \tag{5.26}$$

Using our (x,t) plane we can compute the flows into this square and out of this square, which together define the net shift in f in the sense of a differential equation,

$$\delta f_{\text{in}} - \delta f_{\text{out}} = \delta f(x,-t) \; . \tag{5.27}$$

We compute the incoming and outgoing flows from the history of the (x,t) evolution. At this stage our picture becomes a little subtle; the way we define the path between two splittings in Figure 1 it can enter and leave the square either vertically or horizontally. Because we want to arrive at a differential equation in t we choose the vertical drop, such that the area the incoming and outgoing flows see is given by δt. If we define a splitting as a vertical drop in x at the target value t_{j+1}, an incoming path hitting the square can come from any x-value above the square. Using this convention and following the fat solid lines in Figure 1 the vertical flow into (and out of) the square (x,t) is proportional to δt as the size of the covered interval

$$\begin{aligned}
\delta f_{\text{in}}(-t) &= \delta t \, \left(\frac{\alpha_s \hat{P}}{2\pi t} \otimes f \right) (x,-t) \\
&= \frac{\delta t}{t} \int_x^1 \frac{dz}{z} \, \frac{\alpha_s}{2\pi} \, \hat{P}(z) f\left(\frac{x}{z},-t\right) \\
&\equiv \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \, \frac{\alpha_s}{2\pi} \, \hat{P}(z) f\left(\frac{x}{z},-t\right) \qquad \text{assuming } f(x',-t)=0 \text{ for } x'>1 \; .
\end{aligned} \tag{5.28}$$

We use the definition of a <u>convolution</u>

$$(f\otimes g)(x) = \int_0^1 dx_1 dx_2 f(x_1) g(x_2) \, \delta(x - x_1 x_2) = \int_0^1 \frac{dx_1}{x_1} f(x_1) g\left(\frac{x}{x_1}\right) = \int_0^1 \frac{dx_2}{x_2} f\left(\frac{x}{x_2}\right) g(x_2) \; . \tag{5.29}$$

The outgoing flow we define as leaving the infinitesimal square vertically. Following the fat solid line in Figure 1 it is also proportional to δt

$$\delta f_{\text{out}}(-t) = \delta t \, \int_0^1 dy \frac{\alpha_s \hat{P}(y)}{2\pi t} \, f(x,-t) = \frac{\delta t}{t} f(x,-t) \int_0^1 dy \, \frac{\alpha_s}{2\pi} \, \hat{P}(y) \; . \tag{5.30}$$

The y-integration is not a convolution, because we know the starting condition and integrate over all final configurations. Combining Eq.(5.28) and Eq.(5.30) we can compute the change in the quark pdf as

$$\begin{aligned}
\delta f(x,-t) = \delta f_{\text{in}} - \delta f_{\text{out}} &= \frac{\delta t}{t} \, \left[\int_0^1 \frac{dz}{z} \, \frac{\alpha_s}{2\pi} \, \hat{P}(z) \, f\left(\frac{x}{z},-t\right) - \int_0^1 dy \, \frac{\alpha_s}{2\pi} \, \hat{P}(y) \, f(x,-t) \right] \\
&= \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \, \frac{\alpha_s}{2\pi} \, \left[\hat{P}(z) - \delta(1-z) \int_0^1 dy \hat{P}(y) \right] \, f\left(\frac{x}{z},-t\right) \\
&\equiv \frac{\delta t}{t} \int_x^1 \frac{dz}{z} \, \frac{\alpha_s}{2\pi} \, \hat{P}(z)_+ \, f\left(\frac{x}{z},-t\right) \\
\Leftrightarrow \qquad \frac{\delta f(x,-t)}{\delta(-t)} &= \frac{1}{(-t)} \int_x^1 \frac{dz}{z} \, \frac{\alpha_s}{2\pi} \, \hat{P}(z)_+ \, f\left(\frac{x}{z},-t\right)
\end{aligned} \tag{5.31}$$

Strictly speaking, we require α_s to only depend on t and introduce the so-defined <u>plus subtraction</u>

$$F(z)_+ \equiv F(z) - \delta(1-z) \int_0^1 dy \; F(y) \qquad \text{or} \qquad \int_0^1 dz \, \frac{f(z)}{(1-z)_+} = \int_0^1 dz \, \left(\frac{f(z)}{1-z} - \frac{f(1)}{1-z} \right) \; . \tag{5.32}$$

For the second definition we choose $F(z) = 1/(1-z)$, multiply it with an arbitrary test function $f(z)$ and integrate over z. The plus–subtracted integral is by definition finite in the soft limit $z \to 1$, where some splitting kernels diverge. The plus prescription is related to dimensional regularization, defined as

$$\int_0^1 dz \, \frac{1}{(1-z)^{1-\epsilon}} = \int_0^1 dz \, \frac{1}{z^{1-\epsilon}} = \left. \frac{z^\epsilon}{\epsilon} \right|_0^1 = \frac{1}{\epsilon} \qquad\qquad \text{with } \epsilon > 0 \; , \tag{5.33}$$

corresponding to $4+2\epsilon$ dimensions. We can relate the dimensionally regularized integral to the plus subtraction as

