Symmetries and Representations

The Lie algebra su(2) has Hermitian generators

→ representations in Hilbert spaces, e.g. 2D: $|1\rangle, |2\rangle$

(orthonormal basis)

$\psi \doteq \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

raising, lowering } "ladder operators", sometimes

T_+, T_-, T_3 simplest non-trivial case

$T_+ \leftarrow |1\rangle\langle 2|$, $T_- \leftarrow |2\rangle\langle 1|$, $T_3 \leftarrow \frac{1}{2}(|1\rangle\langle 1| - |2\rangle\langle 2|)$

or alternatively T_1, T_2, T_3 where $\begin{cases} T_1 = \frac{1}{2}(T_+ + T_-) \\ T_2 = \frac{i}{2}(T_- - T_+) \end{cases}$

The defining commutation relation is $[T_a, T_b] = i\epsilon_{abc} T_c$

The generators also obey: $\{T_a, T_b\} = \frac{1}{2}\delta_{ab}$

$T_a T_b = \frac{1}{4}\delta_{ab} + \frac{i}{2}\epsilon_{abc} T_c$

[compare to the internal-space Pauli matrices τ_a]

Thus, $[T_+, T_-] = 2T_3$ and $[T_3, T_\pm] = \pm T_\pm$ and

$T_+|1\rangle = 0$ $\quad T_-|1\rangle = |2\rangle$ $\quad \boxed{T_3|1\rangle = \frac{1}{2}|1\rangle}$

$T_+|2\rangle = |1\rangle$ $\quad T_-|2\rangle = 0$ $\quad \boxed{T_3|2\rangle = -\frac{1}{2}|2\rangle}$

eigenstates of T_3 with eigenvalue $\pm\frac{1}{2}$

Exponentiation takes the Lie algebra → Lie group SU(2)

abstract

let $\vec{\theta} = \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix}$ be real, $\quad U(\vec{\theta}) = e^{i\theta^a T_a} = \sum_{n=0}^{\infty} \frac{(i\theta^a T_a)^n}{n!}$

summed

So using this concrete representation and writing $\vec{\theta} = |\vec{\theta}|\hat{\theta} = \theta\hat{\theta}$, the group elements are:

2D irreducible representation "spinor"/"fundamental"

$$U(\vec{\theta}) = e^{\frac{i}{2}\vec{\sigma}\cdot\vec{\theta}} = \underbrace{\cos\frac{|\vec{\theta}|}{2}\,\mathbb{1}_2 + i(\hat{\theta}\cdot\vec{\sigma})\sin\frac{|\vec{\theta}|}{2}}$$

eigenstates of $\vec{S} = \frac{1}{2}\vec{\sigma}$ along the chosen axis $\hat{\theta}$

"spin $\frac{1}{2}$"

$\vec{\sigma}$ ~ vector of Pauli (spin) matrices

Intuition: $T_3 \sim S_z$

If we let $\theta \to \theta + 2\pi$ then $U \to -U$ (4π symmetry of fermions)

Unitarity, $\underline{\det U = 1}$, $\Rightarrow$ in general for complex α, β we can write U as

$|\alpha|^2 + |\beta|^2 = 1$

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

i.e., decomposed as $U = \mathrm{Re}\{\alpha\} \mathbb{1}_2 + i\sigma_1 \mathrm{Im}\{\beta\} + i\sigma_2 \mathrm{Re}\{\beta\} + i\sigma_3 \mathrm{Im}\{\alpha\}$

<u>But</u> the Hilbert space can also be larger... there are other representations. To distinguish them, we need <u>Casimir operators</u>.

