Particle Physics 1+2
Theory Chapters

Tilman Plehn

Institut für Theoretische Physik

Contents

1 A simple scattering process

When we compute transition amplitudes for collider like LEP or LHC, we usually combine building blocks defined by Feynman rules in a way which does not make it obvious that we are dealing with a quantum field theory. One of the easiest processes we can look at is

$$e^+e^- \to \gamma^* \to q\bar{q} \; , \tag{1.1}$$

through a photon, all starting from these Feynman rules. Let is see what we start from and how we can compute this process using so-called Feynman rules. For this scattering process we need to describe four external fermions, their coupling to a photon, and the propagation of this boson from the e^+e^- annihilation to the point where is splits into a quark and antiquark pair.

From theoretical mechanics we remember that there are several ways to describe a physical system and calculate its time evolution. Assuming one degree of freedom or a real scalar field ϕ, we can for instance start with the action

$$S = \int d^4x \, \mathscr{L}(\phi, \partial_\mu \phi) \qquad \text{with} \qquad x = \begin{pmatrix} x_0 \\ \vec{x} \end{pmatrix} \; . \tag{1.2}$$

The position x is given by a space-time 4-vector, as we know if from special relativity. Under the integral there is a Lagrange density, which works exactly like the Lagrange function in mechanics, just that the object ϕ now is a quantum field, and that in particle physics we use space-time and the Minkoswki metric (+ - - -). The action has to be invariant under a variation $\delta S = 0$. We can translate this condition into the Euler-Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathscr{L}}{\partial \phi} \qquad \text{with} \qquad \partial_\mu = \frac{\partial}{\partial x^\mu} \; . \tag{1.3}$$

The second field for our switch from Lagrangian to the Hamiltonian is the (conjugate) momentum, which we can calculate just like in a classical field theory. It is

$$\pi(x) = \frac{\partial \mathscr{L}}{\partial(\partial_0 \phi)} = \dot{\phi} \; . \tag{1.4}$$

With these two field we define the third object which we can use to describe the dynamics of a system, the Hamiltonian or energy functional

$$\mathcal{H}(t) = \int d^3x \, \left(\pi \dot{\phi} - \mathscr{L} \right) \; . \tag{1.5}$$

While for example in quantum mechanics this Hamiltonian is the basis of most calculations, in field theory we usually start from the Lagrangian. This also means that at the end of the day we never really use the time-dependence given by the conjugate momentum.

Boson field

We already know that for our scattering process we need to compute a transition amplitude between two kinds of matter particles, namely incoming electrons and outgoing quarks, interacting via their electric charges. The interaction is classically described by the electromagnetic Lagrangian based on the abelian $U(1)$ field theory,

$$\mathscr{L}_{\text{photon}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \text{with} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \; , \tag{1.6}$$

in terms of a photon 4-vector field A_μ. This is exactly what we know from classical electrodynamics written in a covariant way. The Maxwell equations

$$0 = \partial^\mu F_{\mu\nu} = \partial^\mu \partial_\mu A_\nu - \partial^\mu \partial_\nu A_\mu = \Box A_\nu \qquad \text{with} \qquad \Box = \partial_\mu \partial^\nu \tag{1.7}$$

are the equations of motion for this photon field. In the last step we assume the Lorentz gauge condition $\partial_\mu A^\mu = 0$ and find the d'Alembert equation for the vector potential A_μ.

To omit the vector index of the photon field, let us instead use the real scalar ϕ from Eq.(1.2) to illustrate bosonic fields. Including a mass for this real scalar field we can write down its equation of motion which is the same for a spin-zero scalar boson as for the spin-one vector boson of Eq.(1.7)

$$\left(\Box + m^2\right)\, \phi(x) = 0 \; . \tag{1.8}$$

This Klein–Gordon equation corresponds to the d'Alembert equation for the electromagnetic vector potential in Lorentz gauge. This equation of motion for a scalar field with a mass m corresponds to a Lagrangian

$$\mathscr{L}_{\text{scalar}} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2 \; , \tag{1.9}$$

which we can confirm using the Euler-Lagrange equation Eq.(1.3). If we want to compute the scattering amplitude in momentum space, we need to Fourier-transform the scalar field into momentum space and then quantize it, *i.e.* define commutation properties for the field in position and momentum space.

For our scattering process we need an object that describes the propagation of the (virtual) photon from its production from an e^+e^- pair to its splitting into a $q\bar{q}$ pair. This so-called propagator in position space is defined as a time–ordered product of two field operators sandwiched between vacuum states. We can think of it as describing a photon starting from its birth out of an e^+e^- pair to its death as a $q\bar{q}$ pair,

$$\Delta(x - x') \equiv i \, \langle 0 | T\left(\phi(x)\phi(x')\right) |0\rangle \; . \tag{1.10}$$

The time–ordered product of two operators is defined as

$$T\left(A(x)B(x')\right) = \begin{cases} A(x)B(x') & x_0 > x_0' \\ B(x')A(x) & x_0' > x_0 \end{cases} \; . \tag{1.11}$$

We can transform this propagator into Fourier space and find

$$\Delta(x - x') = -\int \frac{d^4k}{(2\pi)^4} \; e^{-ik(x-x')} \; \frac{1}{k^2 - m^2 + i\varepsilon} \; . \tag{1.12}$$

The propagator is the Green function for the Klein–Gordon equation Eq.(1.8), as we can explicitly confirm

$$\begin{aligned} \left(\Box + m^2\right)\Delta(x - x') &= -\int \frac{d^4k}{(2\pi)^4} \; \left(\Box + m^2\right) e^{-ik\cdot(x-x')} \; \frac{1}{k^2 - m^2} \\ &= \int \frac{d^4k}{(2\pi)^4} \; \left((-ik)^2 + m^2\right) e^{-ik\cdot(x-x')} \; \frac{(-1)}{k^2 - m^2} \\ &= \int \frac{d^4k}{(2\pi)^4} \; e^{-ik\cdot(x-x')} \\ &= \delta^4(x - x') \; . \end{aligned} \tag{1.13}$$

All these properties we will later use for the photon field A^μ, a vector field, where each component obeys the Klein–Gordon equation. The propagator and quantization aspects like commutation relations for the field operators will not change. The propagator only gets dressed by factors $g_{\mu\nu}$ where appropriate. For the propagator this generalization is strictly speaking gauge dependent, $g^{\mu\nu}$ corresponds to Feynman gauge.

Fermion field

Next, we need to describe (external) fermion fields. Matter particles or fermions, like leptons or quarks, have a different equation of motion and a different contribution to the Lagrangian. A field describing a fermionic particle has to include two spin states of this particles. Moreover, in quantum field theory every fermion $\chi^\dagger$ has an antiparticle with the same mass. The altogether four degrees of freedom naturally combine to one equation with the same mass and the particle and the antiparticle described by one field ψ.

