QED: The Casimir Effect
(force)

?

Quantum Field Theory and Effective Field Theory (EFT)

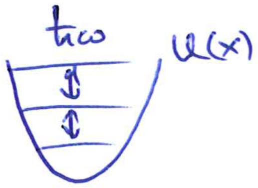
- correlations
- fluctuations
- vacuum structure

$\mathcal{L}$

- valid within some scale/range
- quantitatively captures effects of the neglected degrees of freedom

In QED we can consider field operators as creating/annihilating EM modes.

waves ~ harmonic motion ~ harmonic oscillator



(massive particle) (general)

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega(\underbrace{a^\dagger a}_{\hat{N}\,\sim\,\#\text{ of excitations (photons)}} + \tfrac{1}{2})$$

Eigen-energies for a given mode: $E_n = \hbar\omega_n(N_n + \frac{1}{2})$

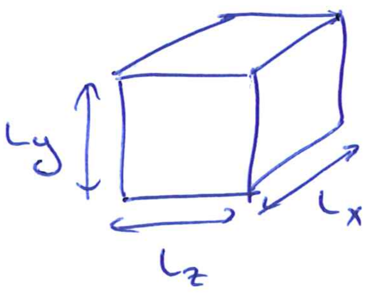
Total energy in the EM field: $E_{tot} = \sum_n E_n = \cancel{\hbar\sum_n \omega_n N_n}$ $\xrightarrow{\;T=0\;(N_n=0)\;}$

$$+ \frac{\hbar}{2}\boxed{\sum_n \omega_n}$$

problem?

Free space: ω allowed with any value (continuous → integral)

Boundary conditions: only specific modes can exist...



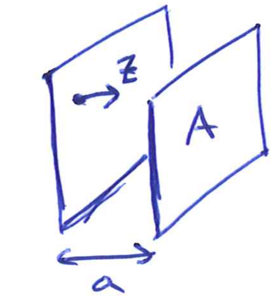
conducting box: $\hat{n}\times\vec{E} = 0$
$\hat{n}\cdot\vec{B} = 0$

$\sin(k_x x) \to \sin\left(\frac{m_x\pi}{L_x}x\right)$ etc.

i.e., $|\vec{k}| = \sqrt{\left(\frac{m_x\pi}{L_x}\right)^2 + \left(\frac{m_y\pi}{L_y}\right)^2 + \left(\frac{m_z\pi}{L_z}\right)^2}$

Energy for a given mode: $\hbar\omega = \hbar|\vec{k}|c$
$= \hbar c\sqrt{k_x^2 + k_y^2 + k_z^2}$

We can treat the 3D case (see Itzykson + Zuber) but let's look at a simplified 1D example. Boundaries only in z, neglect x and y entirely (for now).



3D parallel conducting plates

1D Simplification:

$$k = k_z = \frac{n\pi}{a} \qquad n = 1, 2, 3, \dots$$

mode index

Include 2 polarizations for each mode, and keep zero-temperature limit:

$$E_{tot}(T=0) = 2\,\frac{\hbar}{2}\sum_{n=1}^{\infty}\omega_n = \frac{\hbar c\pi}{a}\sum_{n=1}^{\infty} n$$

$$\overset{\hbar=c=1}{=} \frac{\pi}{a}\underbrace{(1+2+3+\cdots)}_{=-\frac{1}{12}}$$

Plausibility argument (math):

Euler, Grandi's series $1-1+1-1+\cdots = \frac{1}{2}$

$r=1$ is outside the convergence interval, but we can abuse the result $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ $(|r|<1)$

Less trivial: $1+1+1+\cdots$ $(r=+1)$ $\longrightarrow -\frac{1}{2}$

analytic continuation allows extension of functions into a domain where their series representation diverges (see Terry Tao's blog, 10.04.2010, for a discussion in relation to these series)

Plausibility argument (physics): does it make sense to take the upper limit to ∞?