$$\begin{aligned}
\int_0^1 dz\, \frac{f(z)}{(1-z)^{1-\epsilon}} &= \int_0^1 dz\, \frac{f(z)-f(1)}{(1-z)^{1-\epsilon}} + f(1)\int_0^1 dz\, \frac{1}{(1-z)^{1-\epsilon}} \\
&= \int_0^1 dz\, \frac{f(z)-f(1)}{1-z}\,(1+\mathcal{O}(\epsilon)) + \frac{f(1)}{\epsilon} \\
&= \int_0^1 dz\, \frac{f(z)}{(1-z)_+}\,(1+\mathcal{O}(\epsilon)) + \frac{f(1)}{\epsilon} \qquad\qquad \text{by definition} \\
\Leftrightarrow \qquad \int_0^1 dz\, \frac{f(z)}{(1-z)^{1-\epsilon}} - \frac{f(1)}{\epsilon} &= \int_0^1 dz\, \frac{f(z)}{(1-z)_+}\,(1+\mathcal{O}(\epsilon)) \; . \qquad (5.34)
\end{aligned}$$

The dimensionally regularized integral minus the pole, *i.e.* the finite part of the dimensionally regularized integral, is the same as the plus–subtracted integral modulo terms of the order ϵ. The difference between a dimensionally regularized splitting kernel and a plus–subtracted splitting kernel manifests itself as terms proportional to $\delta(1-z)$. They represent contributions to a soft–radiation phase space integral.

To regularize our splitting kernel $\hat{P}_{q\leftarrow q}$ in Eq.(5.22) we can define two subtraction schemes,

$$\begin{aligned}
\left(\frac{1+z^2}{1-z}\right)_+ - (1+z^2)\left(\frac{1}{1-z}\right)_+ &= \frac{1+z^2}{1-z} - \delta(1-z)\int_0^1 dy\, \frac{1+y^2}{1-y} - \frac{1+z^2}{1-z} + \delta(1-z)\int_0^1 dy\, \frac{1+z^2}{1-y} \\
&= -\delta(1-z)\int_0^1 dy\, \left(\frac{1+y^2}{1-y} - \frac{2}{1-y}\right) \\
&= \delta(1-z)\int_0^1 dy\, \frac{y^2-1}{y-1} = \delta(1-z)\int_0^1 dy\,(y+1) = \frac{3}{2}\delta(1-z)\; . \qquad (5.35)
\end{aligned}$$

This means we can write the quark splitting kernel in two equivalent ways

$$P_{q\leftarrow q}(z) \equiv C_F \left(\frac{1+z^2}{1-z}\right)_+ = C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right] \; . \qquad (5.36)$$

Going back to our differential equation, the infinitesimal Eq.(5.31) is the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi or DGLAP equation. For now it describes the virtuality or scale dependence of the quark parton density, and we need to generalize is to quarks and gluons. For the quark density on the left hand side it is

$$\boxed{\frac{df_q(x,-t)}{d\log(-t)} = -t\,\frac{df_q(x,-t)}{d(-t)} = \sum_{j=q,g}\int_x^1 \frac{dz}{z}\,\frac{\alpha_s}{2\pi}\,P_{q\leftarrow j}(z)\, f_j\left(\frac{x}{z},-t\right)} \; . \qquad (5.37)$$

Modifying Eq.(5.31) the relevant splittings on the right hand side are

$$\begin{aligned}
\delta f_q(x,-t) = \frac{\delta t}{t}\,\Bigg[&\int_0^1 \frac{dz}{z}\,\frac{\alpha_s}{2\pi}\,\hat{P}_{q\leftarrow q}(z)\, f_q\left(\frac{x}{z},-t\right) + \int_0^1 \frac{dz}{z}\,\frac{\alpha_s}{2\pi}\,\hat{P}_{q\leftarrow g}(z)\, f_g\left(\frac{x}{z},-t\right) \\
&- \int_0^1 dy\,\frac{\alpha_s}{2\pi}\,\hat{P}_{q\leftarrow q}(y)\, f_q(x,-t)\Bigg] \; . \qquad (5.38)
\end{aligned}$$

Of the three terms the first and the third together define the plus–subtracted splitting kernel $P_{q\leftarrow q}(z)$, just following the argument above. The second term is a convolution proportional to the gluon pdf. Quarks can be produced in gluon splitting but cannot vanish into it. Therefore, the second term in Eq.(5.38) includes $P_{q\leftarrow g}$, without a plus–regulator

$$P_{q\leftarrow g}(z) \equiv \hat{P}_{q\leftarrow g}(z) = T_R\left[z^2+(1-z)^2\right] \; . \qquad (5.39)$$

This kernel is indeed missing a soft-radiation divergence for $z\to 1$.