The form of the fermion field is given by the transformation property under the Lorentz transformation. We remind ourselves that a scalar field $\phi(x)$ transforms under a Lorentz transformation via a unitary operator $U(\Lambda)$ as

$$U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x) \; . \tag{1.14}$$

The fermion field has to live in a different, the so-called spinor representation. It transforms under the Lorentz transformation as

$$U(\Lambda)^{-1}\psi(x)U(\Lambda) = \Lambda_{1/2}\,\psi(\Lambda^{-1}x) \; , \tag{1.15}$$

where $\Lambda_{1/2}$ is this special representation of the Lorentz transformation. We can define it using the four Dirac matrices γ^μ with their anti–commutator Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{1} \; . \tag{1.16}$$

The unit matrix has the same size as the γ matrices. That we usually write them as 4×4 matrices has nothing to do with the number of — also four — γ matrices. The explicit form of the γ_μ matrices is not relevant because it never appears in actual calculations. All we need is a few trace relations arising from their commutators. A representation of the Lorentz algebra in terms of the Dirac matrices is

$$\Lambda_{1/2} = \exp\left(\frac{\omega_{\mu\nu}}{8}\;[\gamma^\mu, \gamma^\nu]\right) \; . \tag{1.17}$$

This give us the transformation rule for the Dirac matrices

$$\Lambda_{1/2}^{-1}\;\gamma^\mu\;\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu \; . \tag{1.18}$$

We now postulate an equation of motion for the fermions, the Dirac equation, which describes fermions in Nature perfectly. One way to motivate this form is by taking some kind of square root of the Klein-Gordon equation. Another way is to write 2-spinors separately, for the two-spin states of the particle or of the antiparticle. Because of the spin, each of these two Dirac equation is then written with the help of the Pauli matrices as generators of the spin group $SU(2)$. For the full Dirac spinor it reads

$$(i\gamma^\mu\partial_\mu - m\mathbb{1})\;\psi(x) \equiv (i\partial\!\!\!/ - m\mathbb{1})\;\psi(x) = 0 \; . \tag{1.19}$$

The unit matrix in the mass term is a four-by-four matrix, just like the Dirac matrices. We want to check that this equation is invariant under Lorentz transformations, keeping in mind that $\Lambda_{1/2}$ commutes with everything except for the Dirac matrices

$$\begin{aligned}
(i\gamma^\mu\partial_\mu - m\mathbb{1})\;\psi(x) &\to \left(i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\mathbb{1}\right)\;\Lambda_{1/2}\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\Lambda_{1/2}^{-1}\left(i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\mathbb{1}\right)\;\Lambda_{1/2}\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\left(i\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2}(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\mathbb{1}\right)\;\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\left(i\Lambda^\mu{}_\rho\gamma^\rho(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\mathbb{1}\right)\;\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\left(ig^\nu{}_\rho\gamma^\rho\partial_\nu - m\mathbb{1}\right)\;\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\left(i\gamma^\nu\partial_\nu - m\mathbb{1}\right)\;\psi(\Lambda^{-1}x) = 0 \; .
\end{aligned} \tag{1.20}$$

We also see that we can multiply the Dirac equation with $(-i\gamma^\mu\partial_\mu - m\mathbb{1})$ and obtain

$$(-i\gamma^\mu\partial_\mu - m\mathbb{1})\,(i\gamma^\nu\partial_\nu - m\mathbb{1})\;\psi(x) = \left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2\mathbb{1}\right)\;\psi(x) = \left(\partial^2 + m^2\mathbb{1}\right)\;\psi(x) \; , \tag{1.21}$$

which means that every fermion field that obeys the Dirac equation also fulfills a Klein–Gordon equation. We will see this similarity when we construct the fermion propagator.

To define a mass term in the Lagrangian we need to form Lorentz scalars or invariants out of the fermion fields ψ. Naively, $(\psi^\dagger \psi)$ would work if the Lorentz transformations in $(\psi^\dagger \Lambda^\dagger_{1/2} \Lambda_{1/2} \psi)$ cancelled. Unfortunately $\Lambda_{1/2}$ is not a unitary transformation. Instead, one can show that the Dirac adjoint

$$\overline{\psi} = \psi^\dagger \gamma^0 \qquad \text{with} \quad \overline{\psi}\psi \to \overline{\psi}\psi \tag{1.22}$$

has the correct transformation property. This allows us to write down the Lagrangian which corresponds to the Dirac equation for a fermion field

$$\mathscr{L}_{\text{fermion}} = \overline{\psi}(i\partial\!\!\!/ - m\mathbb{1})\psi \; . \tag{1.23}$$

Because we will later need the fermion–photon interaction in the form of a Hamiltonian or Lagrangian we introduce the convenient form of the covariant derivative

$$\mathscr{L}_{\text{fermion-photon}} = \overline{\psi}\left(iD\!\!\!\!/ - m\mathbb{1}\right)\psi \equiv \overline{\psi}\left(i(\partial\!\!\!/ + ieQA\!\!\!/) - m\mathbb{1}\right)\psi = \overline{\psi}\left(i(\partial\!\!\!/ - m\mathbb{1}\right)\psi + eQA_\mu\,\overline{\psi}\gamma^\mu\psi \tag{1.24}$$

The last term describes the coupling of a vector photon field A_μ to a vector-like expression $\overline{\psi}\gamma^\mu\psi$ which we call a vector current of a spinor field.

Just like in the bosonic case we now Fourier-transform the Dirac field operators, which we know have to include four degrees of freedom, particle and anti-particle with two spins each,

$$\begin{aligned}
\psi(x) &= \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{\text{spin } s} \left(e^{ikx} v_s(k) b_s^\dagger(\vec{k}) + e^{-ikx} u_s(k) a_s(\vec{k}) \right) \\
\overline{\psi}(x) &= \int \frac{d^3k}{(2\pi)^2 2k_0} \sum_{\text{spin } s} \left(e^{ikx} \bar{u}_s(k) a_s^\dagger(\vec{k}) + e^{-ikx} \bar{v}_s(k) b_s(\vec{k}) \right) \; ,
\end{aligned} \tag{1.25}$$

where the fermion spin can be $s = \pm 1/2$. The 4-dimensional spinors u and v create or annihilate the particle, while $\bar{u}$ and $\bar{v}$ create or annihilate the antiparticle in Fourier space, as listed in Tab. 1. As before, we skip the quantization steps and directly jump to the spin sums for the spinors u and v and their Dirac adjoints

$$\begin{aligned}
\sum_{\text{spin}} u_s(k)\bar{u}_s(k) &= k\!\!\!/ + m\mathbb{1} \\
\sum_{\text{spin}} v_s(k)\bar{v}_s(k) &= k\!\!\!/ - m\mathbb{1} \; .
\end{aligned} \tag{1.26}$$

As before, $k\!\!\!/$ is a (4×4) matrix, so we include a unit matrix which is often omitted. This means the spin sums we can use in our calculations do not combine spinors to scalars, but to Dirac matrices. Most of the time this is not a problem, unless we for example compute traces of chains of Dirac matrices and need to remember that $\text{Tr}\,\mathbb{1} \neq 1$.

symbol	diagram
$u_s(k)$	incoming fermion (e^-, q) with momentum k and spin s
$\bar{v}_s(k)$	incoming anti–fermion $(e^+, \bar{q})$
$\bar{u}_s(k)$	outgoing fermion (e^-, q)
$v_s(k)$	outgoing anti–fermion $(e^+, \bar{q})$

Table 1: Assignment of generation and annihilating operators for particle and antiparticle spinors in momentum space.

Scattering

Now we have everything we need to compute a transition amplitude for our scattering process

$$e^-(k_1,s_1)+e^+(k_2,s_2)\to q(p_1,s_3)+\bar q(p_2,s_4)\ , \tag{1.27}$$

where k_j, p_j and s_j are the four-momenta and spin orientations of the external fermions. In the future, or more specifically asymptotically for $t\to+\infty$, the initial state $\lim_{t\to-\infty}|t\rangle\equiv|i\rangle$ will have evolved into the final state $\lim_{t\to\infty}|t\rangle=\mathcal{S}|i\rangle$ via a yet unknown linear operator $\mathcal{S}$. To describe this scattering into a final state $\langle f|$ we need to compute the transition amplitude

$$S\equiv\langle f|\mathcal{S}|i\rangle=\langle q_3\bar q_4|\mathcal{S}|e_1^+e_2^-\rangle=\langle 0|a_3b_4\,\mathcal{S}\,a_1^\dagger b_2^\dagger|0\rangle\ . \tag{1.28}$$

We use one index to indicate the momenta and spins of the external particles. This transition amplitude is not a vacuum expectation value, but the operator $\mathcal{S}$ sandwiched between physically measurable states made from the vacuum using the generation and annihilation field operators $a, a^\dagger, b, b^\dagger$.