The Casimir operators:

- commute with generators of the Lie algebra
- ↳ are constructed from a combination of operators, e.g., for a given representation
- tell us that a representation is <u>irreducible</u> iff every Casimir is a multiple of the identity ($\mathbb{1}$)
- quadratic: $C_2 = T^a T_a$ (and $[C_2, T_a] = 0$)

For our 3 generators of su(2), $C_2 = T_1^2 + T_2^2 + T_3^2$

(note $T_a^\dagger = T_a \Rightarrow C_2^\dagger = C_2$) $\quad [C_2, T_a] = 0$ cf. S^2 for spin

→ here there is <u>only</u> the quadratic Casimir C_2

The states of a Hilbert space can be labelled by the <u>eigenvalues</u> of a set of <u>commuting</u> operators

"good quantum numbers" ↳ simultaneous diagonalization is possible

Note that the <u>generators</u> do <u>not</u> commute (cf. spin S_x vs. S_y)

We can take C_2 and <u>one</u> generator, typically T_3, and represent the algebra via their eigenstates (Hermitian $\Rightarrow$ real eigenvalues)

$$C_2 |c, m\rangle = c |c, m\rangle \quad \text{and} \quad T_3 |c, m\rangle = m |c, m\rangle$$

↑ two labels sufficient in this case

Now since $C_2 \sim T_3^2$, for finite c the spectrum of T_3 is bounded: $-\sqrt{c} \le m \le \sqrt{c}$

range of eigenvalues

Using our initial construction, and $[T_3, T_+] = T_+$

$$\underbrace{(T_3 T_+ - T_+ T_3)|c,m\rangle}_{} = T_+|c,m\rangle$$

$$= T_3(T_+|c,m\rangle) - m(T_+|c,m\rangle) \quad \text{so} \quad T_3(T_+|c,m\rangle) = (m+1)(T_+|c,m\rangle)$$

i.e., $T_+|c,m\rangle$ is an eigenstate of T_3 with eigenvalue $m+1$ $\Rightarrow$ it is at most a number multiplying $|c,m+1\rangle$

$$T_+|c,m\rangle = f_+(m)\,|c,m+1\rangle$$

(some function of m) $\times \mathbb{1}$

So now,

$$T_3(T_+|c,m\rangle) = f_+(m)\cdot(m+1)|c,m+1\rangle$$

$$C_2(T_+|c,m\rangle) = f_+(m)\cdot c\,|c,m+1\rangle$$

and since T_3 is bounded, for a given c, there is some maximum eigenvalue of T_3

$\Rightarrow$ there is a corresponding "maximum-weight" state,

$$|c,j\rangle = |c, m_{max}\rangle$$

Intuition: maximum projection of spin onto quantization axis

$T_+|c,j\rangle = 0$ as for $T_+|1\rangle = 0$, spin "up" cannot be further raised

$T_3|c,j\rangle = j|c,j\rangle$

For the lowering operator T_- it is similar: there exists a "minimum weight" state $|c,k\rangle$ where $T_-|c,m\rangle = f_-(m)|c,m-1\rangle$

$\Rightarrow T_-|c,k\rangle = 0$ cf. spin "down" cannot be further lowered

$T_3|c,k\rangle = k|c,k\rangle$

Now we can actually determine C_2 from the maximum weight state
↳ i.e., its action on our labelled states

$$C_2|c,j\rangle = \underbrace{(T_1^2+T_2^2+T_3^2)}|c,j\rangle = c|c,j\rangle \,, \quad \text{also } C_2|c,k\rangle = c|c,k\rangle$$

$$= \frac{1}{2}(T_+T_- + T_-T_+) + T_3^2$$

using $[T_+,T_-]=2T_3$, $\{T_+,T_-\}=\mathbb{1}_2$

$$= T_3^2 - T_3 + \cancel{T_+T_-}^{\,0 \text{ for min}} \longrightarrow C_2|c,k\rangle = (k^2-k)|c,k\rangle$$

$$= T_3^2 + T_3 + \cancel{T_-T_+}^{\,0 \text{ for max}} \longrightarrow C_2|c,j\rangle = (j^2+j)|c,j\rangle$$

Now in both cases the eigenvalue c must be the same!

