2.7 Critical Behaviour of He-II at T_λ

Properties near T_c are determined by quantities that go to zero like the order parameter and quantities that diverge like susceptibilities

Landau theory of continuous phase transitions (1937, 1965)

- idea: expansion of free energy in T in terms of the order parameter
- near T_c one should find the following laws with the reduced temperature $t = (T - T_\mathrm{c})/T_\mathrm{c}$

Quantity	Power Law	Critical Exponent
specific heat	$C_V \propto \lvert t\rvert^{\alpha}$	$\alpha = 0$
order parameter	$\Phi \propto \lvert t\rvert^{\beta}$	$\beta = 1/2$
susceptibility	$\chi \propto \lvert t\rvert^{-\gamma}$	$\gamma = 1$
correlation length	$\xi \propto \lvert t\rvert^{-\nu}$	$\nu = 1/2$

Landau type theories:
– van der Waals theory for liquid – gas transition
– Curie-Weiss theory of ferromagnetism
– Ginzburg-Landau theory of superconductivity

2.7 Critical Behaviour of He-II at T_λ

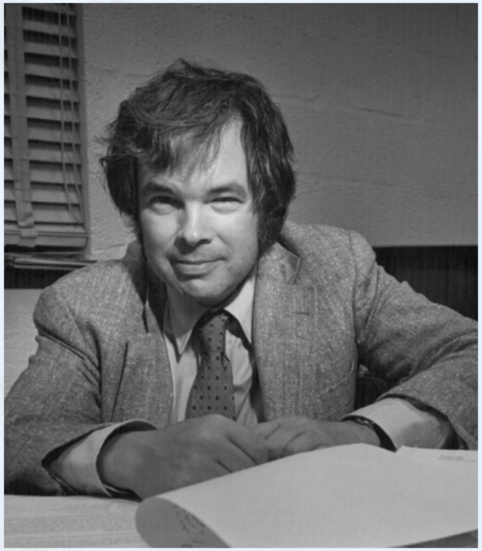
Problem: fluctuations are not included, but they are increasingly important towards T_c

➡ every Landau-type theory breaks down near T_c

Ginzburg criterion

The condition under which a Landau-type theory holds is that fluctuations
of the order parameter are small in comparison of the mean value of the order parameter

for He-II: coherence length is very small ⟶ Ginzburg criterion is "always" violated



Kenneth G. Wilson

Renormalization group

Despite of the short-comings of the Landau universal theory of phase transitions, it was realized that it is possible to assign different physical systems to universality classes, characterized by a set of critical exponents

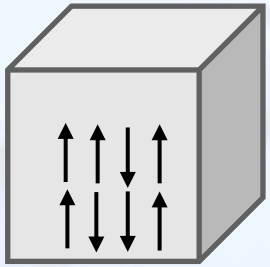
The larger framework is: renormalization group and quantum field theory

different classes are defined by: dimension of system d,
degrees of freedom of order parameter n,
interaction length compared to coherence length

2.7 Critical Behaviour of He-II at T_λ

a few examples:

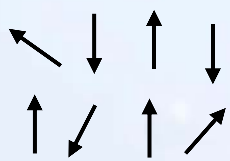
Ising 3 D



$d = 3$
$n = 1$

in this universality class liquid-solid transition fall as well

Heisenberg 2 D



$d = 2$
$n = 3$

at each lattice point each spin can point in 3 direction

x–y 3 D

He-II
superconductors

$d = 3$
$n = 2$

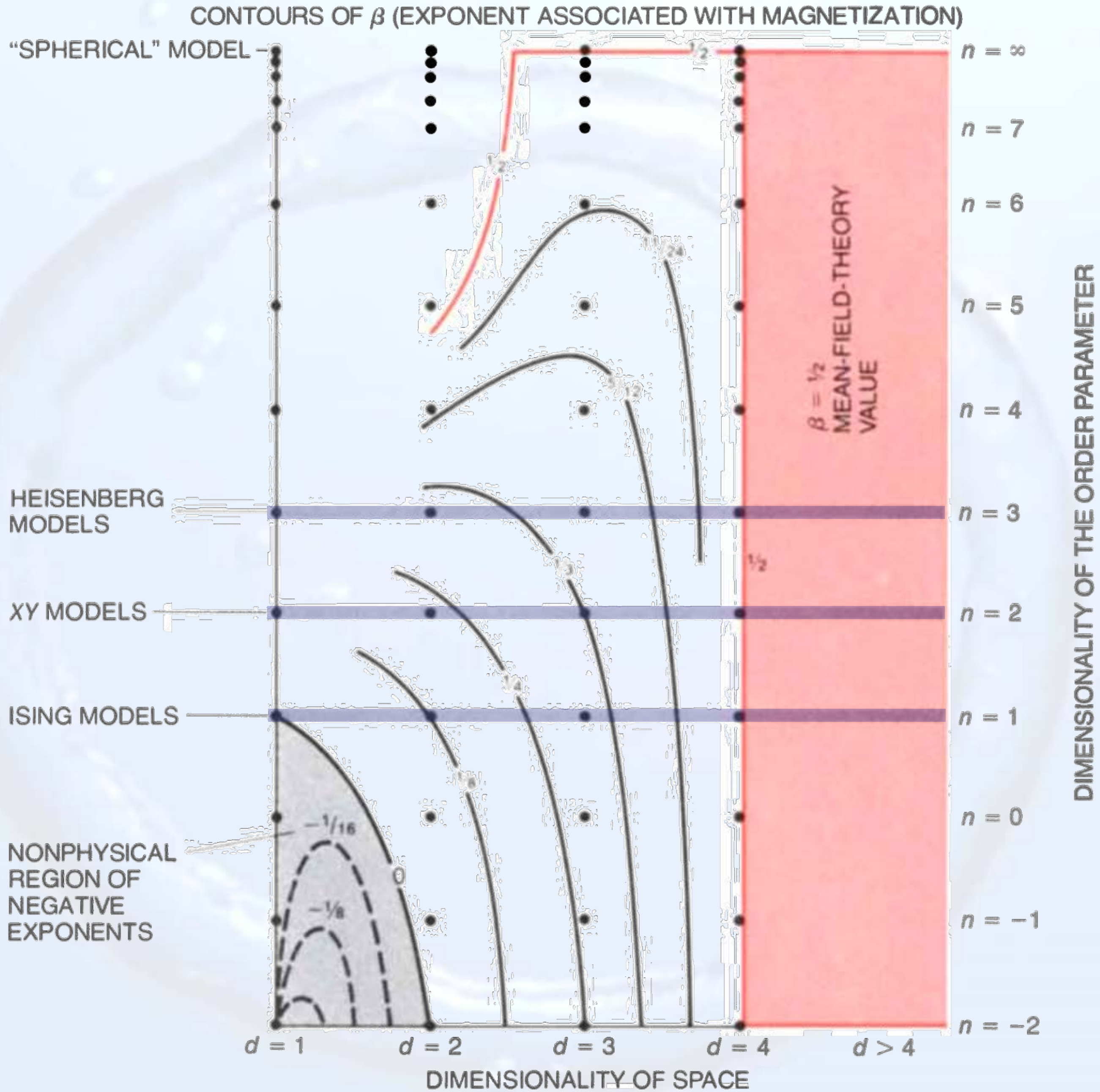
magnitude and phase of wave function

each universality class is described by a set of critical exponents and are connected by sum rules like $\alpha + 2\beta + \gamma = 2$