The transition matrix element $\mathcal{S}$ can be computed from the time evolution of the initial state $i\partial_t|t\rangle=\mathcal{H}(t)|t\rangle$ in the interaction picture with a time-dependent Hamilton operator,

$$\mathcal{S}=T\left(e^{-i\int dt\,\mathcal{H}(t)}\right)\ , \tag{1.29}$$

again with time ordering T. This form ensures that it generates a unitary transformation. For our computation we will be fine with the interaction Hamiltonian for two fermionic currents each involving a different particle species j with charge Q_j,

$$\mathcal{H}_{\text{int}}(t)=-\int d^3x\,\mathscr{L}_{\text{int}}(x)\supset\sum_j -eQ_j\int d^3x\,A_\mu\,\overline{\psi}_j\gamma^\mu\psi_j\ , \tag{1.30}$$

in terms of the four-vector x including its first entry $t=x_0$. We skip the corresponding calculation and just give the result for the numbering of the incoming and outgoing particles defined in Eq.(1.27)

$$S=\sum_{\text{spins}} i(2\pi)^4\delta^4(k_1+k_2-p_1-p_2)\;e^2Q_eQ_q\;\bar u_3\gamma_\mu v_4\frac{1}{(k_1+k_2)^2}\bar v_2\gamma^\mu u_1\ . \tag{1.31}$$

Stripping off unwanted prefactors we can define the transition matrix element for quark–antiquark production in QED as

$$\mathcal{M}=e^2Q_eQ_q\;(\bar u_3\gamma_\mu v_4)\;\frac{1}{(k_1+k_2)^2}\;(\bar v_2\gamma^\mu u_1)\ , \tag{1.32}$$

with $(k_1+k_2)^2=(p_1+p_2)^2$. This matrix element or transition amplitude we have to square to compute the transition probability. Part of the squaring is the sum over all spins which uses the spin sums Eq.(1.26) to get rid of the spinors and then some trace rules to get rid of all Dirac matrices. Neither for the spinors nor for the Dirac matrices we need to know their explicit form

$$\begin{aligned}|\mathcal{M}|^2&=\sum_{\text{spin, color}} e^4Q_e^2Q_q^2\;\frac{1}{(k_1+k_2)^4}\;(\bar v_4\gamma_\nu u_3)(\bar u_1\gamma^\nu v_2)\;(\bar u_3\gamma_\mu v_4)\;(\bar v_2\gamma^\mu u_1)\\&=e^4Q_e^2Q_q^2N_c\;\frac{1}{(k_1+k_2)^4}\;\sum_{\text{spin}}(\bar v_4\gamma_\nu u_3)(\bar u_1\gamma^\nu v_2)\;(\bar u_3\gamma_\mu v_4)\;(\bar v_2\gamma^\mu u_1)\ .\end{aligned} \tag{1.33}$$

The color factor N_c is the number of outgoing color singlet states we can form out of a quark and an antiquark with opposite color charges. Because color only appears in the final state we sum over all possible color states or multiply by N_c. In the next step we can observe how the crucial structure of transition amplitudes with external fermions, namely traces of chains of Dirac matrices, magically form:

$$\begin{aligned}
|\mathcal{M}|^2 =& e^4 Q_e^2 Q_q^2 N_c \, \frac{1}{(k_1+k_2)^4} \, \sum_{\text{spin}} (\bar{v}_4)_i (\gamma_\nu)_{ij} (u_3)_j (\bar{u}_3)_k (\gamma_\mu)_{kl} (v_4)_l \cdots && \text{for one trace} \\
=& e^4 Q_e^2 Q_q^2 N_c \, \frac{1}{(k_1+k_2)^4} \left(\sum_{\text{spin}} (v_4)_l (\bar{v}_4)_i \right) \left(\sum_{\text{spin}} (u_3)_j (\bar{u}_3)_k \right) (\gamma_\nu)_{ij} (\gamma_\mu)_{kl} \cdots \\
=& e^4 Q_e^2 Q_q^2 N_c \, \frac{1}{(k_1+k_2)^4} \, (\not{p}_4)_{li} (\not{p}_3)_{jk} (\gamma_\nu)_{ij} (\gamma_\mu)_{kl} \cdots && \text{using Eq.(1.26)} \\
=& e^4 Q_e^2 Q_q^2 N_c \, \frac{1}{(k_1+k_2)^4} \, \text{Tr} \left(\not{p}_4 \gamma_\nu \not{p}_3 \gamma_\mu \right) \, \text{Tr} \left(\not{p}_1 \gamma^\nu \not{p}_2 \gamma^\mu \right) && \text{both traces again.} \qquad (1.34)
\end{aligned}$$

In the final step we need to use a know expression for the Dirac trace. More complicated and longer traces become very complicated very fast and we evaluate them using symbolic manipulation on the computer. We find

$$\begin{aligned}
|\mathcal{M}|^2 =& e^4 Q_e^2 Q_q^2 N_c \, \frac{1}{(k_1+k_2)^4} \, 4 \left(p_{3\nu} p_{4\mu} + p_{3\mu} p_{4\nu} - g_{\mu\nu} (p_1 p_2) \right) \, 4 \left(k_1^\nu k_2^\mu + k_1^\mu k_2^\nu - g_{\mu\nu} (k_1 k_2) \right) \\
=& 16 e^4 Q_e^2 Q_q^2 N_c \, \frac{1}{(k_1+k_2)^4} \, \left[2 (k_1 p_1)(k_2 p_2) + 2 (k_1 p_2)(k_2 p_1) + 0 \times (p_1 p_2)(k_1 k_2) \right] && \text{with} \quad g_{\mu\nu} g^{\mu\nu} = 4 \\
=& 32 e^4 Q_e^2 Q_q^2 N_c \, \frac{1}{(k_1+k_2)^4} \, \left[(k_1 p_1)(k_2 p_2) + (k_1 p_2)(k_2 p_1) \right] \, , && (1.35)
\end{aligned}$$

To evalue this matrix element we first introduce Mandelstam variables as squares of sums of 4-vectors,

$$s = (k_1 + k_2)^2 > 0 \qquad\qquad t = (k_1 + p_1)^2 < 0 \qquad\qquad u = (k_1 + p_2)^2 \, , \qquad (1.36)$$

where in this sign convention all momenta are incoming, $k_1 + k_2 + p_1 + p_2 = 0$. The second mandelstam variable can be expressed through the polar scattering angle $t = s/2 \times (-1 + \cos\theta)$. Allowing that the external particles have a finite mass we can use this 4-momentum conservation to show

$$s + t + u = p_1^2 + p_2^2 \equiv m_1^2 + m_2^2 \, . \qquad (1.37)$$

This gives us the compact form

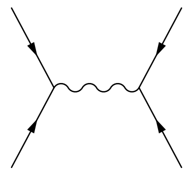
$$\begin{aligned}
|\mathcal{M}|^2 &= 32 e^4 \, Q_e^2 Q_q^2 N_c \, \frac{1}{s^2} \, \left[\frac{t^2}{4} + \frac{u^2}{4} \right] \\
&= 8 e^4 \, Q_e^2 Q_q^2 N_c \, \frac{1}{s^2} \, \left[s^2 + 2st + 2t^2 \right] \\
&= 8 e^4 \, Q_e^2 Q_q^2 N_c \, \left[1 + 2 \frac{t}{s} + 2 \frac{t^2}{s^2} \right] \, . \qquad (1.38)
\end{aligned}$$

We can briefly check if this number is indeed positive, using the definition of the Mandelstam variable t for massless external particles in terms of the polar angle $t = s(-1 + \cos\theta)/2 = -s \cdots 0$: the upper phase space boundary $t = 0$ inserted into the brackets in Eq.(1.38) gives $[\cdots] = 1$, just as the lower boundary $t = -s$ with $[\cdots] = 1 - 2 + 2 = 1$. For the central value $t = -s/2$ the minimum value of the brackets is $[\cdots] = 1 - 1 + 0.5 = 0.5$.