Boundary conditions arise from conductivity
$\rightarrow$ penetration depth drops for large ω (large n)

Short wavelengths (large $k = \frac{2\pi}{\lambda}$) eventually smaller than atomic size
$\rightarrow$ hard x-rays or γ-rays will penetrate easily

Conclusion: our theory breaks down above some energy/frequency scale $\Lambda = k_c = \frac{n_c\pi}{a}$

We could impose a hard cut-off (simply truncate the sum), but it is more likely that higher modes simply have diminishing contributions.

So let's impose an exponential factor with a parameter ϵ that we can later take as small (acknowledging that to do this properly requires additional math...)

$$\text{let} \quad \sum_{n=1}^{\infty} n \longrightarrow \sum_{n=1}^{\infty} n e^{-\epsilon n} = -\frac{\partial}{\partial \epsilon}\left(\sum_{n=1}^{\infty} e^{-\epsilon n}\right)$$

So we interpret $\epsilon \sim n_c^{-1}$ via the <u>critical</u> mode index n_c, with the factor $e^{-\epsilon n} \sim e^{-k/k_c}$ or $e^{-E/\Lambda}$ damping certain modes. We recover the full divergent sum for $\begin{cases}\epsilon \to 0 \\ n_c \to \infty\end{cases}$

$$\text{Thus,} \quad E_{tot}(\epsilon) = -\frac{\pi}{a}\frac{\partial}{\partial \epsilon}\sum_{n=1}^{\infty}\left(e^{-\epsilon}\right)^n \qquad \text{[note sum from } n=1\text{]}$$

$$= -\frac{\pi}{a}\frac{\partial}{\partial \epsilon}\left[\underbrace{\frac{1}{1-e^{-\epsilon}}}_{\frac{1}{1-r}} - \underbrace{1}_{n=0 \text{ term}}\right] = -\frac{\pi}{a}\left(\frac{-e^{-\epsilon}}{\left(e^{\epsilon/2}-e^{-\epsilon/2}\right)^2 \cdot e^{-2\epsilon/2}}\right)$$

$$= +\frac{\pi}{a}\left(2\sinh\frac{\epsilon}{2}\right)^{-2}$$

$$\sinh x = \frac{1}{2}\left(e^{x}-e^{-x}\right) \qquad e^{x} = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$= \frac{\pi}{a}\left[2\left(0 + \frac{\epsilon}{2} + \frac{1}{6}\left(\frac{\epsilon}{2}\right)^3 + \frac{1}{120}\left(\frac{\epsilon}{2}\right)^5 + \dots\right)\right]^{-2}$$

$$= \frac{\pi}{a}\left[\epsilon^{-2}\left(1 + \frac{1}{24}\epsilon^2 + \frac{1}{1920}\epsilon^4 + \dots\right)^{-2}\right]$$

$$= \frac{\pi}{a\epsilon^2}\left[1 - 2\cdot\frac{\epsilon^2}{24} + \mathcal{O}(\epsilon^4)\right] \qquad (1+x)^m = 1 + mx + \dots$$

$$= \frac{\pi}{a}\left[\frac{1}{\epsilon^2} - \frac{1}{12} + \underbrace{\frac{\epsilon^2}{240} - \mathcal{O}(\epsilon^4)}_{\text{cutoff-dependent part}}\right]$$

captures the full divergence in the $\epsilon \to 0$ limit

constant

So we have to re-interpret $\sum_{n=1}^{\infty} n$ a little bit:

$$E_{tot} = \frac{\pi}{a}\underbrace{(1+2+3+\cdots)}_{\text{"}\infty\text{"} - \frac{1}{12}}$$

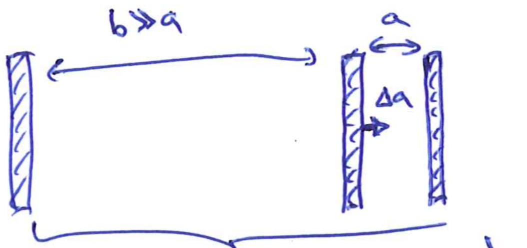
Infinite part: ground-state (zero-point) energy density for the region within the boundaries.