The second parton density we have to study is the gluon density. The incoming contribution to the infinitesimal square is given by the sum of four splitting scenarios each leading to a gluon with virtuality $-t_{j+1}$

$$\begin{aligned}
\delta f_{\text{in}}(-t) &= \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \, \frac{\alpha_s}{2\pi} \, \left[\hat{P}_{g\leftarrow g}(z) \left(f_g\left(\frac{x}{z}, -t\right) + f_g\left(\frac{x}{1-z}, -t\right) \right) + \hat{P}_{g\leftarrow q}(z) \left(f_q\left(\frac{x}{z}, -t\right) + f_{\bar{q}}\left(\frac{x}{z}, -t\right) \right) \right] \\
&= \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \, \frac{\alpha_s}{2\pi} \, \left[2\hat{P}_{g\leftarrow g}(z) f_g\left(\frac{x}{z}, -t\right) + \hat{P}_{g\leftarrow q}(z) \left(f_q\left(\frac{x}{z}, -t\right) + f_{\bar{q}}\left(\frac{x}{z}, -t\right) \right) \right] \; , \qquad (5.40)
\end{aligned}$$

using $P_{g\leftarrow\bar{q}} = P_{g\leftarrow q}$ in the first line and $P_{g\leftarrow g}(1-z) = P_{g\leftarrow g}(z)$ in the second. To leave the volume element in (x,t)-space a gluon can either split into two gluons or radiate one of n_f light-quark flavors. Combining the incoming and outgoing flows we find

$$\begin{aligned}
\delta f_g(x,-t) =& \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \, \frac{\alpha_s}{2\pi} \, \left[2\hat{P}_{g\leftarrow g}(z) f_g\left(\frac{x}{z}, -t\right) + \hat{P}_{g\leftarrow q}(z) \left(f_q\left(\frac{x}{z}, -t\right) + f_{\bar{q}}\left(\frac{x}{z}, -t\right) \right) \right] \\
&- \frac{\delta t}{t} \int_0^1 dy \, \frac{\alpha_s}{2\pi} \, \left[\hat{P}_{g\leftarrow g}(y) + n_f \hat{P}_{q\leftarrow g}(y) \right] f_g(x,-t) \qquad (5.41)
\end{aligned}$$

Unlike in the quark case these terms do not immediately correspond to regularizing the diagonal splitting kernel using the plus prescription.

First, the contribution to δf_{in} proportional to f_q or $f_{\bar{q}}$ which is not matched by the outgoing flow. From the quark case we already know how to deal with it. The corresponding splitting kernel does not need any regularization, so we define

$$P_{g\leftarrow q}(z) \equiv \hat{P}_{g\leftarrow q}(z) = C_F \frac{1+(1-z)^2}{z} \; . \qquad (5.42)$$

We see that the structure of the DGLAP equation implies that the two off-diagonal splitting kernels do not include any plus prescription $\hat{P}_{i\leftarrow j} = P_{i\leftarrow j}$. We could have expected these kernels are finite in the soft limit, $z \to 1$.

Next, we can compute the y-integral describing the gluon splitting into a quark pair directly,

$$\begin{aligned}
-\int_0^1 dy \, \frac{\alpha_s}{2\pi} \, n_f \, \hat{P}_{q\leftarrow g}(y) &= -\frac{\alpha_s}{2\pi} \, n_f \, T_R \int_0^1 dy \, \left[1 - 2y + 2y^2 \right] \qquad \text{using Eq.(5.39)} \\
&= -\frac{\alpha_s}{2\pi} \, n_f \, T_R \left[y - y^2 + \frac{2y^3}{3} \right]_0^1 \\
&= -\frac{2}{3} \, \frac{\alpha_s}{2\pi} \, n_f \, T_R \; . \qquad (5.43)
\end{aligned}$$

Finally, the two terms proportional to the pure gluon splitting $P_{g\leftarrow g}$ in Eq.(5.41) require some work. The y-integral from the outgoing flow has to consist of a finite term and a term we can use to define the plus prescription for $\hat{P}_{g\leftarrow g}$. The integral gives

$$\begin{aligned}
-\int_0^1 dy \, \frac{\alpha_s}{2\pi} \, \hat{P}_{g\leftarrow g}(y) &= -\frac{\alpha_s}{2\pi} \, C_A \int_0^1 dy \, \left[\frac{y}{1-y} + \frac{1-y}{y} + y(1-y) \right] \qquad \text{using Eq.(5.20)} \\
&= -\frac{\alpha_s}{2\pi} \, C_A \int_0^1 dy \, \left[\frac{2y}{1-y} + y(1-y) \right] \\
&= -\frac{\alpha_s}{2\pi} \, C_A \int_0^1 dy \, \left[\frac{2(y-1)}{1-y} + y(1-y) \right] - \frac{\alpha_s}{2\pi} \, C_A \int_0^1 dy \, \frac{2}{1-y} \\
&= -\frac{\alpha_s}{2\pi} \, C_A \int_0^1 dy \, \left[-2 + y - y^2 \right] - \frac{\alpha_s}{2\pi} \, 2C_A \int_0^1 dz \, \frac{1}{1-z} \\
&= -\frac{\alpha_s}{2\pi} \, C_A \left[-2 + \frac{1}{2} - \frac{1}{3} \right] - \frac{\alpha_s}{2\pi} \, 2C_A \int_0^1 dz \, \frac{1}{1-z} \\
&= \frac{\alpha_s}{2\pi} \, \frac{11}{6} \, C_A - \frac{\alpha_s}{2\pi} \, 2C_A \int_0^1 dz \, \frac{1}{1-z} \; . \qquad (5.44)
\end{aligned}$$

The second term in this result is what we need to replace the first term in the splitting kernel of Eq.(5.20) proportional to $1/(1-z)$ by $1/(1-z)_+$. We can see this using $f(z) = z$ and correspondingly $f(1) = 1$ in Eq.(5.32). The two finite terms in Eq.(5.43) and Eq.(5.44) are included in the definition of $\hat{P}_{g\leftarrow g}$ ad hoc. Because the regularized splitting kernel appears in a convolution, the two finite terms require an explicit factor $\delta(1-z)$. Collecting all of them we arrive at

$$P_{g\leftarrow g}(z) = 2C_A \left(\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right) + \frac{11}{6} \, C_A \, \delta(1-z) - \frac{2}{3} \; n_f \, T_R \, \delta(1-z) \; . \tag{5.45}$$

This result concludes our computation of all four regularized splitting kernels.