The azimuthal angle ϕ plays no role at colliders, unless you want to compute gravitational effects on Higgs production at ATLAS and CMS. Any LHC Monte Carlo will either random-generate a reference angle ϕ for the partonic process or pick one and keep it fixed.

Feynman rules

Feynman rules are calculational rules which we can extract from the Lagrangian and which allow us to derive Eq.(1.32) directly. These building blocks representing external and internal particles we combine to construct $\mathcal{M}$. We start by drawing Feynman diagrams representing all ways we can link the given initial and final states through interaction vertices and internal propagators. For our scattering process there exists only one such diagram:



It consist of four external fermions, one internal photon, and two interaction vertices. From Eq.(1.25) and Tab. 1 we know how to describe external fermions in terms of spinors.

Spin sums are the only way to get rid of spinors in the computation. Equation (1.26) shows that as long as we neglect fermion masses the two spinors u and v for particles and antiparticles are identical. To link external particles to each other and to internal propagators we need vertices. If two fermions and a gauge boson interact via a vector current proportional to γ^μ, and adding a conventional factor i, the one vertex rule in QED reads

$$ieQ_f\,\gamma^\mu \qquad\qquad (f-\bar{f}-\gamma). \tag{1.39}$$

This factor i we can consistently change for all three-point and four-point vertices in our theory. Finally, there is the intermediate photon which propagates between the γ^μ and the γ^ν vertices. The wave line in the Feynman diagram corresponds to

$$-i\frac{g^{\mu\nu}}{p^2+i\epsilon}\;. \tag{1.40}$$

Again, the factor $-i$ is conventional. For a bosonic propagator it does not matter in which direction the momentum flows. Blindly combining these Feynman rules gives us directly Eq.(1.32), so all we need to do is square the matrix element, insert the spin sums and compute the Dirac trace.

We do not need it for our QED calculation, but for instance process $e^-\gamma \to e^-\gamma$ is described by an intermediate fermion propagator in the s-channel. This propagator is described by the Feynman rule

$$i\frac{\not{p}+m\mathbb{1}}{p^2-m^2} = i\frac{\not{p}+m\mathbb{1}}{\not{p}^2-m^2} = i\frac{\not{p}+m\mathbb{1}}{(\not{p}+m)(\not{p}-m)} = i\left(\not{p}-m\mathbb{1}\right)^{-1}\;. \tag{1.41}$$

It lives in the same space as gamma matrices. Because the sign of the 4-momentum matters we asign it in parallel to the fermion arrow. This will work fine until we have to deal with Majorana particles in the neutrino sector or supersymmetry (for those who still remember that).

Whenever we compute such a matrix element starting from a Feynman diagram nothing tells us that the lines in the Feynman diagrams are not actual physical states propagating from the left to the right. Even including loop diagrams will still look completely reasonably from a semi–classical point of view. Feynman rules define an algorithm which hides all field theory input in the calculation of scattering amplitudes and are therefore perfectly suited to compute the differential and total cross sections on the computer.

The vector structure of the QED couplings, for example mediated by a covariant derivative Eq.(1.24) we did not actually motivate. It happens to work on the Lagrangian level and agrees with data, so it is correct. We can write a completely general interaction of two fermions with a boson in terms of basis elements

$$g\,\overline{\psi}M\psi = \sum_{\text{basis } j} g_j\,\overline{\psi}M_j\psi\;. \tag{1.42}$$

For a real (4×4) matrix M the necessary 16 basis elements can be organized such that they are easy to keep track of using Lorentz transformation properties. This eventually leads to the so-called Fierz transformation. The vector γ^μ from the QED interaction gives us four such basis elements, the unit matrix a fifth. Another six we already know as well, they are the generators of the spinor representation $[\gamma^\mu, \gamma^\nu]$. We can check that all of them are linearly independent. Five basis elements in a handy form are still missing.

To define them, we need to know that there exists another (4×4) matrix which is invariant under proper Lorentz transformations. We can write it in terms of the four Dirac matrices in two equivalent forms

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\,\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \;, \tag{1.43}$$

using the totally anti–symmetric Levi–Civita tensor $\epsilon_{\mu\nu\rho\sigma}$. This form already shows a major technical complication in dealing with γ_5: in other than four space–time dimensions we do not know how to define the Levi–Civita tensor, which means that for example for regularization purposes we cannot analytically continue our calculation to $n = 4 - 2\epsilon$ dimensions. The main property of γ_5 is equivalent to that fact that it is another basis element of our (4×4) matrices, it commutes with the other four Dirac matrices $[\gamma_\mu, \gamma_5] = 0$. Combining this new object as $(\gamma^\mu\gamma_5)$ and $i\gamma_5$ gives us all 16 basis element for the interaction of two spinors with a third scalar, vector, or tensor field:

	degrees of freedom	basis elements M_j
scalar	1	$\mathbb{1}$
vector	4	γ^μ
pseudoscalar	1	$i\gamma_5$
axialvector	4	$\gamma^\mu\gamma_5$
tensor	6	$\frac{i}{2}[\gamma^\mu, \gamma^\nu]$

The field indices need to contract with the indices of the object $\overline{\psi}M\psi$. Again, the factors i are conventional. In the Standard Model as a fundamental theory, tensor interactions do not play a major role. The reason is the dimensionality of the Lagrangian. The mass dimension of a fermion field ψ or $\overline{\psi}$ is $m^{3/2}$ while the mass dimension of a scalar field, a photon field, or a derivative is m. For example from Eq.(1.24) we see that every term in the QED Lagrangian is of mass dimension four. This is required for a renormalizable fundamental field theory. Introducing a tensor coupling we have to contract two indices, μ and ν, and not with the metric tensor. The only other objects coming to mind have mass dimension m^2, which means that together with the fermion fields the term in the Lagrangian has mass dimension of at least m^5 and is therefore not allowed.

Parity, CP, and Spin

An obvious question from the above discussion is: what does it mean to include a factor γ_5 in the interaction, *i.e.* what distinguishes a scalar from a pseudoscalar and a vector from an axialvector? We can give an easy answer by defining three transformations of our field in space and time. The first one is the parity transformation P which mirrors the three spatial coordinates $(t, \vec{x}) \to (t, -\vec{x})$. The second is charge conjugation C which converts particles into their anti–particles. Both of them leave the Dirac equation intact and can be represented by a unitary transformation. The third transformation is time reversal T which converts $(t, \vec{x}) \to (-t, \vec{x})$, also leaves the Dirac equation intact, but only has an anti–unitary representation. Every single one of them is violated in our Standard Model.