It turns out that the divergent part we have separated out is not of relevance for measurable forces.

The finite part is the measurable Casimir effect.

To see this, recall the cut-off scale $\Lambda = \hbar c k_c = \hbar c \cdot \frac{\pi k_c}{a}$ ϵ^{-1}

and imagine our 1D "box" is embedded in a larger "universe" of fixed size L:



$b + a = L = \text{const.}$

Total energy: $U = E_a + E_b$

$$E_a = \frac{\pi}{a\epsilon^2} - \frac{\pi}{12a} = \frac{\pi}{a}\cdot\frac{a^2 k_c^2}{\pi^2} - \frac{\pi}{12a}$$

$$\Rightarrow U = \underbrace{\frac{k_c^2}{\pi}\underbrace{(a+b)}_{=L}}_{\text{large but constant}} \boxed{-\frac{\pi}{12a}} \underbrace{-\frac{\pi}{12b}}_{\substack{\text{small}\\(b \ll a)}} + \mathcal{O}(k_c^{-2})$$

Casimir energy

cut-off dependence (can also include corrections...)

Now note that we don't measure U directly.. we measure a force associated with changing a slightly:

$F = -\nabla_a U$ gets no contribution from the large constant

Recall $\hbar c \approx 197\ \text{MeV}\cdot\text{fm}$ $\Rightarrow$ Energy: $[\hbar c k] \sim \left[\frac{\hbar c}{a}\right] \sim \text{MeV}$

Force: $[\nabla_a U] \sim \left[\frac{\hbar c}{a^2}\right]$

Pressure: $\left[\frac{\text{Force}}{\text{Area}}\right] \sim \left[\frac{\hbar c}{a^4}\right]$

$F_{1D} = \frac{\pi \hbar c}{12 a^2}$ (attractive)

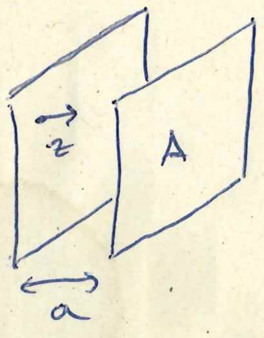
$F_{3D} = \frac{\pi^2 \hbar c}{240 a^4}$ (per unit area, parallel plates $\rightarrow$ "Casimir pressure") ... still attractive, but geometry affects this ...

Casimir Force: Measurements

Theory reminder: $\underbrace{\boxed{\frac{F(a)}{A}}}_{\text{pressure}} = \frac{\pi^2}{240}\frac{\hbar c}{a^4} \approx \underbrace{0.013 \frac{\overset{10\,\mu N}{dyn}}{cm^2}}_{} \cdot \frac{(\mu m)^4}{a^4}$

1.3×10^{-7} N/cm^2 vs. 1 atm ~ 10 N/c[m^2]

$\Rightarrow 1.3 \times 10^{-5}$ mbar for 1 μm separation ($A = 1$ cm^2)



- zero point energy
- vacuum fluctuations
- real photon not needed ($T=0$)

| 2005: >1000 theory
few ~10 expt. |

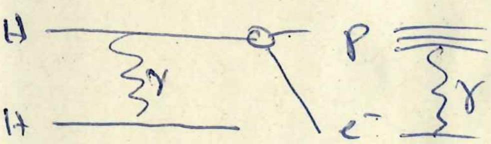
* QED affects macroscopic classical objects

Casimir's "trick": introduce a cutoff

Note that we can view van der Waals, Casimir, Casimir-Polder as specific limits of a unified theory ("dispersion forces")

van der Waals (also London)

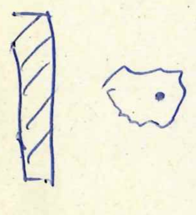
Å/nm-range, rel. retardation not important