$\Rightarrow \; c = j(j+1) = k(k-1)$ which is solved for $\boxed{k=-j}$
(only solution for $j = m_{max}$)

So the spectrum of T_3 has $2j+1$ states: $m \in \{-j, \dots, j\}$

$\Rightarrow$ we can equally well label the states as $|j,m\rangle$
(it's possible to show explicitly via $T_\pm$ that the total number of states is $1+(j-m)+(j+m)$ where 1 counts the state that we start with)

and since $T_+|j,j\rangle = T_-|j,-j\rangle = 0$, also $f_+(j) = f_-(-j) = 0$
the commutator of T_+ and T_- can be used to determine $f_\pm$: $\quad [T_+,T_-]|j,m\rangle = 2T_3|j,m\rangle$

$$T_\pm|j,m\rangle = f_\pm|j,m\pm1\rangle \Rightarrow f_+(m-1)f_-(m) - f_+(m)f_-(m+1) = 2m$$

has the solutions: $\quad f_\pm(m) = \sqrt{(j\mp m)(j\pm m+1)}$

→ cf. raising/lowering operators for spin-½

For a fixed j (fixed c), the states $|m,j\rangle$ form a complete and orthonormal basis set for the $(2j+1)$-dimensional Hilbert space:

$$\langle j,m|j,m'\rangle = \delta_{mm'} \quad \text{and} \quad \mathbb{1}_{2j+1} = \sum_{m=-j}^{j} |j,m\rangle\langle j,m|$$

We can start to understand the connections to physics by considering different representations:

also called: (for su(2)) spin-$\frac{1}{2}$/doublet/spinor

Fundamental representation: $j = \frac{1}{2}$

generators represented by Pauli matrices, $T_a = \frac{1}{2}\tau_a$

$$\tau_a \tau_b = \delta_{ab} \mathbb{1}_2 + i\epsilon_{abc}\tau_c \quad \text{and} \quad \tau_a^* = \tau_a^T = -\tau_2\tau_a\tau_2$$

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

frequently distinguish $\vec{\sigma}$ (spin) from $\vec{\tau}$ (internal spaces)

☆ Commutation: $[T_i, T_j] = i f_{jkl} T_l$ in general

"structure constants" here, ϵ_{jkl} (always antisym.)

fundamental

→ Fundamental rep: $T_i^{(f)} = \tau_i$ (traceless, Hermitian, $N\times N$ for general $SU(N)$)

The Standard Model matter fields transform in fundamental representation

Adjoint representation: set generators to the structure constants (here ϵ_{ijk})

# of Lie algebra generators ~ dimension, here 3 for su(2)

rep. by 3 Hermitian matrices (traceless, 3x3, antisymmetric)

adjoint

$$\left(T_k^{(A)}\right)_{ab} = i f_{kba} = -i\epsilon_{kab}$$

spin-1/triplet/vector representation

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The Standard Model gauge fields (force-carrying bosons) transform in the adjoint representation of the gauge group.

not group of gauge transf.

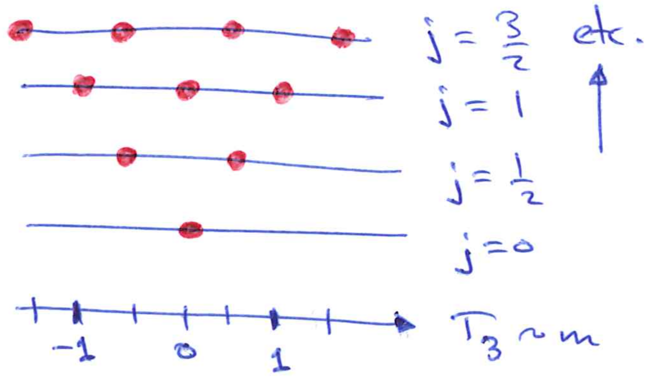
→ we already know under a gauge transformation U, $A'_\mu \to U A U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}$

Trivial representation: $j=0$, $T_a=0$ (cf. commutation relation)

this is also called spin-0/singlet/scalar

Now one Casimir was enough to organize the various su(2) representations (→ "rank 1") and $C_2 \sim j(j+1)$

For a given j, the states are fully labelled by the second identifying quantum number m: (# labels = # casimirs)



So higher spin simply corresponds to different representations of su(2)
→ infinitely many, and distinguished by Casimir operator (or equivalently j)

This also applies to physics cases with the same underlying structure: isospin, any 2-state quantum system, ...