Instead of writing out the representation of these transformations in terms of Dirac matrices we characterize them using the basic interactions from Eq.(1.42). Parity symmetry does not allow any interaction including γ_5, which means it forbids pseudoscalars and axialvectors. Time reversal symmetry does not allow any complex couplings g_j. Because any field theory described by a Lagrangian not including some kind of external field is invariant under CPT, and we have never observed CPT violation, a combined CP-invariance is essentially the same as T invariance.

To look at the parity and CP symmetry more systematically, we rotate the $\{\mathbb{1}, \gamma_5\}$ plane and define the two (4×4)

matrix valued objects

$$\mathbb{P}_{R,L} = \frac{1}{2}\,(\mathbb{1} \pm \gamma_5)\ . \tag{1.44}$$

It is easy to show that the two are orthogonal projectors,

$$\begin{aligned}\mathbb{P}_L\mathbb{P}_R &= \frac{1}{4}\,(\mathbb{1}-\gamma_5)\,(\mathbb{1}+\gamma_5) = \frac{1}{4}\,\left(\mathbb{1}-\gamma_5^2\right) = 0 \qquad\qquad \text{using} \quad \gamma_5^2 = \mathbb{1} \\ \mathbb{P}_{R,L}^2 &= \frac{1}{4}\,\left(\mathbb{1} \pm 2\gamma_5 + \gamma_5^2\right) = \frac{1}{4}\,(2\mathbb{1} \pm 2\gamma_5) = \frac{1}{2}\,(\mathbb{1} \pm \gamma_5) = \mathbb{P}_{R,L}\ .\end{aligned} \tag{1.45}$$

We first look at what happens when we write a kinetic term with left-handed and right-handed projectors or fermion fields,

$$\begin{aligned}\overline{\psi}\,\partial\!\!\!/\psi &= \overline{\psi}\,\partial\!\!\!/\,\left(\mathbb{P}_L^2 + \mathbb{P}_R^2\right)\psi \\ &= \overline{\psi}\,\left(\mathbb{P}_R\partial\!\!\!/\mathbb{P}_L + \mathbb{P}_L\partial\!\!\!/\mathbb{P}_R\right)\psi \\ &= (\overline{\mathbb{P}_L\psi})\partial\!\!\!/(\mathbb{P}_L\psi) + (\overline{\mathbb{P}_R\psi})\partial\!\!\!/(\mathbb{P}_R\psi) \\ &= \overline{\psi}_L\,\partial\!\!\!/\psi_L + \overline{\psi}_R\,\partial\!\!\!/\psi_R\ .\end{aligned} \tag{1.46}$$

The general kinetic term covers left-handed and right-handed fields. Their effect on a mass term is different,

$$\begin{aligned}\overline{\psi}\,\mathbb{1}\psi &= \overline{\psi}\,\left(\mathbb{P}_L + \mathbb{P}_R\right)\psi \\ &= \overline{\psi}\,\left(\mathbb{P}_L^2 + \mathbb{P}_R^2\right)\psi \\ &= \psi^\dagger\gamma_0\,\left(\mathbb{P}_L^2 + \mathbb{P}_R^2\right)\psi && \text{with} \quad \overline{\psi} = \psi^\dagger\gamma^0 \\ &= \psi^\dagger\left(\mathbb{P}_R\gamma^0\mathbb{P}_L + \mathbb{P}_L\gamma^0\mathbb{P}_R\right)\psi && \text{with} \quad \{\gamma_5,\gamma_\mu\} = 0 \\ &= (\mathbb{P}_R\psi)^\dagger\gamma^0(\mathbb{P}_L\psi) + (\mathbb{P}_L\psi)^\dagger\gamma^0(\mathbb{P}_R\psi) && \text{with} \quad \gamma_5^\dagger = \gamma_5, \mathbb{P}_{L,R}^\dagger = \mathbb{P}_{L,R} \\ &= (\overline{\mathbb{P}_R\psi})\mathbb{1}(\mathbb{P}_L\psi) + (\overline{\mathbb{P}_L\psi})\mathbb{1}(\mathbb{P}_R\psi) \\ &= \overline{\psi}_R\,\mathbb{1}\psi_L + \overline{\psi}_L\,\mathbb{1}\psi_R\ .\end{aligned} \tag{1.47}$$

To include a fermion mass we need to combine left-handed and right-handed projectors and fermion fields,

$$\begin{aligned}\overline{\psi}\,\partial\!\!\!/\,\psi &= \overline{\psi}_R\,\partial\!\!\!/\,\psi_R + \overline{\psi}_L\,\partial\!\!\!/\,\psi_L \\ \overline{\psi}\,\mathbb{1}\,\psi &= \overline{\psi}_R\,\mathbb{1}\,\psi_L + \overline{\psi}_L\,\mathbb{1}\,\psi_R\ .\end{aligned} \tag{1.48}$$

In other words, we can write for example QED in terms of independent left and right handed fields as long as we neglect all fermion masses. This defines the chiral limit where the Lagrangian is symmetric under $\psi_L \leftrightarrow \psi_R$. Introducing fermion masses breaks this chiral symmetry, or turning the argument around, to introduce fermion masses we need to combine a left handed and a right handed fermion fields and give them one common Dirac mass.

Moving on to interactions, we define a combined vector–axialvector coupling as $\gamma^\mu \pm \gamma^\mu\gamma_5 = 2\gamma_\mu\mathbb{P}_{R,L}$. Sandwiching this coupling between fermion fields gives for example

$$\begin{aligned}\overline{\psi}\,\gamma_\mu\mathbb{P}_L\,\psi &= \overline{\psi}\,\gamma_\mu\mathbb{P}_L^2\,\psi \\ &= \psi^\dagger\mathbb{P}_L\,\gamma_0\,\gamma_\mu\,\mathbb{P}_L\psi && \text{with} \quad \{\gamma_5,\gamma_\mu\} = 0 \\ &= (\mathbb{P}_L\psi)^\dagger\,\gamma_0\,\gamma_\mu\,\mathbb{P}_L\psi && \text{with} \quad \gamma_5^\dagger = \gamma_5 \\ &= \overline{\psi}_L\,\gamma_\mu\,\psi_L && \text{with} \quad \psi_{L,R} \equiv \mathbb{P}_{L,R}\,\psi\ .\end{aligned} \tag{1.49}$$

If we call the eigenstates of $\mathbb{P}_{R,L}$ right handed and left handed fermions $\psi_{L,R}$ this chirality allows us to define a vector coupling between only left handed fermions by combining the vector and the axialvector couplings with a relative minus sign. The same is of course true for right handed couplings. We can now describe the $\gamma\bar{f}f$ coupling in QED using the Feynman rule

$$-i\gamma^\mu\,(\ell\mathbb{P}_L + r\mathbb{P}_R) \qquad\qquad \text{with} \quad \ell = r = Qe\ . \tag{1.50}$$

with $T_3 = \pm 1/2$.

At this stage it is not obvious at all what chirality means in physics terms. However, we will see that in the Standard Model the left handed fermions play a special role: the massive W bosons only couple to them and not to their right handed counter parts. So chirality is a property of fermions known to one gauge interaction of the Standard Model as part of the corresponding charge. The Higgs mechanism breaks it and only leaves the QCD–like gauge symmetry intact.