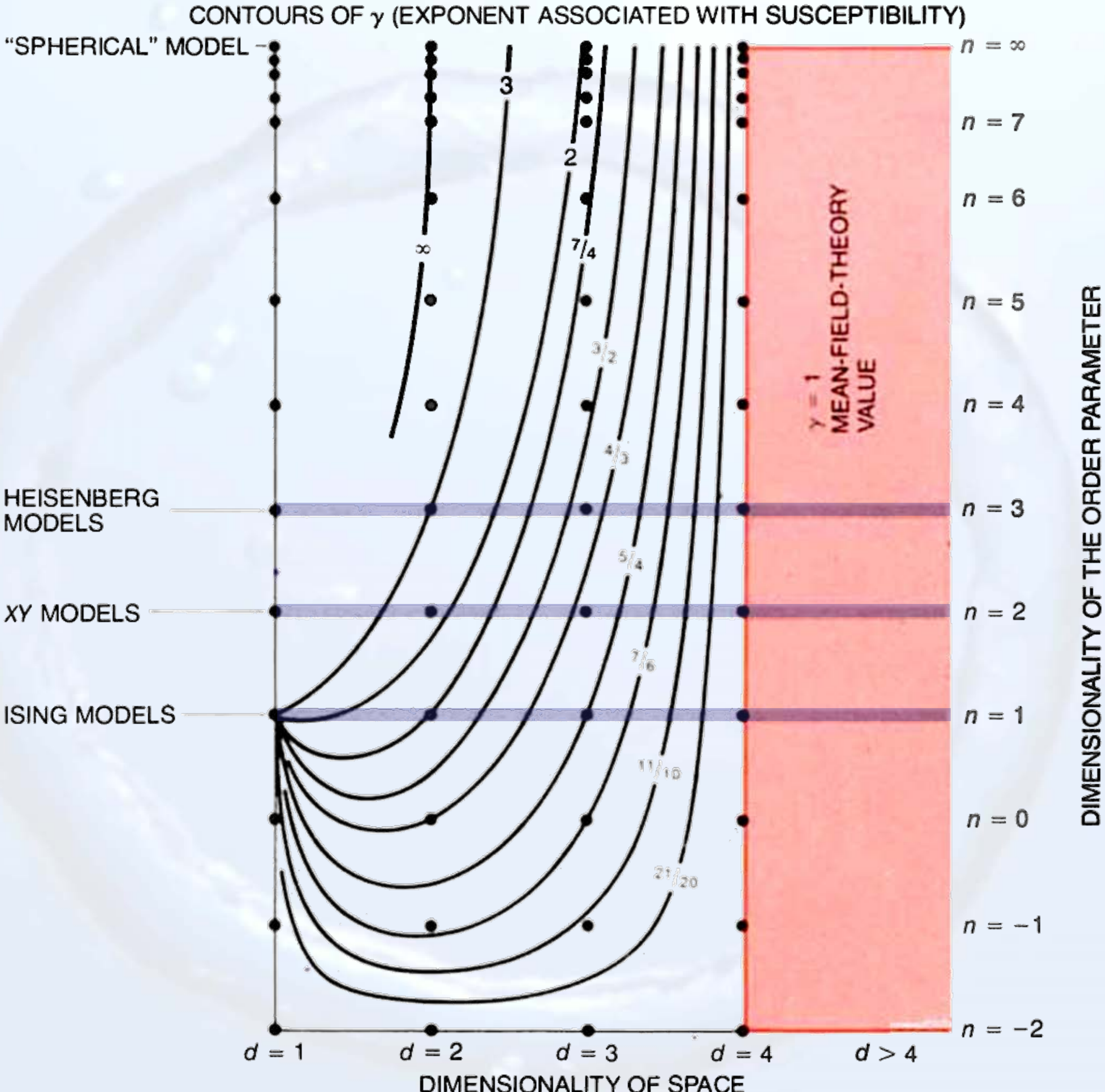
2.7 Critical Behaviour of He-II at T_λ

UNIVERSALITY CLASS		THEORETICAL MODEL	PHYSICAL SYSTEM	ORDER PARAMETER
$d = 2$	$n = 1$	Ising model in two dimensions	Adsorbed films	Surface density
	$n = 2$	XY model in two dimensions	Helium-4 films	Amplitude of superfluid phase
	$n = 3$	Heisenberg model in two dimensions		Magnetization
$d > 2$	$n = \infty$	"Spherical" model	None	
$d = 3$	$n = 0$	Self-avoiding random walk	Conformation of long-chain polymers	Density of chain ends
	$n = 1$	Ising model in three dimensions	Uniaxial ferromagnet	Magnetization
			Fluid near a critical point	Density difference between phases
			Mixture of liquids near consolute point	Concentration difference
			Alloy near order-disorder transition	Concentration difference
	$n = 2$	XY model in three dimensions	Planar ferromagnet	Magnetization
			Helium 4 near superfluid transition	Amplitude of superfluid phase
	$n = 3$	Heisenberg model in three dimensions	Isotropic ferromagnet	Magnetization
$d \leqslant 4$	$n = -2$		None	
	$n = 32$	Quantum chromodynamics	Quarks bound in protons, neutrons, etc.	