Before discussing and solving the DGLAP equation, let us briefly recapitulate: for the full quark and gluon particle content of QCD we have derived the DGLAP equation which describes a factorization scale dependence of the quark and gluon parton densities. The universality of the splitting kernels is obvious from the way we derive them — no information on the n-particle process ever enters the derivation.

The DGLAP equation is formulated in terms of four splitting kernels of gluons and quarks which are linked to the splitting probabilities, but which for the DGLAP equation have to be regularized. With the help of a plus–subtraction all kernels $P_{i\leftarrow j}(z)$ become finite, including in the soft limit $z \to 1$. However, splitting kernels are only regularized when needed, so the finite off-diagonal quark–gluon and gluon–quark splittings are unchanged. This means the plus prescription really acts as an infrared renormalization, moving universal infrared divergences into the definition of the parton densities. The original collinear divergence has vanished as well.

The only approximation we make in the computation of the splitting kernels is that in the y-integrals the running coupling α_s does not depend on the momentum fraction. In its standard form and in terms of the factorization scale $\mu_F^2 \equiv -t$ the DGLAP equation reads

$$\boxed{\frac{df_i(x,\mu_F)}{d\log\mu_F^2} = \sum_j \int_x^1 \frac{dz}{z} \, \frac{\alpha_s}{2\pi} \, P_{i\leftarrow j}(z) \, f_j\left(\frac{x}{z},\mu_F\right) = \frac{\alpha_s}{2\pi} \, \sum_j \, \left(P_{i\leftarrow j} \otimes f_j\right)(x,\mu_F)} \; . \tag{5.46}$$

5.5 Solving the DGLAP equation

While it is hard to solve the DGLAP equation in Eq.(5.46) in general, we can simplify our life by solving it for eigenvalues in parton space. This gets rid of the sum over partons on the right hand side, and one such parton density is the non–singlet parton density,

$$f_q^{\text{NS}} = (f_q - f_{\bar{q}}) \; . \tag{5.47}$$

Since gluons cannot distinguish between quarks and antiquarks, the gluon contribution to their evolution cancels, at least in the massless limit, at arbitrary loop order. The corresponding DGLAP equation with leading order splitting kernels is

$$\frac{df_q^{\text{NS}}(x,\mu_F)}{d\log\mu_F^2} = \int_x^1 \frac{dz}{z} \, \frac{\alpha_s}{2\pi} \, P_{q\leftarrow q}(z) \, f_q^{\text{NS}}\left(\frac{x}{z},\mu_F\right) \; . \tag{5.48}$$

To solve it we need a transformation which simplifies a convolution, leading us to the Mellin transform. Starting from a function $f(x)$ of a real variable x we define the Mellin transform into moment space m

$$\mathcal{M}[f](m) \equiv \int_0^1 dx \, x^{m-1} f(x) \qquad\qquad f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dm \, \frac{\mathcal{M}[f](m)}{x^m} \; , \tag{5.49}$$

where for the back transformation we choose an arbitrary appropriate constant $c > 0$, such that the integration contour for the inverse transformation lies to the right of all singularities of the analytic continuation of $\mathcal{M}[f](m)$. The important property for us is that the Mellin transform of a convolution is the product of the two Mellin transforms, which gives us the transformed DGLAP equation

$$\begin{aligned}
\frac{d\mathcal{M}[f_q^{\text{NS}}](m,\mu_F)}{d\log\mu_F^2} &= \frac{\alpha_s}{2\pi}\,\mathcal{M}\left[\int_0^1 \frac{dz}{z} P_{q\leftarrow q}\left(\frac{x}{z}\right) f_q^{\text{NS}}(z)\right](m) \\
&= \frac{\alpha_s}{2\pi}\,\mathcal{M}[P_{q\leftarrow q}\otimes f_q^{\text{NS}}](m) \\
&= \frac{\alpha_s}{2\pi}\,\mathcal{M}[P_{q\leftarrow q}](m) \quad \mathcal{M}[f_q^{\text{NS}}](m,\mu_F)\;,
\end{aligned} \tag{5.50}$$

with the simple solution

$$\begin{aligned}
\mathcal{M}[f_q^{\text{NS}}](m,\mu_F) &= \mathcal{M}[f_q^{\text{NS}}](m,\mu_{F,0})\;\exp\left(\frac{\alpha_s}{2\pi}\,\mathcal{M}[P_{q\leftarrow q}](m)\log\frac{\mu_F^2}{\mu_{F,0}^2}\right) \\
&= \mathcal{M}[f_q^{\text{NS}}](m,\mu_{F,0})\;\left(\frac{\mu_F^2}{\mu_{F,0}^2}\right)^{\frac{\alpha_s}{2\pi}\mathcal{M}[P_{q\leftarrow q}](m)} \\
&\equiv \mathcal{M}[f_q^{\text{NS}}](m,\mu_{F,0})\;\left(\frac{\mu_F^2}{\mu_{F,0}^2}\right)^{\frac{\alpha_s}{2\pi}\gamma(m)}\;,
\end{aligned} \tag{5.51}$$

defining $\gamma(m) = \mathcal{M}[P](m)$.

