4 Quantum Chromodynamics

After introducing QED as a $U(1)$-gauge interaction and the weak force as an $SU(2)$-gauge interaction, the step to the strong interaction or QCD with based on $SU(3)$ gauge invariance is structurally simple. In particular, we assume, in agreement with all existing measurements, that the gluons as the QCD gauge bosons as massless and that their interaction is not affected by the weak quantum numbers or the fermion doublet structure. This means we can build QCD as a non-abelian version of QED. We will see that QCD, in spite of the structural similarity to QED, has especially interesting properties and equally interesting experimental consequences.

4.1 QCD Lagrangian

We start with the $SU(3)$-version of the same argument we made for $SU(2)$ from Eq.(3.17) on. The $SU(3)$ transformations are given by

$$U = e^{i\alpha_a T_a} \qquad \text{with} \qquad T_{1,2,3} = \frac{1}{2} \begin{pmatrix} \tau_{1,2,3} & 0 \\ 0 & 0 \end{pmatrix} \cdots \tag{4.1}$$

with $a = 1 \,...\, N_c^2 - 1 = 1 \,...\, 8$ and the number of colors in the fundamental representation, $N_c = 3$. Please note the conventional factor $1/2$ relative to the Pauli matrices. The (3×3) matrices T_a are the traceless, hermitian, and unitary Gell-Mann matrices. We give the first three of them in terms of the Pauli matrices, but there is no point in writing them down because we only need their algebraic properties to compute the color factors of scattering amplitudes,

$$[T_a, T_b] = i f_{abc} T_c \qquad \text{and} \qquad \text{Tr}(T_a T_b) = T_R \delta_{ab} \equiv \frac{1}{2} \delta_{ab} \; . \tag{4.2}$$

Here, f_{abc} are the antisymmetric structure constants of $SU(3)$ with

$$f_{acd} f_{bcd} = N_c \delta_{ab} \; . \tag{4.3}$$

The one formula we need to compute color factors for quark processes will be

$$(T_a)_{ij} (T_a)_{k\ell} = \frac{1}{2} \left(\delta_{i\ell} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{k\ell} \right) \; . \tag{4.4}$$

The QCD Lagrangian can be constructed as a non-abelian massless version of the QED Lagrangian. For the dimension-4 Lagrangian we follow Eq.(3.23), limit ourselves to the quarks as fermions charged under $SU(3)$, and write

$$\mathscr{L}_\text{QCD} = \bar{q}_i i \not{D} q_j - \frac{1}{4} F_{a,\mu\nu} F_a^{\mu\nu} \; , \tag{4.5}$$

with the gluon field strength tensor

$$F_{a,\mu\nu} = \partial_\mu A_{a,\nu} - \partial_\nu A_{a,\mu} - i g_s [A_\mu, A_\nu]_a \; , \tag{4.6}$$

and the $SU(3)$-covariant derivative with the appropriate color indices

$$(D_\mu)_{ij} = \partial_\mu \mathbb{1}_{ij} - i g_s A_{\mu,a} (T_a)_{ij} \tag{4.7}$$

From these formulas we see immediately that the quark propagators are the same as for QED, and unlike for the weak interaction we do not have to keep track of the chirality. The quark-quark-gluon interaction is very similar to the quark-quark-photon vertex in Eq.(1.40),

$$-i g_s \, \gamma^\mu \, (T_a)_{ij} \qquad\qquad (q_i - \bar{q}_j - g_{a,\mu}). \tag{4.8}$$

The only additional element is the $SU(3)$-matrix, and we already see that all color matrices in a scattering process will form traces along the quark line(s). This means that color factors factorize from the Dirac algebra and can be computed independently.

Let us start with the electroweak processes

$$e^+e^- \to q\bar{q} \qquad\qquad \text{and} \qquad\qquad q\bar{q} \to e^+e^- \tag{4.9}$$

Here no gluons appear at leading order, which means we do not have any matrices T^a in our calculations. Nevertheless, a color factor appears because we sum over the (identical) colors of the two external quarks. We can write this color sum for the squared matrix element formally as

$$\delta_{ij}\delta_{ji} = \delta_{ii} = N_c \tag{4.10}$$

The different between the process $e^+e^- \to q\bar{q}$ and $q\bar{q} \to e^+e^-$ is that for the former we sum over the color states in the final state and for the latter we average over the color states in the initial state, giving us another factor $1/N_c$. Let us now radiate a gluon from any of these processes, for instance

$$e^+e^- \to q\bar{q}g \qquad\qquad \text{and} \qquad\qquad q\bar{q} \to e^+e^-g \tag{4.11}$$

First, we have to deal with another gluon in the Dirac traces, just like when radiating a photon. In addition, there is the color contribution in Eq.(4.8),

$$(T_a)_{ij}(T_a)_{ji} = \text{Tr}(T_aT_a) = \frac{1}{2}\delta_{aa} = \frac{N_c^2-1}{2} \; . \tag{4.12}$$

It can be expressed in terms of the fundamental Casimir as

$$\text{Tr}(T_aT_a) = N_cC_F \qquad \text{with} \qquad C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3} \; , \tag{4.13}$$

Another example is the successive radiation of two gluons from a hard quark,

$$q \to qg_ag_b \; . \tag{4.14}$$

Depending on the order of the two gluons, we first find the planar color factor,

$$\begin{aligned}
\text{Tr}(T^aT^aT^bT^b) &= (T^aT^a)_{il}(T^bT^b)_{li} \\
&= \frac{1}{4}\left(\delta_{il}\delta_{jj} - \frac{\delta_{ij}\delta_{jl}}{N_c}\right)\left(\delta_{il}\delta_{jj} - \frac{\delta_{ij}\delta_{jl}}{N_c}\right) \\
&= \frac{1}{4}\left(\delta_{il}N_c - \frac{\delta_{il}}{N_c}\right)\left(\delta_{il}N_c - \frac{\delta_{il}}{N_c}\right) \\
&= N_c\,\left(\frac{N_c^2-1}{2N_c}\right)^2 = N_cC_F^2 = \frac{16}{3} = \mathcal{O}(N_c^3) \; .
\end{aligned} \tag{4.15}$$

When we cross the gluon lines between the diagram and its complex conjugate we get the same way

$$\text{Tr}(T_aT_bT_aT_b) = -\frac{C_F}{2} = -\frac{2}{3} = \mathcal{O}(N_c) \; . \tag{4.16}$$

That contribution is suppressed by a factor eight, which means that two gluons have a significant preference for ordered emission.