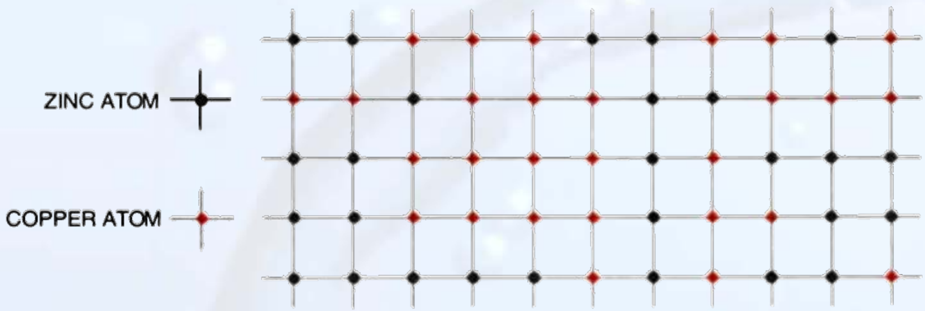
2.7 Critical Behaviour of He-II at T_λ



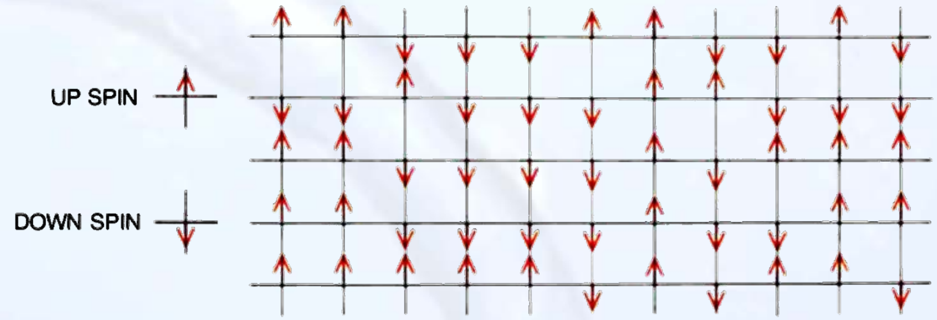
2.7 Critical Behaviour of He-II at T_λ



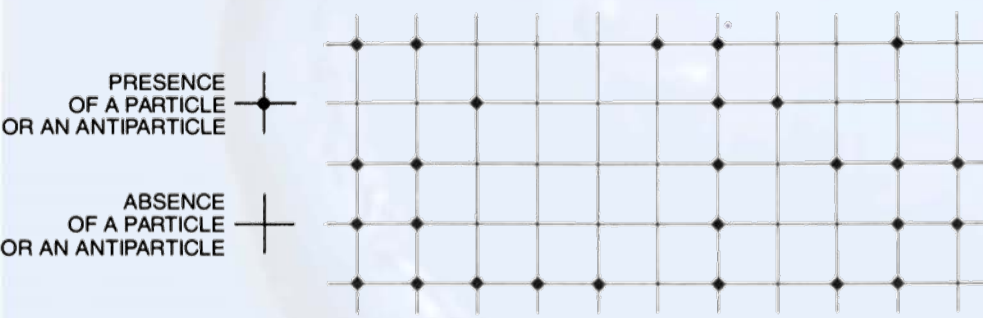
2.7 Critical Behaviour of He-II at T_λ



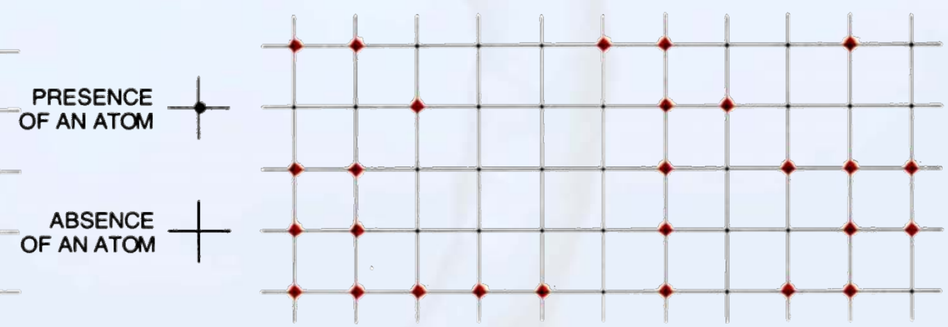
BRASS NEAR ORDER-DISORDER TRANSITION



UNIAXIAL FERROMAGNET



QUANTUM FIELD THEORY



FLUID NEAR CRITICAL POINT

2.7 Critical Behaviour of He-II at T_λ

critical exponents expected for X-Y 3D model:

$\alpha = -0.0146(8)$ ⟵

$\beta = 0.3485(2)$ ⟵

$\gamma = 1.3177(5)$

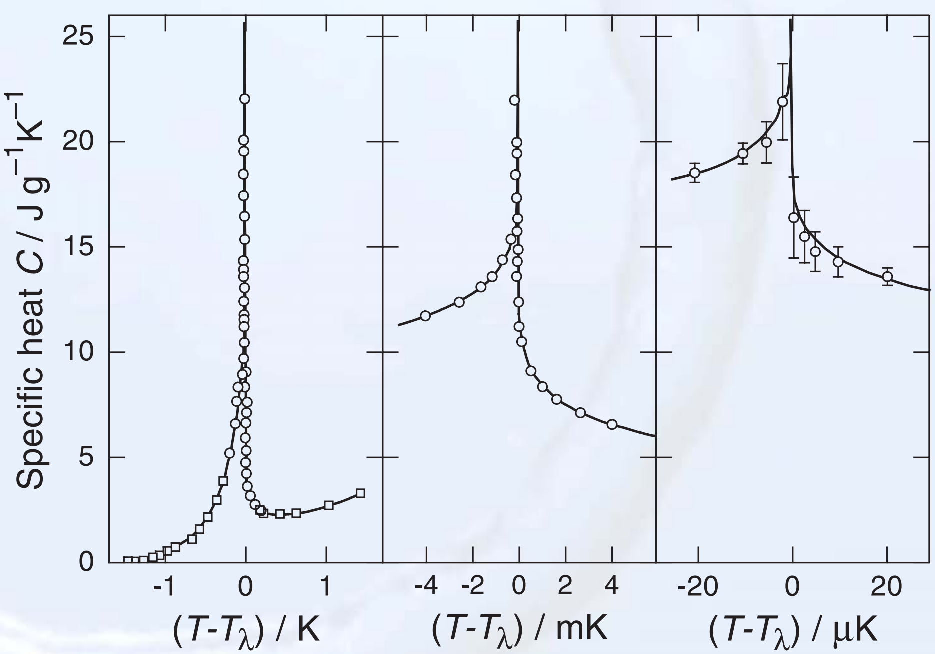
$\delta = 4.780(2)$

$\nu = 0.67155(27)$ ⟵

$\eta = 0.0380(4)$

Experiments near T_λ

a) specific heat



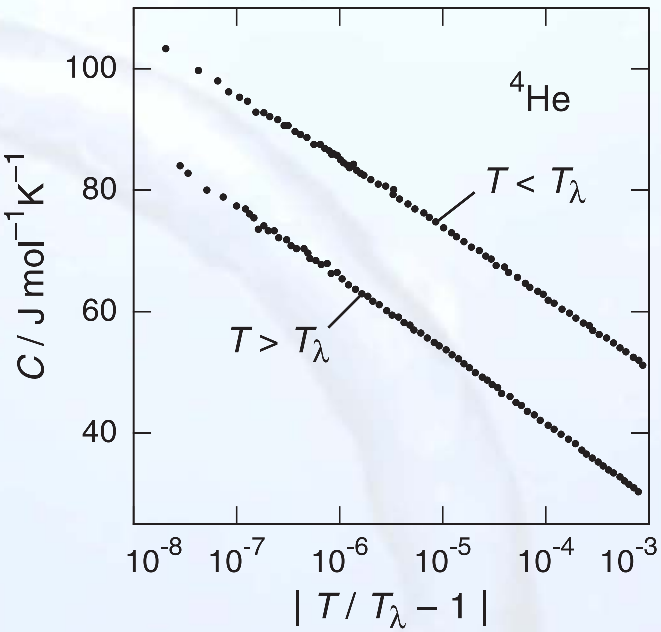
scale going from K to μK

2.7 Critical Behaviour of He-II at T_λ

power law in the vicinity of T_λ ?

data plotted a C_V vs $\log t = \log|T/T_\lambda - 1|$

data can be approximated by $C_V \propto \log t$



logarithmic divergences?