We can also generate representations by direct products.
Note that sum representations, however, are not analogous:

particle in a superposition state $\psi \sim \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$

consider a 6D vector $\vec{q} = (\underbrace{x_1, x_2, x_3}_{\vec{x}}, \underbrace{y_1, y_2, y_3}_{\vec{y}})$

composed of two 3D vectors:

rotations can map $\vec{q} \to \vec{q}\,' = D\vec{q}$, $D \sim \underbrace{\begin{pmatrix} R_3 & 0 \\ 0 & R_3 \end{pmatrix}}_{6\times 6 \text{ reducible}}$

so $6 = 3 \oplus 3$ leaves invariant subspaces!

Products however, $\psi \sim \psi_1 \otimes \psi_2 \sim |1\rangle|2\rangle$

→ sum of irreducible representations, e.g. $2 \otimes 2 = 3 \oplus 1$

Let's consider two identical copies of $su(2)$, with the generators $T_i^{(1)}$ and $T_j^{(2)}$ such that $[T_i^{(1)}, T_j^{(2)}] = 0$ and $[T_i^{(1)}, T_j^{(1)}] = i\epsilon_{ijk} T_k^{(1)}$ as usual (also for $1 \to 2$)

Then the sums of generators fulfill the same algebra:

$$[T_i^{(1)} + T_i^{(2)}, T_j^{(1)} + T_j^{(2)}] = i\epsilon_{ijk}(T_k^{(1)} + T_k^{(2)})$$

and act on the product states of the form $|\,\rangle_1 |\,\rangle_2$ where each operator acts only on the respective state label.

each corresponding to some Hilbert space

A product of two representations is understood as being built out of irreducible representations, as for coupling of angular momenta: e.g., (spin-$\frac{1}{2}$) and (spin-$\frac{1}{2}$)

	$s=\frac{1}{2}$	$s=\frac{1}{2}$		$s=1$		$s=0$
				$\|\uparrow\uparrow\rangle$	$m=+1$	—
$m=+\frac{1}{2}$	$\|\uparrow\rangle_1$	$\|\uparrow\rangle_2$	$\Longrightarrow$	$\frac{1}{\sqrt{2}}(\|\uparrow\downarrow\rangle + \|\downarrow\uparrow\rangle)$	$m=0$	$\frac{1}{\sqrt{2}}(\|\uparrow\downarrow\rangle - \|\downarrow\uparrow\rangle)$ $m=0$
$m=-\frac{1}{2}$	$\|\downarrow\rangle_1$	$\|\downarrow\rangle_2$		$\|\downarrow\downarrow\rangle$	$m=-1$	—

watch out for different labelling conventions... $\rightarrow$ $2 \otimes 2 = 3 \oplus 1$

(Related by Clebsch-Gordon coefficients...)

The maximum-weight state for $(2j+1)\otimes(2k+1)$ is $|j,j\rangle|k,k\rangle$

$$T_3 |j,j\rangle|k,k\rangle = (T_3^{(1)} + T_3^{(2)})|j,j\rangle|k,k\rangle = (j+k)|j,j\rangle|k,k\rangle$$

for (spin-$\frac{1}{2}$) coupled to (spin-$\frac{1}{2}$), this is a statement of <u>angular momentum conservation</u> for the maximum state of the coupled system

Now let's consider SU(3), from su(3)