This solution still includes μ_F and α_s as two free parameters. To simplify this form we can include $\alpha_s(\mu_R^2)$ in the running of the DGLAP equation and identify the renormalization scale μ_R of the strong coupling with the factorization scale

$$\mu_F \equiv \mu_R \equiv \mu\;. \tag{5.52}$$

Physically, this identification is clearly correct for all one-scale problems where we have no freedom to choose either of the two scales. In the DGLAP equation it allows us to replace $\log\mu^2$ by α_s as

$$\frac{d}{d\log\mu^2} = \frac{d\log\alpha_s}{d\log\mu^2}\,\frac{d}{d\log\alpha_s} = \frac{1}{\alpha_s}\,\frac{d\alpha_s}{d\log\mu^2}\,\frac{d}{d\log\alpha_s} = -\alpha_s b_0\,\frac{d}{d\log\alpha_s}\;. \tag{5.53}$$

The additional factor of α_s will cancel the factor α_s on the right hand side of the DGLAP equation Eq.(5.50)

$$\begin{aligned}
\frac{d\mathcal{M}[f_q^{\text{NS}}](m,\mu)}{d\log\alpha_s} &= -\frac{1}{2\pi b_0}\gamma(m)\;\mathcal{M}[f_q^{\text{NS}}](m,\mu) \\
\mathcal{M}[f_q^{\text{NS}}](m,\mu) &= \mathcal{M}[f_q^{\text{NS}}](m,\mu_0)\;\exp\left(-\frac{1}{2\pi b_0}\,\gamma(m)\log\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right) \\
&= \mathcal{M}[f_q^{\text{NS}}](m,\mu_{F,0})\;\left(\frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)}\right)^{\frac{\gamma(m)}{2\pi b_0}}\;.
\end{aligned} \tag{5.54}$$

Among other things, in this derivation we neglect that some splitting functions have singularities and therefore the Mellin transform is not obviously well defined. Our convolution is not really a convolution either, because we cut it off at Q_0^2 etc; but the final structure in Eq.(5.54) really holds.

Instead of the non-singlet parton densities we find the same kind of solution in pure Yang–Mills theory, *i.e.* in QCD without quarks. Looking at the different color factors in QCD this limit can also be derived as the leading terms in N_c. In that case there also exists only one splitting kernel defining an anomalous dimension γ. We find in complete analogy to Eq.(5.54)

$$\mathcal{M}[f_g](m,\mu) = \mathcal{M}[f_g](m,\mu_0)\;\left(\frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)}\right)^{\frac{\gamma(m)}{2\pi b_0}}\;. \tag{5.55}$$

The solutions to the DGLAP equation are not completely determined, because it an integration constant in terms of μ_0. The DGLAP equation does not determine parton densities, it only describes their evolution from one scale μ_F to another, just like a renormalization group equation for the strong coupling.

Remembering how we arrive at the DGLAP equation we notice an analogy to the case of ultraviolet divergences and the running coupling. We start from universal infrared divergences. We describe them in terms of splitting functions which we regularize using the plus prescription. The DGLAP equation plays the role of a renormalization group equation for example for the running coupling. It links parton densities evaluated at different scales μ_F. In analogy to the scaling logarithms considered in Section 4.7 we should test if we can point to a type of logarithm the DGLAP equation resums by reorganizing our perturbative series of parton splitting.

5.6 Resumming collinear logarithms

In our discussion of the DGLAP equation and its solution we for instance encounter the splitting probability in the exponent. To make sense of such a structure we remind ourselves that such ratios of α_s values to some power can appear as a result of a resummed series. Such a series would need to include powers of $(\mathcal{M}[\hat{P}])^n$ summed over n which corresponds to a sum over splittings with a varying number of partons in the final state. Parton densities cannot be formulated in terms of a fixed final state because they include effects from any number of collinear partons summed over the number of such partons. For the processes we can evaluate using parton densities fulfilling the DGLAP equation this means that they always have the form

$$\boxed{pp \to \mu^+\mu^- + X} \qquad\qquad \text{where } X \text{ includes any number of collinear jets.} \tag{5.56}$$

The same argument leads us towards the logarithms the running parton densities re-sum. To identify them we build a physical model based on collinear splitting, but without using the DGLAP equation. We then solve it to see the resulting structure of the solutions and compare it to the structure of the DGLAP solutions in Eq.(5.55).