comparison with RGT

expected scaling for He-II

$$C = B + A\,\frac{t^{-\alpha}}{\alpha}\left(1 - D\sqrt{t}\right)$$

$A,\ B$ and D are constants

with critical exponent expected $\alpha = -0.0146(8)$

➡ expansion in α $\quad t^{-\alpha} = \mathrm{e}^{-\alpha \ln t} \approx 1 - \alpha \ln t \quad$ expansion justified because of small α

experimental result $\alpha \approx -0.013 \pm 0.003$

2.7 Critical Behaviour of He-II at T_λ

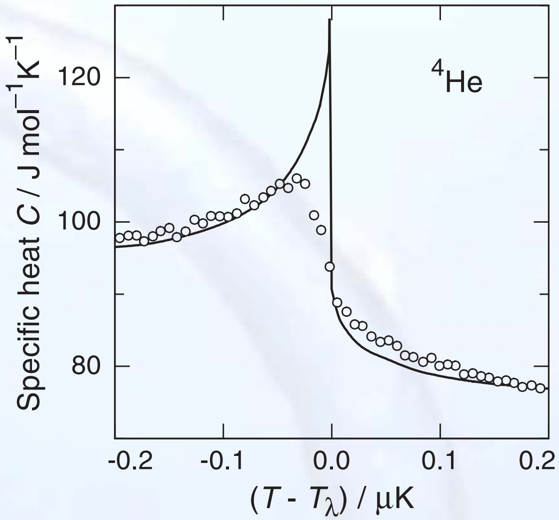
Higher precision experiments near T_λ are needed

measurement on earth

Problems:

gravitation → level height dependence

walls of vessel → first layer solid and healing length diverges with diverges near T_λ with $\xi = \xi_0\, t^{-\nu}$ with $\nu = 0.67155(27)$

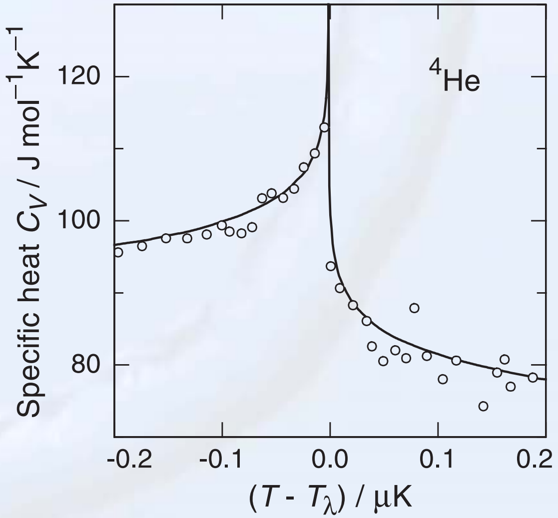


measurement on space shuttle

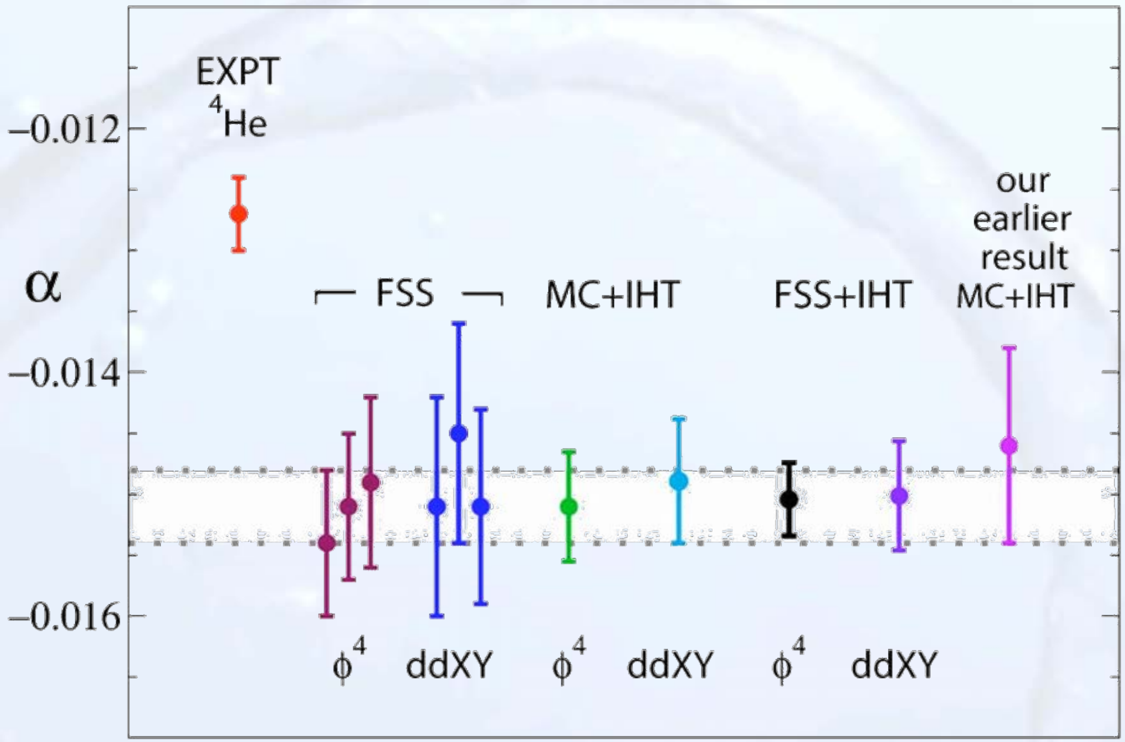
Problems:
cosmic rays → time varying background (heating of thermometer)