Pauli matrices (3) $\longrightarrow$ Gell-Mann matrices (8)
can be understood as 3 overlapping copies of su(2):

1-2 sector $\lambda_1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \end{pmatrix}$ $\quad \lambda_2 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & \end{pmatrix}$ $\quad \lambda_3 = \begin{pmatrix} 1 & 0 & \\ 0 & -1 & \\ & & \end{pmatrix}$

1-3 sector $\lambda_4 = \begin{pmatrix} 0 & & 1 \\ & & \\ 1 & & 0 \end{pmatrix}$ $\quad \lambda_5 = \begin{pmatrix} 0 & & -i \\ & & \\ i & & 0 \end{pmatrix}$

diagonal $\Rightarrow [\lambda_3, \lambda_8] = 0$

2-3 sector $\lambda_6 = \begin{pmatrix} & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$ $\quad \lambda_7 = \begin{pmatrix} & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}$ $\quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$

Note that

$$\begin{pmatrix} 1 & & 0 \\ & & \\ 0 & & -1 \end{pmatrix} = \frac{1}{2}\left(\lambda_3 + \sqrt{3}\,\lambda_8\right) = \frac{1}{2i}[\lambda_4, \lambda_5]$$

$$\begin{pmatrix} & & \\ & 1 & 0 \\ & 0 & -1 \end{pmatrix} = \frac{1}{2}\left(-\lambda_3 + \sqrt{3}\,\lambda_8\right) = \frac{1}{2i}[\lambda_6, \lambda_7]$$

They obey a normalization condition $\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}$
and again the commutation relation involves structure
constants: $[T_a, T_b] = i f_{abc} T_c$ where $T_i = \frac{1}{2}\lambda_i$ as for su(2)
in the fundamental representation

$\{T_a, T_b\} = \frac{1}{3}\delta_{ab} + d_{abc} T_c$

symmetric

antisymmetric, $f_{123} = 1 \quad f_{345} = \frac{1}{2} \quad f_{678} = \frac{\sqrt{3}}{2}$
and all others are either zero or identical to one of these

We can also define a set of six raising and lowering operators:

$$I_\pm = T_1 \pm i T_2 \qquad V_\pm = T_4 \pm i T_5 \qquad U_\pm = T_6 \pm i T_7$$

which, as before, relate states within a representation to each other.

But this time there are two Casimir operators:

$$C_2 = \sum_{i=1}^{8} T_i^2 \qquad \text{quadratic}$$

$$C_3 = \sum_{i,j,k} f_{ijk} T_i T_j T_k \qquad \text{cubic}$$

So now there is an additional label needed. We can label states <u>within</u> each irreducible representation by the eigenvalues of T_3 and T_8 (which commute): $|m_3, m_8\rangle$

The fundamental representation "3" can be used to describe (approximately) the lightest quark flavor states:

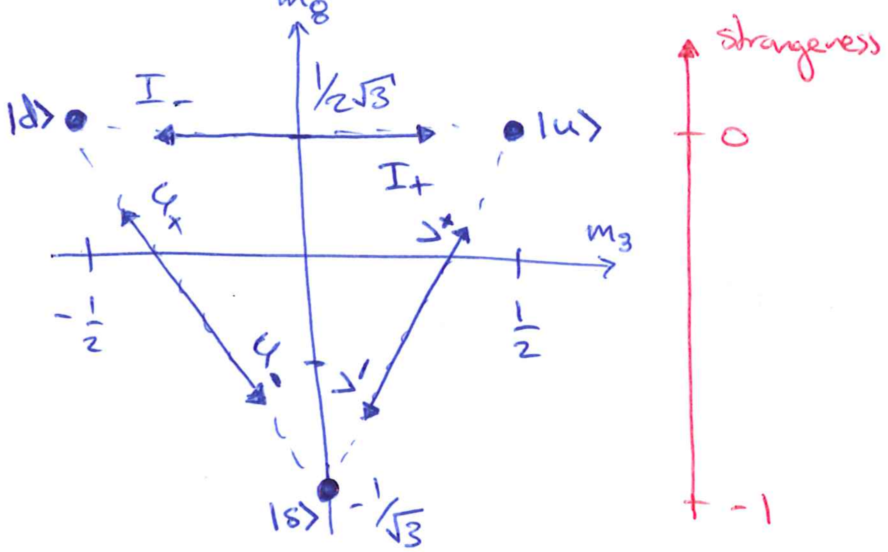
$$|u\rangle = |\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}\rangle$$