We can also compute the color factor for the purely gluonic theory, *i.e.* radiating gluons off two hard gluons in the final state. For instance, planar double gluon emission with the exchanged gluon indices b and f gives us the largest color factor

$$f^{abd}f^{abe}f^{dfg}f^{efg} = N_c\delta^{de}\,N_c\delta^{de} = \mathcal{O}(N_c^3) \; , \tag{4.17}$$

now independent of the ordering.

4.2 Ghosts

The main complication of QCD compared to QED comes from the kinetic term of the gluons, where we keep in mind that for non-abelian gauge groups the term $F_{a,\mu\nu}F_a^{\mu\nu}$ gives rise to the gauge boson propagator and to the self-interactions. We immediately see this when we insert the formula in Eq.(4.6) into the Lagrangian of Eq.(4.5) and encounter up to four powers of the gluon field, and no derivative.

Let us go back to QED, where in Sec. 3.2 we emphasize that photons have only two transverse degree of freedom and then still write the gluon field in terms of a 4-vector A_μ, effectively consisting of four degrees of freedom. How does QED ensure that in the actual calculation the longitudinal and the scalar degrees of freedom do not contribute?

The way to tackle this question is through the gauges introduced in Eq.(3.54), simplified in the massless limit to

$$\Delta^{\mu\nu}(p) = \frac{-i}{p^2+i\epsilon}\left[g^{\mu\nu} + (\xi-1)\frac{p^\mu p^\nu}{p^2}\right] = \begin{cases} \dfrac{-i}{p^2+i\epsilon}\, g^{\mu\nu} & \text{Feynman gauge } \xi = 1 \\ \dfrac{-i}{p^2+i\epsilon}\left[g^{\mu\nu} - \dfrac{p^\mu p^\nu}{p^2}\right] & \text{Landau/Lorenz gauge } \xi = 0 \;. \end{cases} \tag{4.18}$$

Here we need to be careful with the word ‘gauge’, because the different forms of the propagator have nothing to do with the gauge symmetry of the Lagrangian. The unitary gauge makes no sense for massless particles. For the weak bosons the unitary gauge was the only way to decouple and get right of the Goldstone modes.

To learn how to remove the unwanted degrees of freedom we write the photon Lagrangian from Eq.(3.1) such that it gives us the photon propagator from Eq.(4.18) in the R_ξ gauge. We only quote the result as

$$\mathscr{L}_\text{photon, gf} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \;. \tag{4.19}$$

In analogy to Eq.(1.7) this Lagrangian gives us the equation of motion

$$\partial^\mu\partial_\mu A_\nu - \left(1+\frac{1}{\xi}\right)\partial^\mu\partial_\nu A_\mu = 0 \;. \tag{4.20}$$

It is equivalent to the version without the gauge fixing term, in that it requires the d’Alembert equation for the photon and the Lorenz gauge condition.

From the massive photon example we know how to turn Eq.(4.19) gauge-invariant. From the weak gauge bosons we also know how to switch from one gluon propagator to another — again we need to introduce another field. For the massless gauge bosons we will refer to these new fields as ghosts. Let us start with the QED Lagrangian in Eq.(4.19), including the gauge fixing term corresponding to the general photon propagator. Because we know from Eq.(3.2) that $F_{\mu\nu}$ is $U(1)$-gauge invariant, we also know that the Lagrangian in R_ξ gauge is not locally $U(1)$-symmetric. From Eq.(3.2) we remember the gauge transformation of the photon as

$$\begin{aligned} A^\mu &\to A^\mu - \frac{1}{e}\partial^\mu\alpha \\ -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 &\to -\frac{1}{2\xi}\left(\partial_\mu A^\mu - \frac{1}{e}\partial^2\alpha\right)^2 \approx -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{e\xi}(\partial_\mu A^\mu)(\partial^2\alpha) \;. \end{aligned} \tag{4.21}$$

In the last step we ignore higher powers of α, because we are working with infinitesimal gauge transformations. To turn this Lagrangian gauge invariant we add an auxiliary field $\bar{c}$ such that

$$\mathscr{L}_\text{photon-ghost} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{c}\,\partial^2\alpha \;, \tag{4.22}$$

We see that the combination of the gauge fixing term and this new term is gauge-invariant if

$$\bar{c} \to \bar{c} - \frac{1}{e\xi}(\partial_\mu A^\mu) \;. \tag{4.23}$$

To give a meaning to this auxiliary term we move one of the derivatives through an integration by parts. In this form the additional term in the Lagrangian makes sense if we upgrade α to another field, $\alpha \to c$, such that the auxiliary $\bar{c}$ and the upgraded c form a complex scalar,

$$\begin{aligned}\mathscr{L}_{\text{photon-ghost}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + (\partial_\mu \bar{c})\,(\partial^\mu \alpha) \\ &\to -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + (\partial_\mu \bar{c})\,(\partial^\mu c)\,. \end{aligned} \tag{4.24}$$

In Eq.(3.11) we have seen something similar, namely a scalar field added to the QED Lagrangian to make sure the massive degrees of freedom are correctly described. Because the unitary gauge is not defined in Eq.(4.18), there is no gauge choice for which we can decouple and neglect the ghosts, they have to be computed in the Lorenz and in the Feynman gauges. What saves us in QED is that they do not appear elsewhere in the Lagrangian, so they are propagating but non-interacting fields. For our QED calculations this means that they are irrelevant and we can ignore them.