We start from the basic equation defining the physical picture of parton splitting in Eq.(5.19). Only taking into account gluons in pure Yang–Mills theory the starting point of our discussion was a factorization, schematically written as

$$\sigma_{n+1} = \int dz \frac{dt}{t}\, \frac{\alpha_s}{2\pi}\, \hat{P}_{g\leftarrow g}(z)\sigma_n \; . \tag{5.57}$$

For a moment, we forget about the parton densities and assume that they are part of the hadronic cross section σ_n.

To treat initial state splittings, we need a definition of the virtuality t. If we remember that $t = p_b^2 < 0$ we can introduce a positive transverse momentum variable $\vec{p}_T^{\,2}$ such that

$$-t = -\frac{p_T^2}{1-z} = \frac{\vec{p}_T^{\,2}}{1-z} > 0 \qquad\qquad \Rightarrow \quad \frac{dt}{t} = \frac{dp_T^2}{p_T^2} = \frac{d\vec{p}_T^{\,2}}{\vec{p}_T^{\,2}} \; . \tag{5.58}$$

From the definition of p_T in Eq.(**??**) we see that $\vec{p}_T^{\,2}$ is really the transverse three-momentum of of the parton pair after splitting. The factorized form in Eq.(5.57) becomes a convolution in the collinear limit,

$$\sigma_{n+1}(x, \mu_F) = \int_{x_0}^1 \frac{dx_n}{x_n}\, P_{g\leftarrow g}\left(\frac{x}{x_n}\right) \sigma_n(x_n, \mu_0) \int_{\mu_0^2}^{\mu_F^2} \frac{d\vec{p}_{T,n}^{\,2}}{\vec{p}_{T,n}^{\,2}} \frac{\alpha_s(\mu_R^2)}{2\pi} \; . \tag{5.59}$$

Because the splitting kernel is infrared divergent we cut off the convolution integral at x_0. Similarly, the transverse momentum integral is bounded by an infrared cutoff μ_0 and the physical external scale μ_F. This is the range in which an additional collinear radiation is included in σ_{n+1}.

For splitting the two integrals in Eq.(5.59) it is crucial that μ_0 is the only scale the matrix element σ_n depends on. The other integration variable, the transverse momentum, does not feature in σ_n because collinear factorization is defined in the limit $\vec{p}_T^{\,2} \to 0$. All through the argument of this subsection we should keep in mind that we are looking for assumptions which allow us to solve Eq.(5.59) and compare the result to the solution of the DGLAP equation. To develop this physics picture of the DGLAP equation we make three assumptions:

1. If μ_F is the global upper boundary of the transverse momentum integration for collinear splitting, we can apply the recursion formula in Eq.(5.59) iteratively

$$\sigma_{n+1}(x,\mu_F) \sim \int_{x_0}^1 \frac{dx_n}{x_n} \; P_{g\leftarrow g}\left(\frac{x}{x_n}\right) \; \cdots \; \int_{x_0}^1 \frac{dx_1}{x_1} \; P_{g\leftarrow g}\left(\frac{x_2}{x_1}\right) \; \sigma_1(x_1,\mu_0)$$
$$\times \int_{\mu_0}^{\mu_F} \frac{d\vec{p}_{T,n}^{\,2}}{\vec{p}_{T,n}^{\,2}} \frac{\alpha_s(\mu_R^2)}{2\pi} \; \cdots \; \int_{\mu_0} \frac{d\vec{p}_{T,1}^{\,2}}{\vec{p}_{T,1}^{\,2}} \frac{\alpha_s(\mu_R^2)}{2\pi} \; . \tag{5.60}$$

2. We identify the scale of the strong coupling α_s with the transverse momentum scale of the splitting,

$$\mu_R^2 = \vec{p}_T^{\,2} \; . \tag{5.61}$$

   This way we can fully integrate $\alpha_s/(2\pi)$ and link the final result to the global boundary μ_F.

3. Finally, we assume strongly ordered splittings in the transverse momentum. If the ordering of the splitting is fixed externally by the chain of momentum fractions x_j, this means

$$\mu_0^2 < \vec{p}_{T,1}^{\,2} < \vec{p}_{T,2}^{\,2} < \cdots < \mu_F^2 \tag{5.62}$$

Under these three assumptions the transverse momentum integrals in Eq.(5.60) become