$$|d\rangle = |-\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}\rangle$$

$$|s\rangle = |0, -\tfrac{1}{\sqrt{3}}\rangle$$

isospin $T_3 = \frac{1}{2}\lambda_3$

hypercharge $Y = \frac{1}{\sqrt{3}}\lambda_8$



where also $V_+|u\rangle = 0$, etc.

We can work out that,

$$I_\pm |m_3, m_8\rangle \propto |m_3 \pm 1, m_8\rangle$$

$$U_\pm |m_3, m_8\rangle \propto |m_3 \mp \tfrac{1}{2}, m_8 \pm \tfrac{\sqrt{3}}{2}\rangle$$

$$V_\pm |m_3, m_8\rangle \propto |m_3 \pm \tfrac{1}{2}, m_8 \pm \tfrac{\sqrt{3}}{2}\rangle$$

all works the same for SU(3) <u>color</u> which is a better symmetry of the Standard Model

But there is a key difference to SU(2) ...

Under charge conjugation, $T_i^c = -T_i^* = -T_i^T$ (Hermitian)

$\Rightarrow$ commutation relation $[T_i^c, T_j^c] = i f_{ijk} T_k^c$

so there is a <u>complex-conjugate</u> representation as well for every representation!

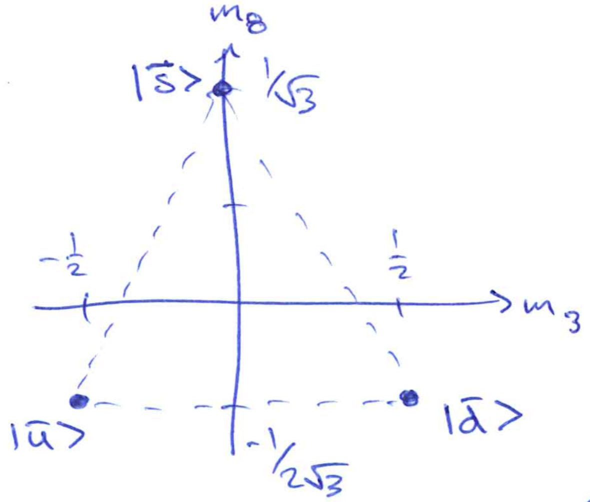
For the fundamental rep. of $SU(3)$ it is also <u>distinct</u>.
(For $SU(2)$, 2^* is the same as 2)

Look at the λ_i $\quad i = 2,5,7 \quad T_i^c = T_i$

$\quad i = 1,3,4,6,8 \quad T_i^c = -T_i$

$\Rightarrow C|m_3, m_8\rangle = |-m_3, -m_8\rangle$ "changes sign of all quantum numbers"

So for "3^*" we identify the states as antiparticles:



$$|\bar{u}\rangle = |-\tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}\rangle$$

$$|\bar{d}\rangle = |\tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}\rangle$$

$$|\bar{s}\rangle = |0, \tfrac{1}{\sqrt{3}}\rangle$$

Question: are the raising/lowering operators the same in 3^* as in 3?

Not quite... they act in different spaces. So operators in a product space can also be written, e.g., as:

$$I_{3\otimes 3^*} = I_3 \otimes \mathbb{1}_{3^*} + \mathbb{1}_3 \otimes I_{3^*}$$

$\rightarrow$ understand from generators + commutation

<u>Singlet</u>

$|0,0\rangle$