The situation changes because of the non-abelian structure of QCD, which replaces the standard derivative with a covariant derivative in the gauge transformation and also in the kinetic term for the ghosts. The gauge-fixed Lagrangian with the compensating ghost fields then reads

$$\mathscr{L}_{\text{gluon-ghost}} = -\frac{1}{4}F_{a,\mu\nu}F_a^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A_a^\mu)^2 - (\partial^\mu \bar{c}_a)(\partial_\mu c_a) + g_s f_{abc} A_a^\mu (\partial_\mu \bar{c}_b) c_c\,. \tag{4.25}$$

The gluon propagator in the R_ξ gauge is the same as the photon propagator in Eq.(4.18), just with a factor δ_{ab} in the numerator. We are skipping the crucial triple and quartic gluon self-interactions, because they are lengthy. However, we can write down the ghost Feynman rules, including the ghost propagator following from Eq.(4.22)

$$-\frac{\delta_{ab}}{p^2 + i\epsilon} \tag{4.26}$$

and the ghost-ghost-gluon interaction

$$-ig f_{abc} p_\mu \qquad\qquad (c - \bar{c} - g)\,, \tag{4.27}$$

where p_μ is the 4-momentum of the incoming ghost field.

Because we derived the ghosts through gauge invariance, we can at least for external gluons follow a slightly different direction to ensure the correct degrees of freedom contribute to the matrix element. For a matrix element with two external gluons $\mathcal{M}^{\mu\nu}$ the explicit condition is

$$p_\mu \mathcal{M}^{\mu\nu} = 0 = p_\nu \mathcal{M}^{\mu\nu}\,, \tag{4.28}$$

and we can enforce it by projecting $\mathcal{M}^{\mu\mu}$ onto the allowed tensor structures. Following the same line of thought for propagators leads us back to Eq.(4.18), where we find for the so-called transverse tensor

$$T^{\mu\nu} = g^{\mu\nu} - \frac{p^\nu p^\mu}{p^2} \qquad \Rightarrow \qquad p_\mu T^{\mu\nu} = 0 = p_\nu T^{\mu\nu}\,. \tag{4.29}$$

It is a projector on the transverse degrees of freedom because

$$\begin{aligned} T^{\mu\nu} T_\nu^\rho &= \left(g^{\mu\nu} - \frac{p^\nu p^\mu}{p^2} \right) \left(g_\nu^\rho - \frac{p_\nu p^\rho}{p^2} \right) \\ &= g^{\mu\rho} - \frac{p^\mu p^\rho}{p^2} - \frac{p^\rho p^\mu}{p^2} + \frac{p^2 p^\mu p^\rho}{p^4} = T^{\mu\rho}\,. \end{aligned} \tag{4.30}$$

This means that the gauge propagator in Lorenz gauge is guaranteed to be transverse in the covariant sense. The problem with this argument is that propagators define individual Feynman diagrams, and gauge invariance only holds for all Feynman diagrams combined. This means that for internal gluons, including loop integrals, even the Lorenz gauge does not ensure the correct gluon polarization and we always have to include the ghosts explicitly.

4.3 Ultraviolet divergences

Renormalization as the proper treatment of ultraviolet divergences is one of the most important things to understand about quantum field theory. It is driven by the appearance of ultraviolet divergences, which we first regularize, *i.e.* describe in a well-defined manner, and then renormalize away through counter terms. This renormalization leads to the appearance of the renormalization scale.

Scales automatically arise from infrared or ultraviolet divergences. We can see this by writing down a simple scalar loop integral, with to two virtual scalar propagators with masses $m_{1,2}$ and an external momentum p flowing through a diagram,

$$B(p^2; m_1, m_2) \equiv \int \frac{d^4 q}{16\pi^2} \, \frac{1}{q^2 - m_1^2} \frac{1}{(q+p)^2 - m_2^2} \; . \tag{4.31}$$

Such two-point functions appear for example in the gluon self energy with virtual gluons, with massless ghost scalars, with a Dirac trace in the numerator for quarks, and with massive scalars for supersymmetric scalar quarks. In those cases the two masses are identical $m_1 = m_2$. The integration measure $1/(16\pi^2)$ is dictated by the Feynman rule for the integration over loop momenta. Counting powers of q, we see that the integral behaves like

$$B(p^2; m_1, m_2) \sim \frac{1}{16\pi^2} \int \frac{d^4 q}{q^4} \tag{4.32}$$

in the ultraviolet, so it is logarithmically divergent.

One regularization scheme is a cutoff into the momentum integral Λ, for example through the so-called Pauli—Villars regularization. Because the ultraviolet behavior of the integrand or integral cannot depend on any parameter living at a small energy scales, the parameterization of the ultraviolet divergence in Eq.(4.31) cannot involve the mass m or the external momentum p^2. The scalar two-point function has mass dimension zero, so its divergence has to be proportional to $\log(\Lambda/\mu_R)$ with a dimensionless prefactor and some scale μ_R^2 which is an artifact of the regularization of such a Feynman diagram. Because it is an artifact, this scale μ_R has to eventually vanish from our theory prediction.

A more elegant regularization scheme is dimensional regularization. It is designed not to break gauge invariance and naively seems to not introduce a mass scale μ_R. When we shift the momentum integration from 4 to $4 - 2\epsilon$ dimensions and use analytic continuation in the number of space–time dimensions to renormalize the theory, a renormalization scale μ_R appears when we ensure the two-point function and with it observables like cross sections keep their correct mass dimension

$$\begin{aligned} \int \frac{d^4 q}{16\pi^2} \, \frac{1}{q^2 - m_1^2} \frac{1}{(q+p)^2 - m_2^2} &\to \mu_R^{2\epsilon} \int \frac{d^{4-2\epsilon} q}{16\pi^2} \, \frac{1}{q^2 - m_1^2} \frac{1}{(q+p)^2 - m_2^2} \\ &= \frac{i \mu_R^{2\epsilon}}{(4\pi)^2} \, \left[\frac{C_{-1}}{\epsilon} + C_0 + C_1 \, \epsilon + \mathcal{O}(\epsilon^2) \right] \; . \end{aligned} \tag{4.33}$$

The constants C_i in the series in $1/\epsilon$ depend on the loop integral. To regularize the ultraviolet divergence we go into the limit $\epsilon > 0$ and find mathematically well defined poles $1/\epsilon$. Defining scalar integrals with the integration measure $1/(i\pi^2)$ will make for example C_{-1} come out as of the order $\mathcal{O}(1)$. This is the reason we usually find factors $1/(4\pi)^2 = \pi^2/(2\pi)^4$ in front of the loop integrals.