Data shown, after sophisticated analysis

→ still somewhat noisy



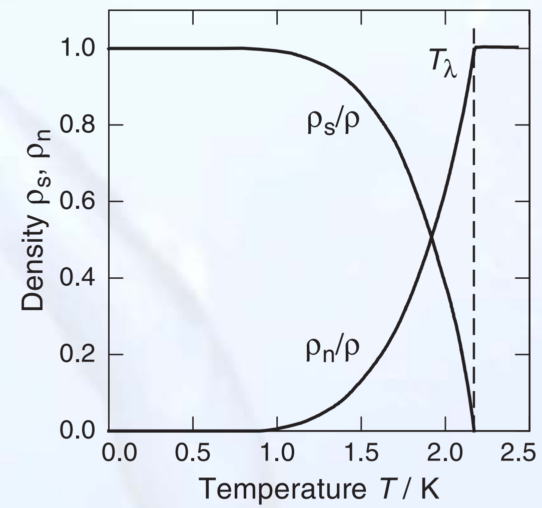
2.7 Critical Behaviour of He-II at T_λ



comparison between space shuttle data and different calculations of α

➡ discrepancy between data and theory outside error bars: reason unknown

2.7 Critical Behaviour of He-II at T_λ



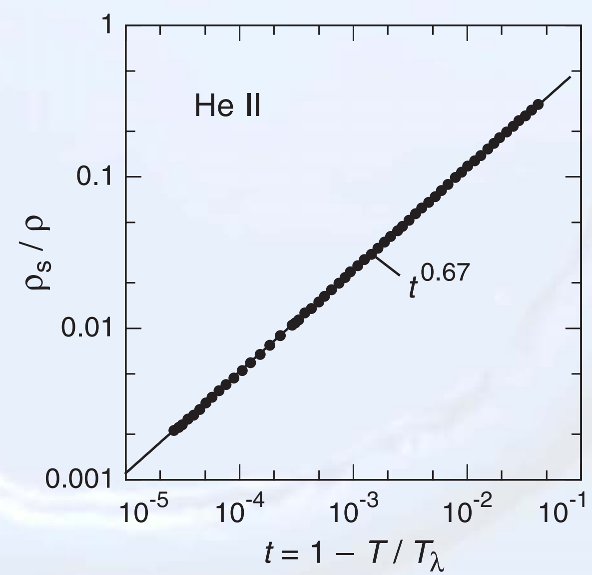
b) Order parameter

amplitude of wave function

$$\psi(\boldsymbol{r}) = \psi_0 \, \mathrm{e}^{\mathrm{i}\varphi(\boldsymbol{r})} \longrightarrow \Psi_0 = \sqrt{\varrho_\mathrm{s}}$$

expected:

$\varrho_\mathrm{s} = t^{2\beta}$ with $\beta = 0.3485(2)$



determined with second sound

$\varrho_\mathrm{s} = t^{0.67}$

➡ excellent agreement

2.7 Critical Behaviour of He-II at T_λ

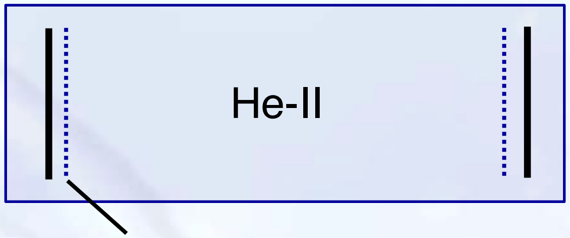
c) Healing length

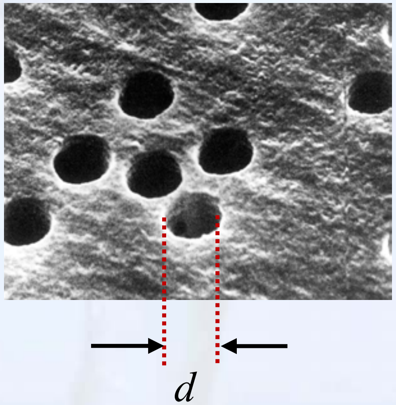
again, second sound measurements
and measurements on thin films