$$\begin{aligned}
&\int_{\mu_0}^{\mu_F} \frac{d\vec{p}_{T,n}^{\,2}}{\vec{p}_{T,n}^{\,2}} \frac{\alpha_s(\vec{p}_{T,n}^{\,2})}{2\pi} \; \cdots \; \int_{\mu_0}^{p_{T,3}} \frac{d\vec{p}_{T,2}^{\,2}}{\vec{p}_{T,2}^{\,2}} \frac{\alpha_s(\vec{p}_{T,2}^{\,2})}{2\pi} \; \int_{\mu_0}^{p_{T,2}} \frac{d\vec{p}_{T,1}^{\,2}}{\vec{p}_{T,1}^{\,2}} \frac{\alpha_s(\vec{p}_{T,1}^{\,2})}{2\pi} \\
&= \int_{\mu_0}^{\mu_F} \frac{d\vec{p}_{T,n}^{\,2}}{\vec{p}_{T,n}^{\,2}} \frac{1}{2\pi b_0 \log \dfrac{\vec{p}_{T,n}^{\,2}}{\Lambda_{\text{QCD}}^2}} \; \cdots \; \int_{\mu_0}^{p_{T,3}} \frac{d\vec{p}_{T,2}^{\,2}}{\vec{p}_{T,2}^{\,2}} \frac{1}{2\pi b_0 \log \dfrac{\vec{p}_{T,2}^{\,2}}{\Lambda_{\text{QCD}}^2}} \; \int_{\mu_0}^{p_{T,2}} \frac{d\vec{p}_{T,1}^{\,2}}{\vec{p}_{T,1}^{\,2}} \frac{1}{2\pi b_0 \log \dfrac{\vec{p}_{T,1}^{\,2}}{\Lambda_{\text{QCD}}^2}} \\
&= \frac{1}{(2\pi b_0)^n} \int_{\mu_0}^{\mu_F} \frac{d\vec{p}_{T,n}^{\,2}}{\vec{p}_{T,n}^{\,2}} \frac{1}{\log \dfrac{\vec{p}_{T,n}^{\,2}}{\Lambda_{\text{QCD}}^2}} \; \cdots \; \int_{\mu_0}^{p_{T,3}} \frac{d\vec{p}_{T,2}^{\,2}}{\vec{p}_{T,2}^{\,2}} \frac{1}{\log \dfrac{\vec{p}_{T,2}^{\,2}}{\Lambda_{\text{QCD}}^2}} \; \int_{\mu_0}^{p_{T,2}} \frac{d\vec{p}_{T,1}^{\,2}}{\vec{p}_{T,1}^{\,2}} \frac{1}{\log \dfrac{\vec{p}_{T,1}^{\,2}}{\Lambda_{\text{QCD}}^2}} \; .
\end{aligned} \tag{5.63}$$

We can solve the individual integrals by switching variables, for example in the last integral

$$\begin{aligned}
\int_{\mu_0}^{p_{T,2}} \frac{d\vec{p}_{T,1}^{\,2}}{\vec{p}_{T,1}^{\,2}} \frac{1}{\log \dfrac{\vec{p}_{T,1}^{\,2}}{\Lambda_{\text{QCD}}^2}} &= \int_{\log\log \mu_0^2/\Lambda^2}^{\log\log p_{T,2}^2/\Lambda^2} d\log\log \frac{\vec{p}_{T,1}^{\,2}}{\Lambda_{\text{QCD}}^2} \qquad \text{with} \qquad \frac{d(ax)}{(ax)\log x} = d\log\log x \\
&= \log \frac{\log \vec{p}_{T,2}^{\,2}/\Lambda_{\text{QCD}}^2}{\log \mu_0^2/\Lambda_{\text{QCD}}^2} \; .
\end{aligned} \tag{5.64}$$

This gives us for the chain of transverse momentum integrals, shifted to get rid of the lower boundaries,

$$\begin{aligned}
&\int^{p_{T,n}\equiv\mu_F} d\log \frac{\log \vec{p}_{T,n}^{\,2}/\Lambda_{\text{QCD}}^2}{\log \mu_0^2/\Lambda_{\text{QCD}}^2} \; \cdots \int^{p_{T,2}\equiv p_{T,3}} d\log \frac{\log \vec{p}_{T,2}^{\,2}/\Lambda_{\text{QCD}}^2}{\log \mu_0^2/\Lambda_{\text{QCD}}^2} \int^{p_{T,1}\equiv p_{T,2}} d\log \frac{\log \vec{p}_{T,1}^{\,2}/\Lambda_{\text{QCD}}^2}{\log \mu_0^2/\Lambda_{\text{QCD}}^2} \\
&= \int^{p_{T,n}\equiv\mu_F} d\log \frac{\log \vec{p}_{T,n}^{\,2}/\Lambda_{\text{QCD}}^2}{\log \mu_0^2/\Lambda_{\text{QCD}}^2} \; \cdots \int^{p_{T,2}\equiv p_{T,3}} d\log \frac{\log \vec{p}_{T,2}^{\,2}/\Lambda_{\text{QCD}}^2}{\log \mu_0^2/\Lambda_{\text{QCD}}^2} \left(\log \frac{\log \vec{p}_{T,2}^{\,2}/\Lambda_{\text{QCD}}^2}{\log \mu_0^2/\Lambda_{\text{QCD}}^2} \right) \\
&= \int^{p_{T,n}\equiv\mu_F} d\log \frac{\log \vec{p}_{T,n}^{\,2}/\Lambda_{\text{QCD}}^2}{\log \mu_0^2/\Lambda_{\text{QCD}}^2} \; \cdots \frac{1}{2} \left(\log \frac{\log \vec{p}_{T,3}^{\,2}/\Lambda_{\text{QCD}}^2}{\log \mu_0^2/\Lambda_{\text{QCD}}^2} \right)^2 \\
&= \int^{p_{T,n}\equiv\mu_F} d\log \frac{\log \vec{p}_{T,n}^{\,2}/\Lambda_{\text{QCD}}^2}{\log \mu_0^2/\Lambda_{\text{QCD}}^2} \left(\frac{1}{2} \cdots \frac{1}{n-1} \right) \left(\log \frac{\log \vec{p}_{T,n}^{\,2}/\Lambda_{\text{QCD}}^2}{\log \mu_0^2/\Lambda_{\text{QCD}}^2} \right)^{n-1} \\
&= \frac{1}{n!} \left(\log \frac{\log \mu_F^2/\Lambda_{\text{QCD}}^2}{\log \mu_0^2/\Lambda_{\text{QCD}}^2} \right)^n = \frac{1}{n!} \left(\log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu_F^2)} \right)^n \; .
\end{aligned} \tag{5.65}$$