4.4 Counter terms

The ultraviolet poles in $1/\epsilon$ will cancel with universal counter terms once we renormalize the theory. We include counter terms by shifting parameters in the Lagrangian and the leading order matrix element. They cancel the poles for example from virtual one-loop diagrams,

$$\begin{aligned} \left| \mathcal{M}_\text{LO}(g) + \mathcal{M}_\text{virt} \right|^2 &= \left| \mathcal{M}_\text{LO}(g) \right|^2 + 2 \, \text{Re} \; \mathcal{M}_\text{LO}(g) \mathcal{M}_\text{virt} + \cdots \\ &\to \left| \mathcal{M}_\text{LO}(g + \delta g) \right|^2 + 2 \, \text{Re} \; \mathcal{M}_\text{LO}(g) \mathcal{M}_\text{virt} + \cdots \\ \text{with} \qquad g \to g^\text{bare} &= g + \delta g \qquad \text{and} \quad \delta g \propto \alpha_s/\epsilon \; . \end{aligned} \tag{4.34}$$

The dots indicate higher orders in α_s.

The counter terms do not come with a factor $\mu_R^{2\epsilon}$, so this factor will not be matched between the actual ultraviolet divergence and the counter term. We can keep track of the renormalization scale best by expanding the prefactor of the regularized but not yet renormalized integral in Eq.(4.33) in a Taylor series in ϵ, no question asked about convergence radii

$$\begin{aligned}\mu_R^{2\epsilon} \; \left[\frac{C_{-1}}{\epsilon} + C_0 + \mathcal{O}(\epsilon)\right] &= e^{2\epsilon \log \mu_R} \; \left[\frac{C_{-1}}{\epsilon} + C_0 + \mathcal{O}(\epsilon)\right] \\ &= \left[1 + 2\epsilon \log \mu_R + \mathcal{O}(\epsilon^2)\right] \; \left[\frac{C_{-1}}{\epsilon} + C_0 + \mathcal{O}(\epsilon)\right] \\ &= \frac{C_{-1}}{\epsilon} + C_0 + C_{-1} \; \log \mu_R^2 + \mathcal{O}(\epsilon) \\ &\to \frac{C_{-1}}{\epsilon} + C_0 + C_{-1} \; \log \frac{\mu_R^2}{M^2} + \mathcal{O}(\epsilon) \; . \end{aligned} \tag{4.35}$$

In the last step we correct by hand for the fact that $\log \mu_R^2$ with a mass dimension inside the logarithm cannot appear in our calculations. From somewhere else in our calculation the logarithm will be matched with a $\log M^2$ where M^2 is the typical mass or energy scale in our process. This little argument shows that also in dimensional regularization we introduce a mass scale μ_R which appears as $\log(\mu_R^2/M^2)$ in the renormalized expression for our observables.

In Eq.(4.35) there appear two finite contributions to a given observable, the expected C_0 and the renormalization–induced C_{-1}. Because the factors C_{-1} are linked to the counter terms in the theory we can often guess them without actually computing the complete loop integral, which is very useful in cases where they numerically dominate.

Counter terms are not uniquely defined. They remove a given divergence to return finite observables, but we are free to add any finite contribution we want. This opens many ways to define a counter term for example based on physical processes where counter terms do not only cancel the pole but also finite contributions at a given order in perturbation theory. An example for such a physical renormalization scheme is the on–shell scheme for masses, where we define a counter term such that external on–shell particles do not receive any corrections to their masses. For the top mass this means

$$\begin{aligned} m_t^{\text{bare}} &= m_t + \delta m_t \\ &= m_t + m_t \frac{\alpha_s C_F}{4\pi} \left(3 \left(-\frac{1}{\epsilon} + \gamma_E - \log(4\pi) - \log \frac{\mu_R^2}{M^2} \right) - 4 + 3 \log \frac{m_t^2}{M^2} \right) \\ &\equiv m_t + m_t \frac{\alpha_s C_F}{4\pi} \left(-\frac{3}{\tilde{\epsilon}} - 4 + 3 \log \frac{m_t^2}{M^2} \right) \qquad \Leftrightarrow \qquad \frac{1}{\tilde{\epsilon}\left(\dfrac{\mu_R}{M}\right)} \equiv \frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi \mu_R^2}{M^2} \; , \end{aligned} \tag{4.36}$$

with the color factor $C_F = (N^2-1)/(2N)$ and the Euler constant $\gamma_E \approx 0.577$ coming from the evaluation of the Gamma function $\Gamma(\epsilon) = 1/\epsilon + \gamma_E + \mathcal{O}(\epsilon)$. The convenient scale dependent pole $1/\tilde{\epsilon}$ includes the universal additional terms like the Euler gamma function and the scaling logarithm. This logarithm is the big problem in this universality argument, since we need to introduce the arbitrary energy scale M to separate the universal logarithm of the renormalization scale and the parameter-dependent logarithm of the physical process.

Another example for a process dependent renormalization scheme is the mixing of γ and Z propagators. There we choose the counter term of the weak mixing angle such that an on–shell Z boson cannot oscillate into a photon, and vice versa.

One common feature of all mass counter terms is $\delta m \propto m$, which means that our renormalization is multiplicative,

$$m^{\text{bare}} = Z_m \, m = (1 + \delta Z_m) \, m = \left(1 + \frac{\delta m}{m}\right) m = m + \delta m \qquad \text{with} \qquad \delta Z_m = \frac{\delta m}{m} \; , \tag{4.37}$$

This form implies that particles with zero mass will not obtain a finite mass through renormalization. If we remember that chiral symmetry protects a Lagrangian from acquiring fermion masses this means that on–shell renormalization does not break this symmetry. A massless theory cannot become massive by mass renormalization. Regularization and renormalization schemes which do not break symmetries of the Lagrangian are ideal.

