# Weak Decay of Charged Pions

## Pseudoscalar mesons

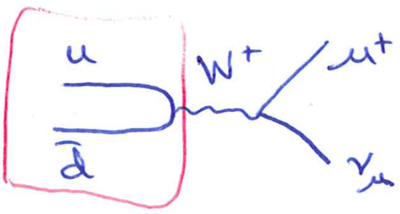
$\pi^\pm$ and $\pi^0$ form a (strong) isospin triplet, with $J^P = 0^-$

Quark content: $\pi^+ \sim u\bar{d}$, $m \approx 140$ MeV, $\tau \approx 26\,\boxed{ns}$ $10^{-9}$s

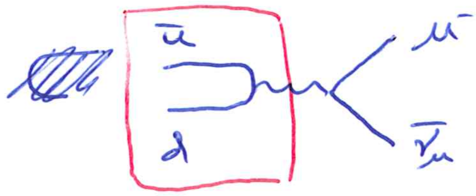
$\boxed{\text{all have spin} = 0}$ $\pi^- \sim \bar{u}d$, " "

$\pi^0 \sim \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$, $m \approx 135$ MeV, $\tau \approx 85\,\boxed{as}$ $10^{-18}$s

Decays:




bound $\pi^+$

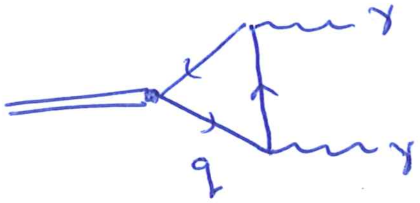



bound $\pi^-$

too light to decay via strong interaction

$\pi^0 \to \gamma\gamma$ (99%)

EM "fast" compared to weak decay




We will take $\pi^- \to \mu^- \bar{\nu}_\mu$ vs $\pi^- \to e^- \bar{\nu}_e$ and compare them.

Phase space: $e^-$ decay should be preferred ($m_{e^-} \ll m_{\mu^-}$)

Experiment: $\mu^-$ more likely by a factor $10^4$

Q: what can we write for the quark current?

→ $u, v, \bar{u}, \bar{v}$ are free particle Dirac spinors: not these

$$\mathcal{M}_F = \frac{G_F}{\sqrt{2}}\, j_\pi^\mu \cdot \bar{u}\gamma_\mu(1-\gamma_5)v$$

$\mu^-/e^-$ $\bar{\nu}_\mu/\bar{\nu}_e$

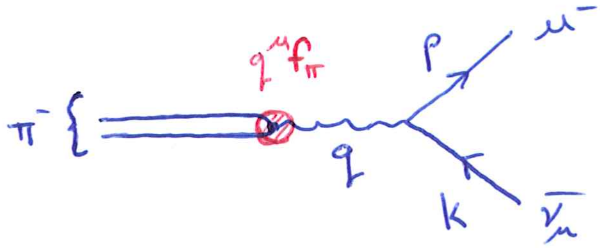
what is this? We can proceed bottom-up or top-down.

$$\mathcal{M}_{SM} = \frac{g_W^2}{8} (?)^\mu \cdot \frac{g_{\mu\nu}}{m_W^2} \cdot \bar{u}\gamma^\nu(1-\gamma_5)v$$

Reason on the basis that $\mathcal{M}$ should be Lorentz-invariant

Note that we do not expect an angular dependence (zero spin $\Rightarrow$ only the transfer momentum $q^\mu$ is available to carry a Lorentz index

single index $\Rightarrow$ vector or axial vector)




$q = p + k$

Now in the $\pi^-$ rest frame, only $m_\pi \approx 140$ MeV is available for the decay.

$$\Rightarrow (\ )^\mu = q^\mu f(q^2) = q^\mu f(m_\pi^2)$$
$$= q^\mu f_\pi \quad \text{where } f_\pi = \text{const.}$$

So noting that $m_\pi^2/m_W^2 \approx 10^{-6}$ we can proceed: (using $m_\mu$ for now)

$$M = \frac{G_F}{\sqrt{2}}(p+k)^\mu f_\pi \bar{u}(p)\gamma_\mu(1-\gamma_5)v(k)$$
$$= \frac{G_F}{\sqrt{2}} f_\pi m_\mu \bar{u}(1-\gamma_5)v$$