There exists a property which is identical to chirality for massless fermions and has an easy physical interpretation: the helicity. It is defined as the projection of the particle spin onto its three-momentum direction

$$h = \vec{s} \cdot \frac{\vec{p}}{|\vec{p}|} = \left(\vec{s} + \vec{L} \right) \cdot \frac{\vec{p}}{|\vec{p}|} = \vec{J} \cdot \frac{\vec{p}}{|\vec{p}|} \qquad \text{with} \quad \vec{p} \perp \vec{L} \; , \tag{1.51}$$

or equivalently the projection of the combined orbital angular momentum and the spin on the momentum direction. From quantum mechanics we know that there exist discrete eigenvalues for the z component of the angular momentum operator, symmetric around zero. Applied to fermions this gives us two spin states with the eigenvalues of h being $\pm 1/2$. Unfortunately, there is no really nice way to show this identity. What we need to know is that the spin operator is in general given by

$$\vec{s} = \gamma_5 \gamma^0 \, \vec{\gamma} \, . \tag{1.52}$$

We can show this by writing it out in terms of Pauli matrices, but we will skip this here and instead just accept this general form. We then write the solution ψ to the massless Dirac equation after transforming it into momentum space $\psi(\vec{x}) = u(\vec{p}) \exp(-ip \cdot x)$

$$\begin{aligned}
\left(\gamma^0 p_0 - \vec{\gamma} \vec{p} \right) \; u(\vec{p}) &= 0 \\
\gamma_5 \gamma^0 \; \gamma^0 p_0 \; u(\vec{p}) &= \gamma_5 \gamma^0 \; \vec{\gamma} \vec{p} \, u(\vec{p}) \\
\gamma_5 p_0 \; u(\vec{p}) &= \vec{s} \cdot \vec{p} \, u(\vec{p}) \qquad \text{with} \;\; \left(\gamma^0 \right)^2 = \mathbb{1} \\
\gamma_5 \; u(\vec{p}) &= \frac{\vec{s} \cdot \vec{p}}{p_0} \; u(\vec{p}) \\
\gamma_5 \; u(\vec{p}) &= \pm \frac{\vec{s} \cdot \vec{p}}{|\vec{p}|} \; u(\vec{p}) = \pm h \; u(\vec{p}) \; .
\end{aligned} \tag{1.53}$$

In other words, the chirality operator γ_5 indeed gives us the helicity of the particle state, modulo a sign depending on the sign of the energy. For the helicity it is easy to argue why for massive particles this property is not Lorentz invariant and hence not a well defined property: massless particles propagate with the speed of light, which means we can never boost into their rest frame or pass them. For massive particles we can do that and this way switch the sign of $\vec{p}$ and the sign of h. Luckily, for almost all Standard Model fermions we can at the LHC neglect their masses.

Cross section measurements

To compute a $2 \to 2$ scattering rate we combine the scattering matrix element from Eq.(1.38) with a two-particle phase space integration for massless particles,

$$s^2 \frac{d\sigma}{dt} \bigg|_{2 \to 2} = \frac{\pi}{(4\pi)^2} \; K_{ij} \; |\mathcal{M}|^2 \tag{1.54}$$

with an averaging factor K_{ij} for initial–state spins and colors, as only the sum is included in Eq.(1.38). For incoming electrons as well as incoming quarks this factor K_{ij} includes $1/4$ for the spins. For an incoming $q\bar{q}$ pair we would also average over the color, $1/N_c^2$.

For our QED process we then find the differential cross section in four space–time dimensions, using $\alpha = e^2/(4\pi)$

$$\begin{aligned}\frac{d\sigma}{dt} &= \frac{1}{s^2}\frac{\pi}{(4\pi)^2}\,\frac{1}{4}\;8\;Q_e^2 Q_q^2 (4\pi\alpha)^2\;N_c\;\left[1+2\frac{t}{s}+2\frac{t^2}{s^2}\right]\\ &= \frac{1}{s^2}2\pi\alpha^2\;N_c\;Q_e Q_q^2\;\left[1+2\frac{t}{s}+2\frac{t^2}{s^2}\right]\;. \end{aligned} \tag{1.55}$$

We integrate this expression over the polar angle or the Mandelstam variable t to compute the total cross section

$$\begin{aligned}\sigma &= \frac{1}{s^2}2\pi\alpha^2\;N_c\;Q_e^2 Q_q^2 \int_{-s}^{0} dt\;\left[1+2\frac{t}{s}+2\frac{t^2}{s^2}\right]\\ &= \frac{1}{s^2}2\pi\alpha^2\;N_c\;Q_e^2 Q_q^2 \left[t+\frac{t^2}{s}+\frac{2t^3}{3s^2}\right]_{-s}^{0}\\ &= \frac{1}{s^2}2\pi\alpha^2\;N_c\;Q_e^2 Q_q^2 \left[s-\frac{s^2}{s}+\frac{2s^3}{3s^2}\right]\\ &= \frac{1}{s}2\pi\alpha^2\;N_c\;Q_e^2 Q_q^2\;\frac{2}{3} \qquad\qquad \Rightarrow \qquad\qquad \sigma(e^+e^-\to q\bar{q})\Bigg|_{\text{QED}} = \frac{4\pi\alpha^2 N_c}{3s}\;Q_e^2 Q_q^2\;. \end{aligned} \tag{1.56}$$

In the history of QCD, this process played a crucial role, namely the production rate of quarks in e^+e^- scattering. For small enough energies we can neglect the Z exchange contribution. At leading order we can then compute the corresponding production cross sections for muon pairs and for quark pairs in e^+e^- collisions.

$$R \equiv \frac{\sigma(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\ell^+\ell^-)} = \frac{\sum_{\text{quarks}}\dfrac{4\pi\alpha^2 N_c}{3s}\;Q_e^2 Q_q^2}{\dfrac{4\pi\alpha^2}{3s}\;Q_e^2 Q_\ell^2} = N_c\left(3\frac{1}{9}+2\frac{4}{9}\right) = \frac{11N_c}{9}\;, \tag{1.57}$$

for example for five quark flavors where the top quark is too heavy to be produced at the given e^+e^- collider energy. For those interested in the details we did take one short cut: hadrons are also produced in the hadronic decays of $e^+e^-\to\tau^+\tau^-$ which we strictly speaking need to subtract. This way, R as a function of the collider energy is a beautiful measurement of the weak and color charges of the quarks in QCD.

Finally, if we face the fact that most particle physicists nowadays work on precision hadron colliders, and high-energy e^+e^--colliders are either a thing of the past or a dream for the future, we want to compute our QED process the other way around. This means we move the quarks into the initial state and include a color-averaging factor. The corresponding process is called the Drell–Yan process

$$\sigma(q\bar{q}\to\ell^+\ell^-)\Bigg|_{\text{QED}} = \frac{4\pi\alpha^2}{3N_c s}\;Q_\ell^2 Q_q^2\;. \tag{1.58}$$

It will be the process which guides us through the discussion of modern collider physics. Obviously, to describe lepton pair production at the LHC, we need to include the massive electroweak gauge bosons, not just the photons. We will get to that later.

2 Extra: modern helicity amplitudes

When we simulate LHC events we do not actually rely on the approach usually described in text books. This is most obvious when it comes to the computation of a transition matrix elements in modern LHC Monte Carlo tools, which you will not even recognize when looking at the codes. In Section **??** we compute the cross section for Z production by writing down all external spinors, external polarization vectors, interaction vertices and propagators and squaring the amplitude analytically. The amplitude itself inherits external indices for example from the polarization vectors, while $|\mathcal{M}|^2$ is a real positive number with a fixed mass dimension depending on the number of external particles.

For the LHC nobody calculates gamma matrix traces by hand anymore. Instead, we use powerful tools like FORM to compute traces of Dirac matrices in the calculation of $|\mathcal{M}|^2$. Nevertheless, a major problem with squaring Feynman diagrams and computing polarization sums and gamma matrix traces is that once we include more than three particles in the final state, the number of terms appearing in $|\mathcal{M}|^2$ soon becomes very large. Moreover, this approach requires symbolic computer manipulation instead of pure numerics. In this section we illustrate how we can transform the computation of $|\mathcal{M}|^2$ at the tree level into a purely numerical problem.