Another way of introducing counter terms is by defining a renormalization point. This can be the energy scale at which the counter terms cancels all higher order contributions, divergent as well as finite. The best known example is the electric charge which we renormalize in the Thomson limit of zero momentum transfer through the photon propagator

$$e \to e^{\text{bare}} = e + \delta e \; . \tag{4.38}$$

Finally, there is a way to define a completely general counter term: if dimensional regularization does not break any of the symmetries of our Lagrangian, we can simply subtract the ultraviolet pole. The only question is: do we subtract $1/\epsilon$ in the MS scheme or $1/\tilde{\epsilon}$ in the $\overline{\text{MS}}$ scheme. In the $\overline{\text{MS}}$ scheme the counter term becomes scale dependent.

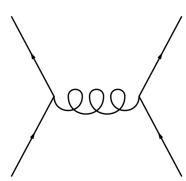
Carefully counting, there are three scales we need to consider:

1. the physical scale in the process, like the top mass m_t in the matrix element for the top decay;
2. the renormalization scale μ_R, a reference scale as part of the definition of the counter term;
3. The scale M separating the counter term from the process dependent result, which we can choose however we want. The role of M will become clear when we go through the example of the running strong coupling α_s.

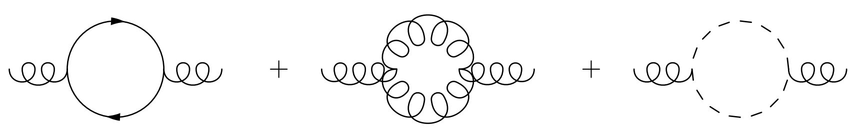
Of course, we would prefer to choose all three scales the same, but in a complex physical process this will not be possible. For example, any massive $(2 \to 3)$ production process naturally involves several external physical scales.

4.5 Running coupling

To get an idea what these different scales mean we compute the running strong coupling $\alpha_s(\mu_R^2)$. A simple process where we can study it is bottom pair production, where at some energy range we will be dominated by valence quarks: $q\bar{q} \to b\bar{b}$. The only Feynman diagram is an s-channel off–shell gluon with momentum $p^2 \equiv s$,



At next–to–leading order this gluon propagator will be corrected by self energy loops, where the gluon splits into two quarks or gluons and re-combines before it produces the two final–state bottoms. Let us for now assume that all quarks are massless. The Feynman diagrams for the gluon self energy include a quark look, a gluon loop, and the ghost loop which removes the unphysical degrees of freedom of the gluon inside the loop:



The gluon self energy correction or so-called vacuum polarization will be a scalar. All fermion lines close in the Feynman diagram and the Dirac trace is computed inside the loop. In color space the self energy will (hopefully) be diagonal, just like the gluon propagator itself, so we can ignore the color indices for now. In Lorenz gauge the gluon propagator is proportional to the transverse tensor defined in Eq.(4.29). The same should be true for the gluon self energy, which we therefore write as $\Pi^{\mu\nu} \equiv \Pi \, T^{\mu\nu}$. Including the gluon, quark, and ghost loops the regularized gluon self energy with a momentum flow p^2 through the propagator reads

$$
\begin{aligned}
-\frac{1}{p^2} \, \Pi \left(\frac{\mu_R^2}{p^2} \right) &= \frac{\alpha_s}{4\pi} \, \left(-\frac{1}{\tilde{\epsilon}} + \log \frac{p^2}{M^2} \right) \, \left(\frac{13}{6} N_c - \frac{2}{3} n_f \right) + \mathcal{O}(\log m_t^2) \\
&\equiv \alpha_s \, \left(-\frac{1}{\tilde{\epsilon}} + \log \frac{p^2}{M^2} \right) \, b_0 + \mathcal{O}(\log m_t^2) \\
&\text{with} \qquad b_0 = \frac{1}{4\pi} \left(\frac{13}{6} N_c - \frac{2}{3} n_f \right) \\
&\text{but really} \qquad b_0 = \frac{1}{4\pi} \left(\frac{11}{3} N_c - \frac{2}{3} n_f \right) \overset{\text{SM}}{>} 0 \; . \qquad (4.39)
\end{aligned}
$$

The number of fermions coupling to the gluons is n_f. The factor b_0 reflects the one-loop corrections. Strictly speaking, it gives the first term in a perturbative series in the strong coupling $\alpha_s = g_s^2/(4\pi)$.

In the last step of Eq.(4.39) we have snuck in additional contributions by replacing the factor 13/6 by a factor 11/3. This is related to the fact that there are actually three types of divergent virtual gluon diagrams in the physical process $q\bar{q} \to b\bar{b}$: the external quark self energies with renormalization factors $Z_f^{1/2}$, the internal gluon self energy Z_A, and the vertex corrections Z_{Aff}. The physical parameters we can renormalize in this process are the strong coupling and the quark masses. Wave function renormalization constants are not physical. The entire divergence which eventually needs to be absorbed in Z_g is given by the combination

$$
Z_{Aff} = Z_g Z_A^{1/2} Z_f \qquad \Leftrightarrow \qquad \frac{Z_{Aff}}{Z_A^{1/2} Z_f} \equiv Z_g \; . \qquad (4.40)
$$

This changes the factor from 13/6 to 11/3 in the running of the strong coupling.

We can check this definition of Z_g by comparing all vertices in which the strong coupling g_s appears, namely the gluon coupling to quarks and ghosts, as well as the triple and quartic gluon vertex. All of them need to have the same divergence structure

$$
\frac{Z_{Aff}}{Z_A^{1/2} Z_f} \overset{!}{=} \frac{Z_{Acc}}{Z_A^{1/2} Z_c} \overset{!}{=} \frac{Z_{3A}}{Z_A^{3/2}} \overset{!}{=} \sqrt{\frac{Z_{4A}}{Z_A^2}} \; . \qquad (4.41)
$$

If we had done the same calculation in QED and looked for a running electric charge, we would have found that the vacuum polarization diagrams for the photon do account for the entire counter term of the electric charge. The other two renormalization constants Z_{Aff} and Z_f cancel because of gauge invariance.