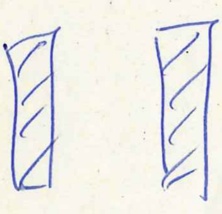
separation < penetration depth

polarizable atom/molecule induced dipole-dipole always attractive

Casimir-Polder



Casimir



larger separations, relativistic retardation is important ⇒ both QM and rel.

can also be repulsive → geom.-dependent

virtual photons do not reach 2nd body but QM fluctuations are correlated:

$$\langle E_k(t,\vec r_{1,2})\rangle = 0 \text{ but } \langle E_k(t,\vec r_1)E_k(t,\vec r_2)\rangle \neq 0$$

Note Casimir imagined an electron w[ith] this force balancing electrostatic repulsion ⇒ falsified by experiment

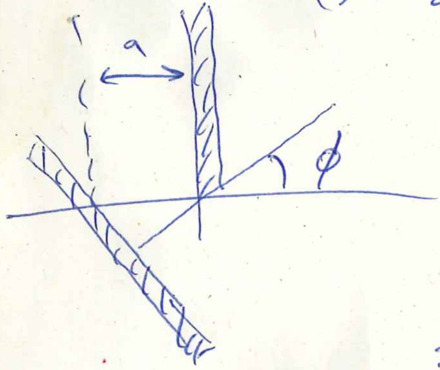
Lifschitz theory: (1956) considers dispersion forces from fluctuations in <u>dissipative</u> media. Ie., E-field fluctuates w/in and outside any medium.

Fluctuation-Dissipation theorem: relates fluctuation spectrum (at equilibrium) to the medium's generalized susceptibility (response to external influences). (cf. $\vec{P} = \epsilon_0 \chi \vec{E}$)

<u>First experiments:</u>

			error
1958,	Sparnaay	(parallel plates)	$\sim 100\%$
1997,	Lamoreaux	(plate-sphere)	5-10%
1998,	Mohideen + Roy	(plate-sphere)	1%

<u>Sparnaay</u>: "...do not contradict Casimir's theoretical prediction" (but does rule out his model of the electron)



<u>Alignment</u> was major challenge

$$\frac{E}{A} = -C \frac{\hbar c}{a^3} \quad \text{at } \phi = 0$$

$$\frac{\pi^2}{720} = C \approx 0.014 \longrightarrow \frac{C(\phi)}{\sin\phi} \quad \text{for } \phi \neq 0$$

$\sim 0.007 - 0.012$

[0906.2313]

Effective distance is <u>also</u> a problem: $F \sim \frac{1}{a^4}$

$\Rightarrow$ dominated by zones of closest approach

also hard to know what the precise distance <u>is</u>

Measurement: capacitance, potential $V \Rightarrow \frac{E}{A} = \frac{1}{2}\left(\frac{C}{A}\right)V^2$

small attractive force $\Rightarrow$ increase C $\quad = \frac{1}{2}\left(\frac{\epsilon_0}{a}\right)V^2$

Geometrical effects depend on surface energy

$\Rightarrow$ similar for Casimir + electrostatic

Surface roughness, let $z_1 = A_1 f_1(x_1, y_1)$, $z_2 = a + A_2 f_2(x_2, y_2)$ $\Big\}$ $\max|f_i(x_i, y_i)| = 1$

[RMP 81.1827 (2009)]

expansion in $\frac{A_i}{a}$ can treat roughness as small angles:

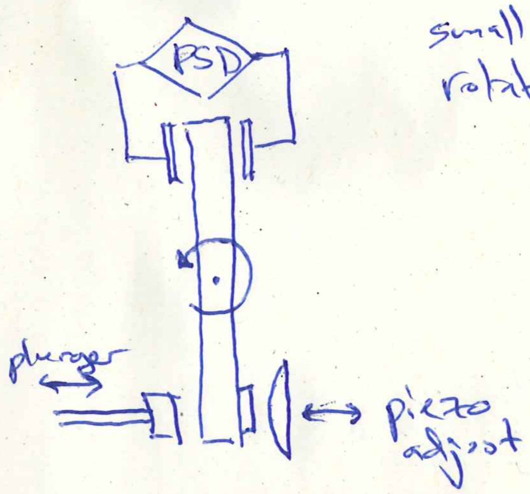
$$\frac{F(a)}{A} \times \underbrace{\left(1 + \frac{10}{a^2}\left[A_1^2\langle f_1^2\rangle + A_2^2\langle f_2^2\rangle - 2\langle f_1 f_2\rangle A_1 A_2\right]\right.}_{} \left. + \mathcal{O}\left(\left(\frac{A_i}{a}\right)^3\right)\right)$$

small angle α : $1 + \frac{10}{3}\left(\frac{\alpha L}{a}\right)^2 + 7\left(\frac{\alpha L}{a}\right)^4 + \dots$ where $2L$ is the plate's characteristic length

Practical challenges:
- surface potentials/static charge
- thin films (insulating oxides)
- roughness, dust
- anharmonic vibrations
- hysteresis

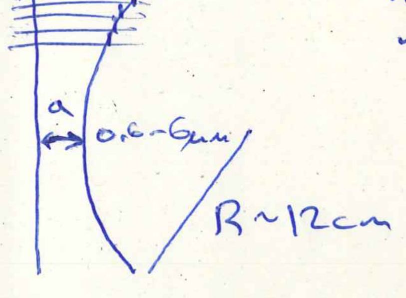
Note: conductive plates $\Rightarrow$ optical alignment methods problematic

<u>Lamoreaux</u>: torsion pendulum, with one curved plate



small Casimir force produces torque $\Rightarrow$ pendulum rotates to finite angle
- adjust the offset voltage to balance the dual capacitor $\Rightarrow$ compensate torque
- measurement records offset voltage
  (capacitors compared/balanced via phase-sensitive AC circuit)

[PRL 78.5 (1997)]



Highly tolerant of misalignment, but $\frac{F(a)}{A}$ no longer uniform $\Rightarrow$ calculate total

Proximity Force Approximation ("theorem")
calculate total force as sum over infinitesimal <u>parallel</u> plates

$$\frac{F(a)}{A} \rightarrow \int r\,dr\,d\phi\,\frac{F(a)}{dA} \Rightarrow F_{tot} = \underbrace{2\pi r}_{\text{factor from geom. curvature}} \cdot \underbrace{\frac{\pi^2}{720}\frac{\hbar c}{a^3}}_{E/A \text{ for } \parallel \text{ plates}}$$

This works surprisingly well, but with the disadvantage that errors are not reliably evaluated

- naively expect deviations of order $\frac{a}{R} \sim 10^{-5}$ (here)
  but need exact/numerical calculation to confirm
  (may not be available) have this, e.g. for plate-cylinder

Finite-temperature: large ($\sim 100\%$) correction!

$$k_B T \cdot \frac{a}{\hbar c} \approx 0.126 \frac{a}{\mu m} \text{ at } 300\,K \left\{ \begin{array}{l} \lesssim \frac{1}{2} \Rightarrow a \lesssim 4\mu m \\ > \frac{1}{2} \Rightarrow a > 4\mu m \end{array} \right.$$

$$F_{tot} \to F_{tot} \times \begin{cases} 1 + \frac{720}{\pi^2}\left(\frac{k_B T a}{\hbar c}\right)^3 \frac{\zeta(3)}{2\pi} - 16\left(\frac{k_B T a}{\hbar c}\right)^4 + \mathcal{O}(a^5) \\ \frac{720}{\pi^2}\left(\frac{k_B T a}{\hbar c}\right) \frac{\zeta(3)}{8\pi} + \mathcal{O}(a^{-1}) \end{cases} \qquad \zeta(3) \approx 1.2$$