This is the final result for the chain of transverse momentum integrals in Eq.(5.60). After integrating over the transverse momenta, the strong coupling is evaluated at $\mu_R \equiv \mu_F$. This leaves us with the convolution integrals from Eq.(5.59),

$$\sigma_{n+1}(x,\mu) \sim \frac{1}{n!} \left(\frac{1}{2\pi b_0} \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right)^n \int_{x_0}^1 \frac{dx_n}{x_n} P_{g\leftarrow g}\left(\frac{x}{x_n}\right) \cdots \int_{x_0}^1 \frac{dx_1}{x_1} P_{g\leftarrow g}\left(\frac{x_2}{x_1}\right) \sigma_1(x_1,\mu_0) \; . \tag{5.66}$$

As before, we Mellin-transform the equation into moment space

$$\begin{aligned}
\mathcal{M}[\sigma_{n+1}](m,\mu) \sim & \frac{1}{n!} \left(\frac{1}{2\pi b_0} \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right)^n \mathcal{M}\left[\int_{x_0}^1 \frac{dx_n}{x_n} P_{g\leftarrow g}\left(\frac{x}{x_n}\right) \cdots \int_{x_0}^1 \frac{dx_1}{x_1} P_{g\leftarrow g}\left(\frac{x_2}{x_1}\right) \sigma_1(x_1,\mu_0) \right](m) \\
= & \frac{1}{n!} \left(\frac{1}{2\pi b_0} \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right)^n \gamma(m)^n \, \mathcal{M}[\sigma_1](m,\mu_0) \qquad \text{using } \gamma(m) \equiv \mathcal{M}[P](m) \\
= & \frac{1}{n!} \left(\frac{1}{2\pi b_0} \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \, \gamma(m) \right)^n \mathcal{M}[\sigma_1](m,\mu_0) \; .
\end{aligned} \tag{5.67}$$

Finally, we sum the production cross sections for up to n collinear jets,

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{M}[\sigma_{n+1}](m,\mu) = & \mathcal{M}[\sigma_1](m,\mu_0) \sum_n \frac{1}{n!} \left(\frac{1}{2\pi b_0} \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \, \gamma(m) \right)^n \\
= & \mathcal{M}[\sigma_1](m,\mu_0) \, \exp \left(\frac{\gamma(m)}{2\pi b_0} \, \log \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right) \\
= & \mathcal{M}[\sigma_1](m,\mu_0) \, \left(\frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right)^{\frac{\gamma(m)}{2\pi b_0}} \; .
\end{aligned} \tag{5.68}$$

This is the same structure as the DGLAP equation's solution in Eq.(5.55). It means that we can understand the physics of the DGLAP equation using our model calculation of a successive gluon emission, including the generically variable number of collinear jets in the form of $pp \to \mu^+\mu^- + X$, as shown in Eq.(5.56). On the left hand side of Eq.(5.68) we have the sum over any number of additional collinear partons; on the right hand side we see fixed order Drell–Yan production without any additional partons, but with an exponentiated correction factor. Comparing this to the running parton densities we can draw the analogy that any process computed with a scale dependent parton density where the scale dependence is governed by the DGLAP equation includes any number of collinear partons.

We can also identify the logarithms which are resummed by scale dependent parton densities. Going back to Eq.(5.12) reminds us that we start from the divergent collinear logarithms $\log p_T^{\max}/p_T^{\min}$ arising from the collinear phase space

	renormalization scale μ_R	factorization scale μ_F
source	ultraviolet divergence	collinear (infrared) divergence
poles cancelled	counter terms (renormalization)	parton densities (mass factorization)
summation	resum self energy bubbles	resum parton splittings
parameter	running coupling $\alpha_s(\mu_R^2)$	running parton density $f_j(x,\mu_F)$
evolution	RGE for α_s	DGLAP equation
large scales	decrease of σ_{tot}	increase of σ_{tot} for gluons/sea quarks
theory background	renormalizability proven for gauge theories	factorization proven all orders for DIS proven order-by-order DY...

Table 2: Comparison of renormalization and factorization scales appearing in LHC cross sections.

integration. In our model for successive splitting we replace the upper boundary by μ_F. The collinear logarithm of successive initial–state parton splitting diverges for $\mu_0 \to 0$, but it gets absorbed into the parton densities and determines the structure of the DGLAP equation and its solutions. The upper boundary μ_F tells us to what extent we assume incoming quarks and gluons to be a coupled system of splitting partons and what the maximum momentum scale of these splittings is. Transverse momenta $p_T > \mu_F$ generated by hard parton splitting are not covered by the DGLAP equation and hence not a feature of the incoming partons anymore. They belong to the hard process and have to be consistently simulated. While this scale can be chosen freely we have to make sure that it does not become too large, because at some point the collinear approximation $C \simeq$ constant in Eq.(5.12) ceases to hold.