In contrast to QED, the strong coupling diverges in the Thomson limit because QCD is confined towards large distances and weakly coupled at small distances. Lacking a well enough motivated reference point we are lead to renormalize $\alpha_s = g_s^2/(4\pi)$ in the $\overline{\text{MS}}$ scheme. From Eq.(4.39) we know that the ultraviolet pole which needs to be cancelled by the counter term is proportional to the function b_0

$$
\begin{aligned}
\alpha_s^{\text{bare}} = \left(1 + \frac{\delta\alpha_s}{\alpha_s} \right) \alpha_s &\overset{\overline{\text{MS}}}{=} \left(1 - \left. \frac{\Pi}{p^2} \right|_{\text{pole}} \right) \alpha_s(M^2) \\
&\overset{\text{Eq.(4.39)}}{=} \left(1 - \frac{\alpha_s}{\tilde{\epsilon} \left(\frac{\mu_R}{M} \right)} \, b_0 \right) \alpha_s(M^2) \; . \qquad (4.42)
\end{aligned}
$$

Here we explicitly include the scale dependence of the counter term. Because the bare coupling does not depend on any scales, this means that the renormalized α_s depends on an unphysical scale M. We can also evaluate it at the momentum flowing through the gluon propagator p^2 and write according to Eq.(4.39)

$$
\alpha_s^{\text{bare}} = \alpha_s(p^2) \left(1 - \frac{\alpha_s(p^2) b_0}{\tilde{\epsilon}} + \alpha_s(p^2) b_0 \log \frac{p^2}{M^2} \right) \; . \qquad (4.43)
$$

On the right-hand side α_s is evaluated at the physical scale p^2. The logarithm shifts the argument of $\tilde{\epsilon}$ from M^2 to p^2. This way the formula defines a running coupling $\alpha_s(p^2)$ and accounts for shifts between the physical scale p^2 and the general scale M^2 coming out of the $\overline{\text{MS}}$ scheme. Identifying the right-hand sides of Eqs.(4.42) and (4.43) we find

$$\begin{aligned}
& \alpha_s(M^2) = \alpha_s(p^2) + \alpha_s^2(p^2) b_0 \log \frac{p^2}{M^2} = \alpha_s(p^2) \left(1 + \alpha_s(p^2) b_0 \log \frac{p^2}{M^2} \right) \\
\Leftrightarrow \qquad & \frac{d\alpha_s(p^2)}{d \log p^2} = -\alpha_s^2(p^2) b_0 + \mathcal{O}(\alpha_s^3) \; . \qquad (4.44)
\end{aligned}$$

To the given loop order the argument of the strong coupling squared on the right side can be neglected — its effect is of higher order. We nevertheless keep the argument as a higher order effect to later distinguish different approaches to the running coupling. From Eq.(4.39) we know that $b_0 > 0$, which means that towards larger scales the strong coupling has a negative slope, so the ultraviolet limit of the strong coupling is zero and QCD is asymptotically free.

4.6 Resummation

We can do better than fixed order in perturbation theory: instead of simply including the gluon self energy bubble at a given order in perturbation theory we can include chains of one-loop diagrams with Π appearing many times in the gluon propagator. It means we replace the off–shell gluon propagator by

$$\begin{aligned}
\frac{T^{\mu\nu}}{p^2} \to & \frac{T^{\mu\nu}}{p^2} + \left(\frac{T}{p^2} \cdot (-T\,\Pi) \cdot \frac{T}{p^2} \right)^{\mu\nu} \\
& + \left(\frac{T}{p^2} \cdot (-T\,\Pi) \cdot \frac{T}{p^2} \cdot (-T\,\Pi) \cdot \frac{T}{p^2} \right)^{\mu\nu} + \cdots \\
= & \frac{T^{\mu\nu}}{p^2} \sum_{j=0}^{\infty} \left(-\frac{\Pi}{p^2} \right)^j = \frac{T^{\mu\nu}}{p^2} \, \frac{1}{1 + \Pi/p^2} \; , \qquad (4.45)
\end{aligned}$$

schematically written without the factors i. To avoid indices we abbreviate $T^{\mu\nu} T_\nu^\rho = T \cdot T$ and

$$(T \cdot T \cdot T)^{\mu\nu} = T^{\mu\rho} T_\rho^\sigma T_\sigma^\nu = T^{\mu\sigma} T_\sigma^\nu = T^{\mu\nu} \; . \qquad (4.46)$$

This (re-)summation moves the finite shift in α_s from Eqs.(4.39) and (4.43) into the denominator, while we assume that the pole will be properly taken care of at any given order,

$$\alpha_s^{\text{bare}} = \alpha_s(M^2) - \frac{\alpha_s^2 b_0}{\tilde{\epsilon}} \equiv \frac{\alpha_s(p^2)}{1 - \alpha_s(p^2)\; b_0 \; \log \dfrac{p^2}{M^2}} - \frac{\alpha_s^2 b_0}{\tilde{\epsilon}} \; . \qquad (4.47)$$

As before, we can relate the values of α_s at two reference points, *i.e.* we consider it a renormalization group equation (RGE) which evolves physical parameters from one scale to another in analogy to the fixed order version in Eq.(4.44)

$$\frac{1}{\alpha_s(M^2)} = \frac{1}{\alpha_s(p^2)} \left(1 - \alpha_s(p^2)\; b_0 \; \log \frac{p^2}{M^2} \right) = \frac{1}{\alpha_s(p^2)} - b_0 \; \log \frac{p^2}{M^2} + \mathcal{O}(\alpha_s) \; . \qquad (4.48)$$

The factor α_s in the parentheses can be evaluated at either of the two scales, the difference is a higher order effect. If we keep it at p^2 we see how Eq.(4.48) is different from Eq.(4.43) and how resumming the vacuum expectation bubbles differs from the un-resummed result in higher order contributions. When we differentiate $\alpha_s(p^2)$ with respect to p^2 we find with $d/dx(1/\alpha_s) = -1/\alpha_s^2 \; d\alpha_s/dx$

$$\begin{aligned}
p^2 \frac{d\alpha_s}{dp^2} = \frac{d\alpha_s}{d\log p^2} & = -\alpha_s^2 \frac{d}{d\log p^2} \frac{1}{\alpha_s(p^2)} = -\alpha_s^2 \frac{d}{d\log p^2} \; \left[\frac{1}{\alpha_s(M^2)} + b_0 \log \frac{p^2}{M^2} \right] \\
& = -\alpha_s^2 \, b_0 + \mathcal{O}(\alpha_s^2) \\
& \to -\alpha_s^2 \sum_{n=0} b_n \alpha_s^n \equiv \beta \; . \qquad (4.49)
\end{aligned}$$

This is the running of the strong coupling constant including all higher order terms b_n.