(replace $m_\mu \to m_e$ for $\pi^- \to e^- \bar{\nu}_e$)

Dirac equation

$\not{k} v = 0$

$\bar{u} \cdot (\not{p} - m_\mu) = 0$

To get the lifetime, we want to go from a matrix element to a differential decay rate,

$$d\Gamma = W_{fi} \cdot \rho_f$$

phase space: $\prod_f \left( \frac{d^3 p_f}{(2\pi)^3} \cdot V \right)$

So for $\pi \to n$ particles,

$$d\Gamma = (2\pi)^4 \delta^4(p_\pi - \sum_f p_f) \times \frac{|M_{fi}|^2}{2m_\pi} \underbrace{\prod_f \frac{d^3 p_f}{(2\pi)^3 2E_f}}$$

$dLIPS_n$ i.e., Lorentz-invariant phase space for $n$-particle final state

transition probability per unit time:

$$\frac{(2\pi)^4 \delta^4\left(\sum_f p_f - \sum_i p_i\right) V |M_{fi}|^2}{\prod_i (2E_i V) \prod_f (2E_f V)}$$

transition amplitude: $M_{fi}$ from theory, Feynman rules

So simplifying for a 2-particle final state ($\pi^- \to \ell^- \bar{\nu}_\ell$)

$$d\Gamma = \frac{\langle |M|^2 \rangle}{2m_\pi} \frac{d^3p}{(2\pi)^3 2E_\mu} \frac{d^3k}{(2\pi)^3 2E_\nu} (2\pi)^4 \delta(q-(p+k))$$

Note $\langle\,\rangle$ is then just the final-state spin sum

Using the Dirac spinor completeness relations,

$$\langle |M|^2 \rangle = \frac{G_F^2}{2} f_\pi^2 m_\mu^2 \,\mathrm{Tr}\left[(\not{p}+m_\mu)(1-\gamma_5)\not{k}(1+\gamma_5)\right]$$

$$= 2 \times \frac{4}{2} G_F^2 f_\pi^2 m_\mu^2 (p\cdot k)$$

using $\mathrm{Tr}[\not{a}\not{b}] = 4(a\cdot b)$

$\mathrm{Tr}[\gamma_5 \not{a}\not{b}] = 0$

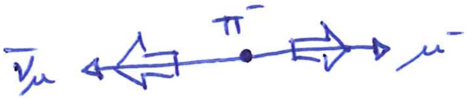
$\mathrm{Tr}[\text{odd}\ \#\ \gamma^\mu] = 0$

$\mathrm{Tr}[\gamma_5] = 0$

and $\gamma_5^2 = \mathbb{1}$

Now note that in the $\pi^-$ rest frame, $\vec{k} = -\vec{p}$

$$\Rightarrow p\cdot k = E_\mu E_\nu + |\vec{k}|^2 = E_\nu (E_\mu + E_\nu)$$




note: implies $\mu^-_R$ (singlet state)

$$|\vec{p}|^2 = |\vec{k}|^2 = E_\nu^2 = E_\mu^2 - m_\mu^2 \Rightarrow E_\mu = \sqrt{E_\nu^2 + m_\mu^2}$$

and $d^3\vec{k} = |\vec{k}|^2 d|\vec{k}|\, d\Omega = 4\pi E_\nu^2 dE_\nu$

since no directional dependence

Putting this together,

$$\Gamma = \int d\Gamma = \frac{G_F^2 f_\pi^2 m_\mu^2}{(2\pi)^2 2m_\pi} \int \frac{d^3p\, d^3k}{E_\mu E_\nu} E_\nu (E_\mu + E_\nu)\, \underbrace{\delta^3(\vec{k}+\vec{p})}_{\text{momentum conservation}}\, \underbrace{\delta(m_\pi - (E_\mu + E_\nu))}_{\text{energy conservation}}$$

easy

Now use $\delta(f(x)) = \sum_{x_0} \frac{\delta(x-x_0)}{\left|\frac{\partial f}{\partial x}\right|_{x_0}}$ where $x_0$ are the zeros of $f$