$$= \begin{cases} \frac{\pi^3 R}{360} \frac{\hbar c}{a^3} + \underbrace{R\zeta(3)\frac{(k_B T)^3}{\hbar^2 c^2}}_{\text{indp of } a} - \underbrace{\frac{2\pi^3 R}{45} a \cdot \frac{(k_B T)^4}{\hbar^3 c^3}}_{\text{linear in } a} + \mathcal{O}(a^2) \\ \underbrace{\frac{R}{4a^2} k_B T \zeta(3)}_{\text{indp of } \hbar c} \end{cases}$$

indp of $\hbar c$
cf. long-wavelength
limit (R-J) of blackbody

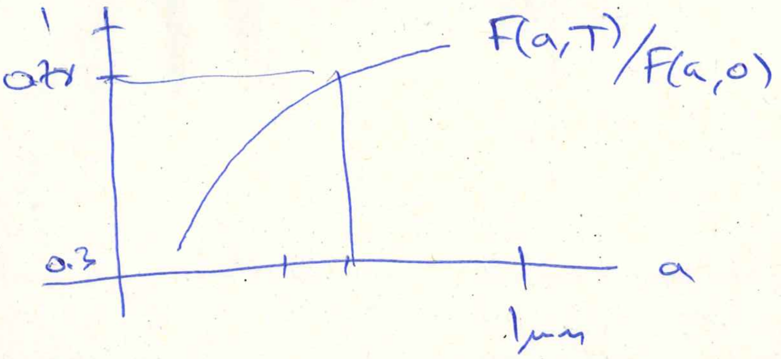
Finite conductivity: recall $\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \Rightarrow$ 10-20% correction

plasma frequency in metals, $\omega_p = \frac{2\pi c}{\lambda_p}$ and for $\left(\frac{\lambda_p}{a}\right)^2 \ll 1$

$$F_{tot} \to F_{tot} \times \Big(\underbrace{1 - \frac{2\lambda_p}{\pi a}}_{\equiv \eta}\Big) = F_{tot} \times \left(1 - \frac{4c}{a\omega_p}\right)$$

In reality the correction to $F(a)$ varies more slowly
$\Rightarrow$ use experimental data for complex index of refraction

Indistinguishable from a calibration error!



For *ideal* metal plates

at smallest separation, $\eta(0.6\,\mu m) \approx 0.78$ (700nm Au)
0.49 Cu

under if Au diffused into Cu, or film was thinner

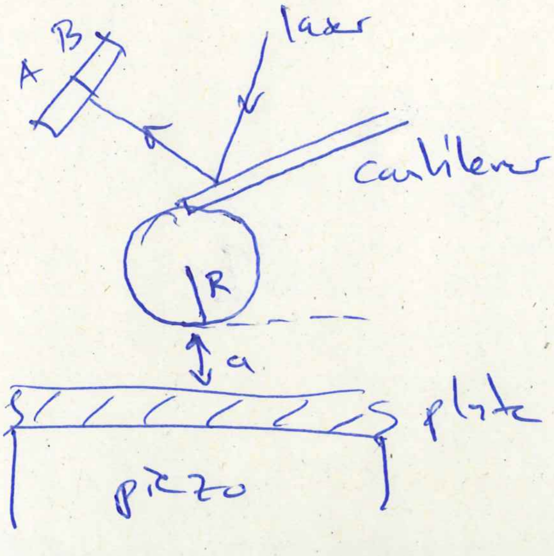
Problems: large residual potential offset, electrostatic force

data not accurate enough to demonstrate finite-T correction

best agreement around 1μm
small thermal correction
skin effect and roughness partially compensate.

measured force (Casimir) up to ~100 μdyn ≈ 1000 pN

Mohideen + Roy AFM: atomic force microscope



bending of cantilever
⇒ laser beam deflects,
split photodiode signal

measured a ~100nm – 1μm
w/ R = 200μm
⇒ forces up to 120pN at closest approach

Required corrections:

- finite conductivity
- roughness
- temperature

1.6 pN deviation (rms) overall, but individual corrections w/ both signs and sizes up to 48 pN