In the running of the strong coupling constant we relate the different scales through multiplicative factors of the kind

$$\left(1 \pm \alpha_s b_0 \log \frac{p^2}{M^2} \right) \; . \tag{4.50}$$

Such factors appear in the un-resummed computation of Eq.(4.44) as well as in Eq.(4.47) after resummation. Because they are multiplicative, these factors can move into the denominator, where they should not vanish. Dependent on the sign of b_0 this becomes a problem for large scale ratios $|\alpha_s \log p^2/M^2| > 1$, leading to a so-called Landau pole. For $b_0 > 0$ and large couplings at small $p^2 \ll M^2$ the combination $(1 + \alpha_s b_0 \log p^2/M^2)$ indeed vanishes. We turn the problem into a feature by replacing the renormalization point of α_s in Eq.(4.47) with a reference scale defined by the Landau pole. At one loop order we define the scale Λ_{QCD} through

$$1 + \alpha_s(M^2)\; b_0 \;\log \frac{\Lambda^2_{\text{QCD}}}{M^2} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \log \frac{\Lambda^2_{\text{QCD}}}{M^2} = - \frac{1}{\alpha_s(M^2) b_0} \quad \Leftrightarrow \quad \log \frac{p^2}{M^2} = \log \frac{p^2}{\Lambda^2_{\text{QCD}}} - \frac{1}{\alpha_s(M^2) b_0} \; , \tag{4.51}$$

and then include it in the running

$$\begin{aligned} \frac{1}{\alpha_s(p^2)} &\stackrel{\text{Eq.(4.48)}}{=} \frac{1}{\alpha_s(M^2)} + b_0 \;\log \frac{p^2}{M^2} \\ &= \frac{1}{\alpha_s(M^2)} + b_0 \log \frac{p^2}{\Lambda^2_{\text{QCD}}} - \frac{1}{\alpha_s(M^2)} = b_0 \log \frac{p^2}{\Lambda^2_{\text{QCD}}} \qquad \Leftrightarrow \qquad \alpha_s(p^2) = \frac{1}{b_0 \log \dfrac{p^2}{\Lambda^2_{\text{QCD}}}} \; . \end{aligned} \tag{4.52}$$

An interesting aspect of Λ_{QCD} is that we introduce a scale into our theory without ever setting it. All we do is renormalize a coupling which becomes strong at large energies and search for the mass scale of this strong interaction. This trick is called dimensional transmutation.

In terms of language, there is a little bit of confusion between field theorists and phenomenologists: we have introduced the renormalization scale μ_R as the renormalization point, for example of the strong coupling constant. In the $\overline{\text{MS}}$ scheme, the subtraction of $1/\tilde{\epsilon}$ shifts the scale dependence of the strong coupling to M^2 and moves the logarithm $\log M^2/\Lambda^2_{\text{QCD}}$ into the definition of the renormalized parameter. This is what we call the renormalization scale in the phenomenological sense, *i.e.* the argument we evaluate α_s at.

4.7 Resumming scaling logarithms

Up to now we have introduced the running strong coupling in a fairly abstract manner and did not link the resummation of diagrams and the running of α_s in Eqs.(4.44) and (4.49) to physics. In what way does the resummation of the one-loop diagrams for the s-channel gluon improve our prediction of the LHC observables?

As an illustration we look at a simple observable which depends on just one physical energy scale p^2. The first observable coming to mind is again the Drell–Yan cross section $\sigma(q\bar{q} \to \mu^+\mu^-)$, but since we are not really sure what to do with the parton densities we resort to simpler e^+e^- collisions. A simple observable which includes α_s at least in the one-loop corrections is

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_{\text{quarks}} Q_q^2 = \frac{11 N_c}{9} \; . \tag{4.53}$$

The numerical value at leading order assumes five quarks. Including higher order corrections we can express the result in a power series in α_s. In the $\overline{\text{MS}}$ scheme we introduce an unphysical scale dependence on M in the individual r_n

$$R\left(\frac{p^2}{M^2}, \alpha_s \right) = \sum_{n=0} r_n \left(\frac{p^2}{M^2} \right) \; \alpha_s^n(M^2) \qquad \text{with} \qquad r_0 = \frac{11 N_c}{9} \; . \tag{4.54}$$

The r_n we can assume to be dimensionless. This implies that the calculated r_n only depend on ratios of two scales, the external p^2 and the artificial M^2.

At the same time R is an observable, so including all orders in perturbation theory it cannot depend on any artificial scale M. Writing this dependence as a total derivative and setting it to zero we find an equation which would be called a Callan–Symanzik equation if instead of the running coupling we had included a running mass

$$
\begin{aligned}
0 &\stackrel{!}{=} M^2 \frac{d}{dM^2} R\left(\frac{p^2}{M^2}, \alpha_s(M^2)\right) \\
&= \left[M^2 \frac{\partial}{\partial M^2} + \beta \frac{\partial}{\partial \alpha_s}\right] \sum_{n=0} r_n \left(\frac{p^2}{M^2}\right) \alpha_s^n && \text{with} \quad \beta = M^2 \frac{\partial \alpha_s}{\partial M^2} \\
&= \sum_{n=1} M^2 \frac{\partial r_n}{\partial M^2} \alpha_s^n + \sum_{n=1} \beta\, r_n\, n \alpha_s^{n-1} && \text{with} \quad r_0 = \frac{11 N_c}{9} = \text{const} \\
&= M^2 \sum_{n=1} \frac{\partial r_n}{\partial M^2} \alpha_s^n - \sum_{n=1} \sum_{m=0} n r_n\, \alpha_s^{n+m+1}\, b_m && \text{with} \quad \beta = -\alpha_s^2 \sum_{m=0} b_m \alpha_s^m \\
&= M^2 \frac{\partial r_1}{\partial M^2} \alpha_s + \left(M^2 \frac{\partial r_2}{\partial M^2} - r_1 b_0\right) \alpha_s^2 + \left(M^2 \frac{\partial r_3}{\partial M^2} - 2 r_2 b_0 - r_1 b_1\right) \alpha_s^3 + \mathcal{O}(\alpha_s^4) \; . && (4.55)
\end{aligned}
$$

This perturbative series in α_s has to vanish in each order of perturbation theory,

$$
\begin{aligned}
\frac{\partial r_1}{\partial \log M^2} &= 0 \\
\frac{\partial r_2}{\partial \log M^2} &= r_1 b_0 \\
\frac{\partial r_3}{\partial \log M^2} &= r_1 b_1 + 2 r_2(M^2) b_0 \\
&\vdots
\end{aligned}
\qquad (4.56)
$$

The mix of r_n derivatives and the perturbative terms in the β function can be seen for α_s^3: first, we have the appropriate NNNLO corrections r_3; next, we have one loop in the gluon propagator b_0 and two loops for example in the vertex r_2; and finally, we need the two-loop diagram for the gluon propagator b_1 and a one-loop vertex correction r_1.