$\Rightarrow$ rewrite the $\delta(m_\pi - (E_\mu + E_\nu))$ as $\delta(E_\nu - E_\nu^{(0)}) \Big/ 1 + \frac{E_\nu^{(0)}}{E_\mu}$ ← also at $E_\nu^{(0)}$

which removes the integrand $\left(1 + \frac{E_\nu}{E_\mu}\right)$

The integral thus simplifies to $4\pi \int dE_\nu\, E_\nu^2\, \delta(E_\nu - E_\nu^{(0)}) = 4\pi (E_\nu^{(0)})^2$

Now $E_\nu^{(0)} = \dfrac{m_\pi^2 - m_\mu^2}{2m_\pi}$ from $E_\nu^{(0)} = m_\pi - \underbrace{\sqrt{m_\mu^2 + E_\nu^2}}_{E_\mu}$

and so $\Gamma = \dfrac{G_F^2 f_\pi^2 m_\mu^2}{(2\pi)^2 2m_\pi} \cdot 4\pi \cdot \dfrac{m_\pi^2}{4}\left(1 - \dfrac{m_\mu^2}{m_\pi^2}\right)^2$

$$= \frac{G_F^2}{8\pi} f_\pi^2 \underbrace{m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2}_{= \frac{m_\mu^2}{m_\pi^3}(m_\pi^2 - m_\mu^2)^2 \text{ as sometimes written}}$$

$G_F$ is universal, but $f_\pi$ still needs to be fixed. The branch $\Gamma(\pi^- \to e^- \bar{\nu}_e)$ should be obtained via $m_\mu \to m_e$, i.e.,

$$\frac{\Gamma(\pi^- \to e^- \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)} = \underbrace{\left(\frac{m_e}{m_\mu}\right)^2}_{\sim 10^{-5}} \left(\frac{\overbrace{m_\pi^2 - m_e^2}^{\sim 20000}}{\underbrace{m_\pi^2 - m_\mu^2}_{\sim 8000}}\right)^2 \approx 1.2 \times 10^{-4}$$

... contrary to the expectation from phase space!

This effect is known as <u>helicity suppression</u>, i.e. the RH antineutrino forces the electron into a <u>positive-helicity</u> state. Since the weak decay mediated by $W^\pm$ couples only to <u>left-handed chiral</u> $SU(2)$ states, this would be the disfavored component in the chiral limit. But chiral symmetry is broken by the lepton mass: this is where $e^-$ and $\mu^-$ become significantly different ($m_\mu/m_e \approx 207$).

The comparatively small electron mass much more nearly approaches the limit ($m_e \to 0$) where helicity and chirality states coincide and the decay would be forbidden.

The muon also must have positive helicity to conserve angular momentum, but due to the larger $m_\mu$ this state has a much larger admixture of the LH chiral state that is required for the weak decay coupling.

$\Rightarrow$ decay of $\pi^-$ into $\mu^-$ is $\sim 10^4$ times more likely than decay to $e^-$

This can also be seen from the decay kinematics. A two-particle final state implies monoenergetic spectra (to conserve both energy and momentum). The electron has $\beta \approx 1$ and so to a good approximation the positive-helicity state corresponds to a RH chiral state. The LH admixture is approximately scaled as $1-\beta \approx 3\times10^{-5}$.

On the other hand, muons from pion decay have $\beta \approx 0.27$ and thus a very large LH chiral component in the positive-helicity state. It is instructive to calculate the energies of the decay products in both reactions, and evaluate the helicity-spinor as a superposition of LH and RH chiral components at the corresponding energies...

Finally note that the pion decay constant $f_\pi$ is related to the wavefunction overlap of the $q\bar{q}$ components that form the meson bound state. Same constant relates to nucleons:

heavier mesons $\Rightarrow$ typically larger decay constants

$$f_\pi^2 m_\pi^2 = -2 m_q \langle q\bar{q} \rangle \quad \text{and} \quad g_A m_N \sim f_\pi \, g_{\pi NN}$$

$f_\pi \approx 130$ MeV, but can be defined w/ $\sqrt{2}$ to get 184 or 92 MeV.

expectation value in the meson state

EFT vertex factor