expected:

$$\xi = \xi_0 \, t^{-\nu} \quad \text{with} \quad \nu = 0.67155(27)$$

Helmholtz resonator

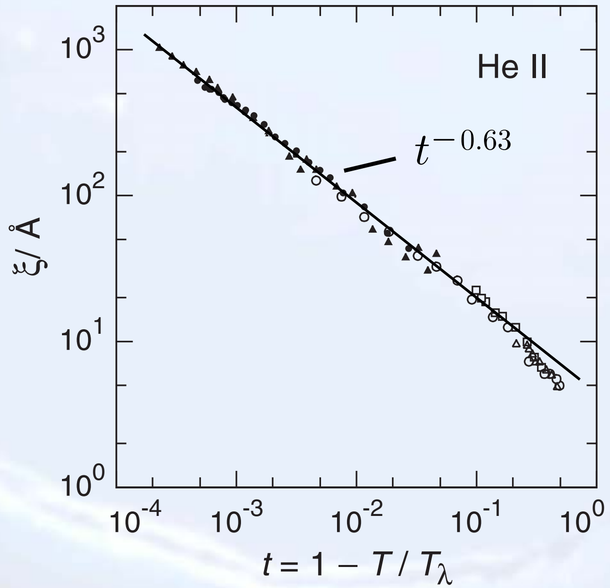




$$\xi_0 = 2.8 \pm 0.5 \, \text{Å}$$

$$\nu \; = \; 0.63$$

➡ excellent agreement



second sound vanishes
for $\xi > d$

3) Normalfluid ^{3}He

First measurements 1949 (from natural abundance)

Landau theory of Fermi liquids 1956-1958 ⟶ prediction of zero sound and collision-less spin waves

3.1 Ideal Fermi-Gas

Schrödinger equation $$-\frac{\hbar^2}{2m}\nabla^2\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r})$$

ansatz: $$\psi(\boldsymbol{r}) = \frac{1}{\sqrt{V}}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \quad \curvearrowright \quad E_k = \frac{\hbar^2 k^2}{2m}$$

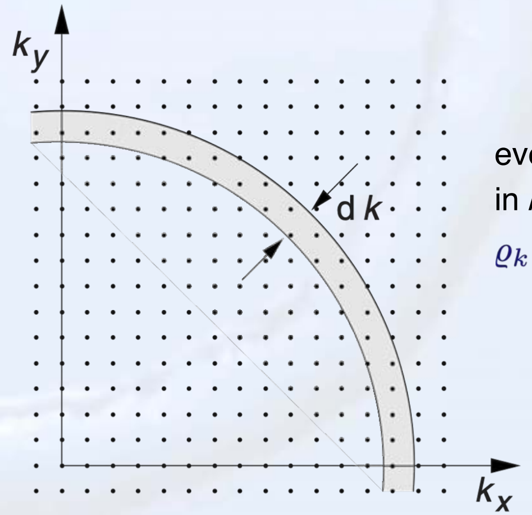
periodic boundary conditions: $\psi = \psi(x+L,y,z) = \psi(x,y+L,z) = \psi(x,y,z+L)$

$$k_x = \frac{2\pi}{L}n_x\,, \quad k_y = \frac{2\pi}{L}n_y\,, \quad k_z = \frac{2\pi}{L}n_z$$ with n_x, n_y, n_z integer values

density of states

$$\mathcal{D}(k)\mathrm{d}k = \varrho_k 4\pi k^2 \mathrm{d}k = \frac{V}{2\pi^2}\,k^2\mathrm{d}k$$

$$D(k) = \frac{2\mathcal{D}(k)}{V} = \frac{k^2}{\pi^2}$$ density k space density per volume for 2 spin states



even distribution in k space density $\varrho_k = (L/2\pi)^3 = V/(2\pi)^3$

$$D(E) = D(k)\frac{\mathrm{d}k}{\mathrm{d}E} = \frac{(2m)^{3/2}\sqrt{E}}{2\pi^2\hbar^3} \propto \sqrt{E}$$