The M^2-dependence vanishes for r_0 and r_1. Keeping in mind the integration constants c_n we find the solutions

$$
\begin{aligned}
r_0 &= c_0 = \frac{11 N_c}{9} \\
r_1 &= c_1 \\
r_2 &= c_2 + r_1 b_0 \log \frac{M^2}{p^2} = c_2 + c_1 b_0 \log \frac{M^2}{p^2} \\
r_3 &= \int d \log \frac{M'^2}{p^2} \left(c_1 b_1 + 2\left(c_2 + c_1 b_0 \log \frac{M'^2}{p^2}\right) b_0\right) = c_3 + (c_1 b_1 + 2 c_2 b_0) \log \frac{M^2}{p^2} + c_1 b_0^2 \log^2 \frac{M^2}{p^2} \\
&\vdots
\end{aligned}
\qquad (4.57)
$$

This chain of r_n values suggests to interpret the fixed-order perturbative series in Eq.(4.54) as implicitly including terms $\log^{n-1} M^2/p^2$ in each r_n. They become problematic if the logarithm becomes large enough to spoil the convergence in terms of $\alpha_s \sim 0.1$, *i.e.* measuring R at scales p^2 far away from the scale choice for the strong coupling constant. M^2.

Interestinglty, we can use Eq.(4.57) to express R in terms of the c_n,

$$R = \sum_n r_n \left(\frac{p^2}{M^2} \right) \; \alpha_s^n(M^2) = c_0 + c_1 \left(1 + \alpha_s(M^2) b_0 \log \frac{M^2}{p^2} + \alpha_s^2(M^2) b_0^2 \log^2 \frac{M^2}{p^2} + \cdots \right) \alpha_s(M^2)$$
$$+ c_2 \left(1 + 2\alpha_s(M^2) b_0 \log \frac{M^2}{p^2} + \cdots \right) \alpha_s^2(M^2) + \cdots \qquad (4.58)$$

We can resum this geometric series to

$$R = c_0 + c_1 \frac{\alpha_s(M^2)}{1 - \alpha_s(M^2) b_0 \log \dfrac{M^2}{p^2}} + c_2 \left(\frac{\alpha_s(M^2)}{1 - \alpha_s(M^2) b_0 \log \dfrac{M^2}{p^2}} \right)^2 + \cdots \equiv \sum c_n \; \alpha_s^n(p^2) \; . \qquad (4.59)$$

In the last step we use Eq.(4.48) with flipped arguments p^2 and M^2, derived from the resummation of the vacuum polarization bubbles. In contrast to the r_n, the c_n are by definition independent of p^2/M^2 and therefore more suitable as a perturbative series in the presence of potentially large logarithms. This re-organization of the perturbation series for R can be interpreted as resumming all logarithms of the kind $\log M^2/p^2$ in a new organization of the perturbative series. Some higher–order factors c_n are known, for example inserting $N_c = 3$ and five quark flavors just as we assume in Eq.(4.53)

$$R = \frac{11}{3} \left(1 + \frac{\alpha_s(p^2)}{\pi} + 1.4 \left(\frac{\alpha_s(p^2)}{\pi} \right)^2 - 12 \left(\frac{\alpha_s(p^2)}{\pi} \right)^3 + \mathcal{O} \left(\frac{\alpha_s(p^2)}{\pi} \right)^4 \right) \; . \qquad (4.60)$$

This alternating series with increasing perturbative prefactors indicates the asymptotic instead of convergent behavior of perturbative QCD. At the bottom mass scale the relevant coupling factor is only $\alpha_s(m_b^2)/\pi \sim 1/14$, so a further increase of the c_n would become dangerous. However, a detailed look into the calculation shows that the dominant contributions to c_n arise from the analytic continuation of logarithms, which are large finite terms for example from $\text{Re}(\log^2(-E^2)) = \log^2 E^2 + \pi^2$. In the literature such π^2 terms arising from the analytic continuation of loop integrals are often phrased in terms of $\zeta_2 = \pi^2/6$.

Before moving on we collect the logic of the argument given in this section: when we regularize an ultraviolet divergence we automatically introduce a reference scale μ_R. Naively, this could be an ultraviolet cutoff scale, but even the seemingly scale invariant dimensional regularization in the conformal limit of our field theory cannot avoid the introduction of a scale. There are several ways of dealing with such a scale: first, we can renormalize our parameter at a reference point. Secondly, we can define a running parameter and this way absorb the scale logarithm into the $\overline{\text{MS}}$ counter term. For the asymptotically free strong coupling Λ_{QCD} leaves us with a compact form of the running coupling $\alpha_s(M^2)$.

Strictly speaking, at each order in perturbation theory the scale dependence should vanish together with the ultraviolet poles, as long as there is only one scale affecting a given observable. However, defining the running strong coupling we sum one-loop vacuum polarization graphs. Even when we compute an observable at a given loop order, we implicitly include higher order contributions. They lead to a dependence of our perturbative result on the artificial scale M^2, which phenomenologists refer to as renormalization scale dependence.

Using the R ratio we see what our definition of the running coupling means in terms of resumming logarithms: reorganizing our perturbative series to get rid of the ultraviolet divergence $\alpha_s(p^2)$ resums the scale logarithms $\log p^2/M^2$ to all orders in perturbation theory.