As an example, we consider our usual toy process

$$u\bar{u} \to \gamma^* \to \mu^+\mu^- \ . \tag{2.1}$$

The structure of the amplitude $\mathcal{M}$ with two internal Dirac indices μ and ν involves one vector current on each side $(\bar{u}_f \gamma_\mu u_f)$ where $f = u, \mu$ are to good approximation massless, so we do not have to be careful with the different spinors u and v. The entries in the external spinors are given by the spin of the massless fermions obeying the Dirac equation. For each value of $\mu = 0 \cdots 3$ each current is a complex number, computed from the four component of each spinor and the respective 4×4 gamma matrix γ^μ. The intermediate photon propagator has the form $g_{\mu\nu}/s$, which is a real number for each value of $\mu = \nu$. Summing over μ and ν in both currents forms the matrix element. To square this matrix element we need to sum $\mathcal{M}^* \times \mathcal{M}$ over all possible spin directions of the external fermions.

Instead of squaring this amplitude symbolically we can follow exactly the steps described above and compute an array of numbers for different spin and helicity combinations numerically. Summing over the internal Dirac indices we compute the matrix element; however, to compute the matrix element squared we need to sum over external fermion spin directions or gauge boson polarizations. The helicity basis we have to specify externally. This is why this method is called helicity amplitude approach. To explain the way this method works, we illustrate it for muon pair production based on the implementation in the Madgraph/Helas package.

Madgraph is a tool to compute matrix elements this way. Other event generators have corresponding codes serving the same purposes. In our case, Madgraph5 automatically produces a Fortran routine which then calls functions to compute spinors, polarization vectors, currents of all kinds, etc. These functions are available as the so-called Helas library. For our toy process Eq.(2.1) the slightly shortened Madgraph5 output reads

```
      REAL*8 FUNCTION MATRIX1(P,NHEL,IC)
C
C     Generated by Madgraph 5
C
C     Returns amplitude squared summed/avg over colors
C     for the point with external lines W(0:6,NEXTERNAL)
C
C     Process: u u~ > mu+ mu- / z WEIGHTED=4 @1
C
      INTEGER    NGRAPHS, NWAVEFUNCS, NCOLOR
      PARAMETER (NGRAPHS=1, NWAVEFUNCS=5, NCOLOR=1)

      REAL*8 P(0:3,NEXTERNAL)
      INTEGER NHEL(NEXTERNAL), IC(NEXTERNAL)

      INCLUDE 'coupl.inc'

      DATA DENOM(1)/1/
      DATA (CF(I,  1),I=  1,  1) /    3/

      CALL IXXXXX(P(0,1),ZERO,NHEL(1),+1*IC(1),W(1,1))
      CALL OXXXXX(P(0,2),ZERO,NHEL(2),-1*IC(2),W(1,2))
      CALL IXXXXX(P(0,3),ZERO,NHEL(3),-1*IC(3),W(1,3))
      CALL OXXXXX(P(0,4),ZERO,NHEL(4),+1*IC(4),W(1,4))
      CALL FFV1_3(W(1,1),W(1,2),GC_2,ZERO, ZERO, W(1,5))
      CALL FFV1_0(W(1,3),W(1,4),W(1,5),GC_3,AMP(1))
      JAMP(1)=+AMP(1)

      DO I = 1, NCOLOR
        DO J = 1, NCOLOR
          ZTEMP = ZTEMP + CF(J,I)*JAMP(J)
        ENDDO
```

```
  MATRIX1 = MATRIX1 + ZTEMP*DCONJG(JAMP(I))/DENOM(I)
ENDDO

END
```

The input to this function are the external four-momenta $p(0:3,1:4)$ and the helicities of all fermions $n_\text{hel}(1:4)$ in the process. Remember that helicity and chirality are identical only for massless fermions because chirality is defined as the eigenvalue of the projectors $(\mathbb{1} \pm \gamma_5)/2$, while helicity is defined as the projection of the spin onto the momentum direction, *i.e.* as the left or right handedness. We give the exact definition of these two properties in Section 1. The entries of n_hel will be either $+1$ or -1. For each point in phase space and each helicity combination the Madgraph subroutine `MATRIX1` computes the matrix element using standard Helas routines.

- `IXXXXX`$(p, m, n_\text{hel}, n_\text{sf}, F)$ computes the wave function of a fermion with incoming fermion number, so either an incoming fermion or an outgoing anti–fermion. As input it requires the four-momentum, the mass and the helicity of this fermion. Moreover, $n_\text{sf} = +1$ marks the incoming fermion u and $n_\text{sf} = -1$ the outgoing anti–fermion μ^+, because by convention Madgraph defines its particles as u and μ^-.

  The fermion wave function output is a complex array $F(1:6)$. Its first two entries are the left–chiral part of the fermionic spinor, *i.e.* $F(1:2) = (\mathbb{1} - \gamma_5)/2\ u$ or $F(1:2) = (\mathbb{1} - \gamma_5)/2\ v$ for $n_\text{sf} = \pm 1$. The entries $F(3:4)$ are the right–chiral spinor. These four numbers can directly be computed from the four-momentum if we know the helicity. The four entries correspond to the size of one γ matrix, so we can compute the trace of the chain of gamma matrices. Because for massless particles helicity and chirality are identical, our quarks and leptons will only have finite entries $F(1:2)$ for $n_\text{hel} = -1$ and $F(3:4)$ for $n_\text{hel} = +1$.

  The last two entries of F contain the four-momentum in the direction of the fermion flow, namely $F(5) = n_\text{sf}(p(0) + ip(3))$ and $F(6) = n_\text{sf}(p(1) + ip(2))$.

- `OXXXXX`$(p, m, n_\text{hel}, n_\text{sf}, F)$ does the same for a fermion with outgoing fermion flow, *i.e.* our incoming $\bar{u}$ and our outgoing μ^-. The left–chiral and right–chiral components now read $F(1:2) = \bar{u}(\mathbb{1} - \gamma_5)/2$ and $F(3:4) = \bar{u}(\mathbb{1} + \gamma_5)/2$, and similarly for the spinor $\bar{v}$. The last two entries are $F(5) = n_\text{sf}(p(0) + ip(3))$ and $F(6) = n_\text{sf}(p(1) + ip(2))$.

- `FFV1_3`$(F_i, F_o, g, m, \Gamma, J_{io})$ computes the (off–shell) current for the vector boson attached to the two external fermions F_i and F_o. The coupling $g(1:2)$ is a complex array with the interaction of the left–chiral and right–chiral fermion in the upper and lower index. For a general Breit–Wigner propagator we need to know the mass m and the width Γ of the intermediate vector boson. The output array J_{io} again has six components which for the photon with momentum q are

$$\begin{aligned} J_{io}(\mu+1) &= -\frac{i}{q^2}\ F_o^T\ \gamma^\mu \left(g(1)\ \frac{\mathbb{1} - \gamma_5}{2}\ + g(2)\ \frac{\mathbb{1} + \gamma_5}{2} \right)\ F_i \qquad \mu = 0,1,2,3 \\ J_{io}(5) &= -F_i(5) + F_o(5) \sim -p_i(0) + p_o(0) + i\left(-p_i(3) - p_o(3)\right) \\ J_{io}(6) &= -F_i(6) + F_o(6) \sim -p_i(1) + p_o(1) + i\left(-p_i(2) + p_o(2)\right)\ . \end{aligned} \tag{2.2}$$

  The first four entries in J_{io} correspond to the index μ or the dimensionality of the Dirac matrices in this vector current. The spinor index is contracted between F_o^T and F_i.

  As two more arguments J_{io} includes the four-momentum flowing through the gauge boson propagator. They allow us to reconstruct q^μ from the last two entries

$$q^\mu = (\text{Re}J_{io}(5), \text{Re}J_{io}(6), \text{Im}J_{io}(6), \text{Im}J_{io}(5))\ . \tag{2.3}$$

- `FFV1_0`(F_i, F_o, J, g, V) computes the amplitude of a fermion–fermion–vector coupling using the two external fermionic spinors F_i and F_o and an incoming vector current J which in our case comes from `FFV1_3`. Again, the coupling $g(1:2)$ is a complex array, so we numerically compute

$$F_o^T\ \not{J} \left(g(1)\ \frac{\mathbb{1} - \gamma_5}{2}\ + g(2)\ \frac{\mathbb{1} + \gamma_5}{2} \right)\ F_i\ . \tag{2.4}$$

All spinor and Dirac indices of the three input arguments are contracted in the final result. Momentum conservation is not enforced by FFV1_0, so we have to take care of it by hand.

Given the list above it is easy to follow how Madgraph computes the amplitude for $u\bar{u} \to \gamma^* \to \mu^+\mu^-$. First, it calls wave functions for all external particles and puts them into the array $W(1:6,1:4)$. The vectors $W(*,1)$ and $W(*,3)$ correspond to $F_i(u)$ and $F_i(\mu^+)$, while $W(*,2)$ and $W(*,4)$ mean $F_o(\bar{u}$ and $F_o(\mu^-)$.

The first vertex we evaluate is the incoming quark–photon vertex. Given the wave functions $F_i = W(*,1)$ and $F_o = W(*,2)$ FFV1_3 computes the vector current for the massless photon in the s-channel. Not much changes if we instead choose a massive Z boson, except for the arguments m and Γ in the FFV1_3 call. Its output is the photon current $J_{io} \equiv W(*,5)$.

The last step combines this current with the two outgoing muons coupling to the photon. Since this number gives the final amplitude, it should return a complex number, not an array. Madgraph calls FFV1_0 with $F_i = W(*,3)$ and $F_o = W(*,4)$, combined with the photon current $J = W(*,5)$. The result AMP is copied into JAMP without an additional sign which could have come from the relative ordering of external fermions in different Feynman diagrams contributing to the same process.

The only remaining sum left to compute before we square JAMP is the color structure, which in our simple case means one color structure with a color factor $N_c = 3$.

As an added bonus Madgraph produces a file with all Feynman diagrams in which the numbering of the external particles corresponds to the second argument of W and the numbering of the Feynman diagrams corresponds to the argument of AMP. This helps us identify intermediate results W, each of which is only computed once, even if is appears several times in the different Feynman diagrams.

As mentioned above, to calculate the transition amplitude Madgraph requires all masses and couplings. They are transferred through common blocks in the file coupl.inc and computed elsewhere. In general, Madgraph uses unitary gauge for all vector bosons, because in the helicity amplitude approach it is easy to accommodate complicated tensors, in exchange for a large number of Feynman diagrams.

The function MATRIX1 described above is not yet the full story. When we square $\mathcal{M}$ symbolically we need to sum over the spins of the outgoing states to transform a spinor product of the kind $u\bar{u}$ into the residue or numerator of a fermion propagator. To obtain the full transition amplitude numerically we correspondingly sum over all helicity combinations of the external fermions, in our case $2^4 = 16$ combinations.

```
      SUBROUTINE SMATRIX1(P,ANS)
C
C     Generated by Madgraph 5
C
C     Returns amplitude squared summed/avg over colors
C     and helicities for the point in phase space P(0:3,NEXTERNAL)
C
C     Process: u u~ > mu+ mu- / z
C
      INTEGER     NCOMB, NGRAPHS, NDIAGS, THEL
      PARAMETER (NCOMB=16, NGRAPHS=1, NDIAGS=1, THEL=2*NCOMB)

      REAL*8 P(0:3,NEXTERNAL)

      INTEGER I,J,IDEN
      INTEGER NHEL(NEXTERNAL,NCOMB),NTRY(2),ISHEL(2),JHEL(2)
      INTEGER JC(NEXTERNAL),NGOOD(2), IGOOD(NCOMB,2)
      REAL*8 T,MATRIX1
      LOGICAL GOODHEL(NCOMB,2)

      DATA NGOOD /0,0/
      DATA ISHEL/0,0/
      DATA GOODHEL/THEL*.FALSE./

      DATA (NHEL(I,   1),I=1,4) /-1,-1,-1,-1/
      DATA (NHEL(I,   2),I=1,4) /-1,-1,-1, 1/
      DATA (NHEL(I,   3),I=1,4) /-1,-1, 1,-1/
      DATA (NHEL(I,   4),I=1,4) /-1,-1, 1, 1/
      DATA (NHEL(I,   5),I=1,4) /-1, 1,-1,-1/
```

```
DATA (NHEL(I,   6),I=1,4) /-1, 1,-1, 1/
DATA (NHEL(I,   7),I=1,4) /-1, 1, 1,-1/
DATA (NHEL(I,   8),I=1,4) /-1, 1, 1, 1/
DATA (NHEL(I,   9),I=1,4) / 1,-1,-1,-1/
DATA (NHEL(I,  10),I=1,4) / 1,-1,-1, 1/
DATA (NHEL(I,  11),I=1,4) / 1,-1, 1,-1/
DATA (NHEL(I,  12),I=1,4) / 1,-1, 1, 1/
DATA (NHEL(I,  13),I=1,4) / 1, 1,-1,-1/
DATA (NHEL(I,  14),I=1,4) / 1, 1,-1, 1/
DATA (NHEL(I,  15),I=1,4) / 1, 1, 1,-1/
DATA (NHEL(I,  16),I=1,4) / 1, 1, 1, 1/
DATA IDEN/36/

DO I=1,NEXTERNAL
  JC(I) = +1
ENDDO

DO I=1,NCOMB
  IF (GOODHEL(I,IMIRROR) .OR. NTRY(IMIRROR).LE.MAXTRIES) THEN
    T = MATRIX1(P ,NHEL(1,I),JC(1))
    ANS = ANS+T
  ENDIF
ENDDO

ANS = ANS/DBLE(IDEN)
END
```

The important part of this subroutine is the list of possible helicity combinations stored in the array $n_{\text{hel}}(1:4,1:16)$. Adding all different helicity combinations means a loop over the second argument and a call of `MATRIX1` with the respective helicity combination. Because of the naive helicity combinations many are not allowed the array `GOODHEL` keeps track of valid combinations. After an initialization to all 'false' this array is only switched to 'true' if `MATRIX1` returns a finite value, otherwise Madgraph does not waste time to compute the matrix element. At the very end, a complete spin–color averaging factor is included as `IDEN` and in our case given by $2 \times 2 \times N_c^2 = 36$.

Altogether, Madgraph provides us with the subroutine `SMATRIX1` and the function `MATRIX1` which together compute $\overline{|\mathcal{M}|^2}$ for each phase space point given as an external momentum configuration. This helicity method might not seem particularly appealing for a simple $(2 \to 2)$ process, but it makes it possible to compute processes with many particles in the final state and typically up to 10000 Feynman diagrams which we could never square symbolically, no matter how many graduate students' live times we throw in.