# 2.3 Properties of He-II described using the two-fluid model

## (i) First sound

with (i) and (iii)

$$v = v_1 = \sqrt{\left(\frac{\partial p}{\partial \varrho}\right)_S}$$

$$\left(\frac{v}{v_1}\right)^2 - 1 = 0$$

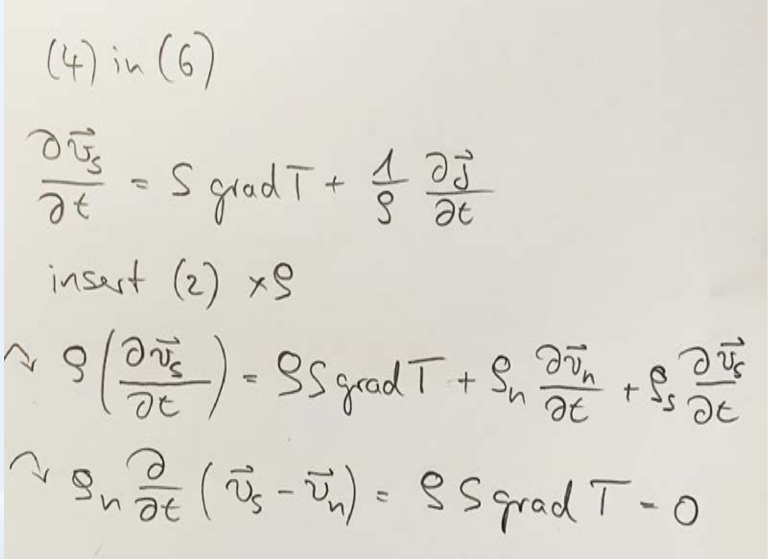
$$\varrho' \neq 0 \quad S' = 0 \longrightarrow \mathrm{grad}\, T = 0$$

as usual for ordinary (first) sound

insert (4) into (6)

$$\varrho_\mathrm{n} \frac{\partial}{\partial t} \underbrace{(\boldsymbol{v}_\mathrm{n} - \boldsymbol{v}_\mathrm{s})}_{\boldsymbol{v}_\mathrm{n} = \boldsymbol{v}_\mathrm{s}} = \varrho S \,\mathrm{grad}\, T = 0$$

→ superfluid and normalfluid component are in phase



(4) in (6)

$$\frac{\partial \vec{v}_s}{\partial t} = S \,\mathrm{grad}\, T + \frac{1}{\varrho} \frac{\partial \vec{j}}{\partial t}$$

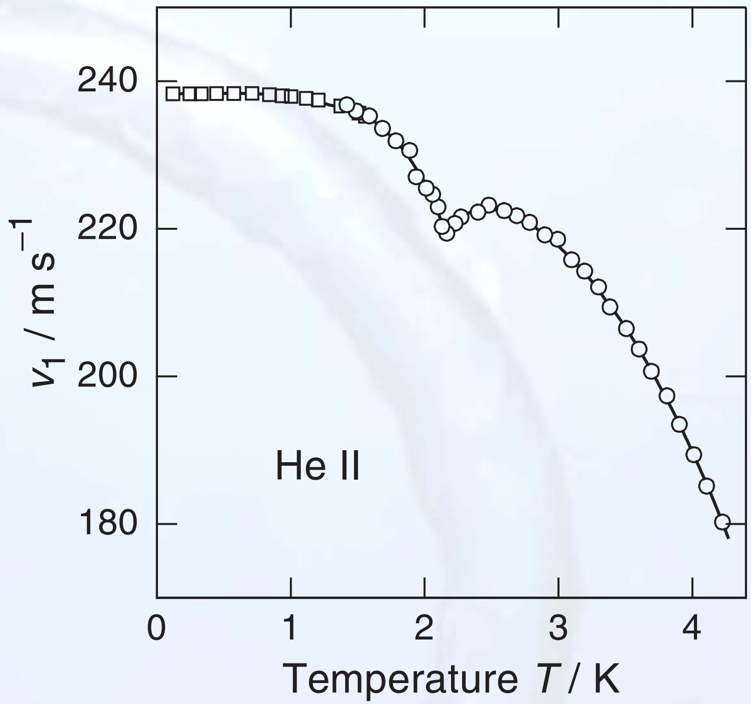
insert (2) $\times \varrho$

$$\rightsquigarrow \varrho \left(\frac{\partial \vec{v}_s}{\partial t}\right) = \varrho S \,\mathrm{grad}\, T + \varrho_n \frac{\partial \vec{v}_n}{\partial t} + \varrho_s \frac{\partial \vec{v}_s}{\partial t}$$

$$\rightsquigarrow \varrho_n \frac{\partial}{\partial t} (\vec{v}_s - \vec{v}_n) = \varrho S \,\mathrm{grad}\, T = 0$$

# 2.3 Properties of He-II described using the two-fluid model

## (i) First sound



- for $T \to 0$: $v_1 \approx 238\,\mathrm{m\,s^{-1}}$.

  ➡ only density variation ➡ almost ordinary sound

- for $T \to T_\lambda$: corrections become important

# 2.3 Properties of He-II described using the two-fluid model

## (ii) Second sound

with (ii) and (iii) we find

$$v = v_2 = \sqrt{\frac{\varrho_\mathrm{s}}{\varrho_\mathrm{n}} S^2 \left(\frac{\partial T}{\partial S}\right)_\varrho}$$

$$\left(\frac{v}{v_2}\right)^2 - 1 = 0$$

$$S' \neq 0 \;,\quad \varrho' = 0$$

$$\hookrightarrow \quad \mathrm{grad}\, p = 0$$

with (4)

$$\frac{\partial \boldsymbol{j}}{\partial t} = -\mathrm{grad}\, p \overset{!}{=} 0 \quad \longrightarrow \quad \frac{\partial \varrho_\mathrm{n} \boldsymbol{v}_\mathrm{n}}{\partial t} + \frac{\partial \varrho_\mathrm{s} \boldsymbol{v}_\mathrm{s}}{\partial t} = 0$$

$$\varrho_\mathrm{n} \boldsymbol{v}_\mathrm{n} + \varrho_\mathrm{s} \boldsymbol{v}_\mathrm{s} = 0$$

no mass flow in closed vessel

↷ $\varrho_\mathrm{n} \uparrow \;,\quad \varrho_\mathrm{s} \downarrow$ counter flow and no density variation

→ temperature wave

# 2.3 Properties of He-II described using the two-fluid model

$$\boxed{v_2 = \sqrt{\frac{\varrho_\mathrm{s}}{\varrho_\mathrm{n}} S^2 \left(\frac{\partial T}{\partial S}\right)_\varrho} = \sqrt{\frac{\varrho_\mathrm{s}}{\varrho_\mathrm{n}} \frac{T\,S^2}{C_p}}}$$

density variation in phonon gas

possibility to determine $\varrho_\mathrm{s}/\varrho_\mathrm{n}$

ultra-low temperatures:

excitations at $T \to 0$ are only longitudinal phonons

Landau

Debye model $A = 2\pi^2 k_\mathrm{B}^4/(45\hbar^3 v_1^3 \varrho)$

$$C_p = 3AT^3$$

$$S = AT^3$$

in addition

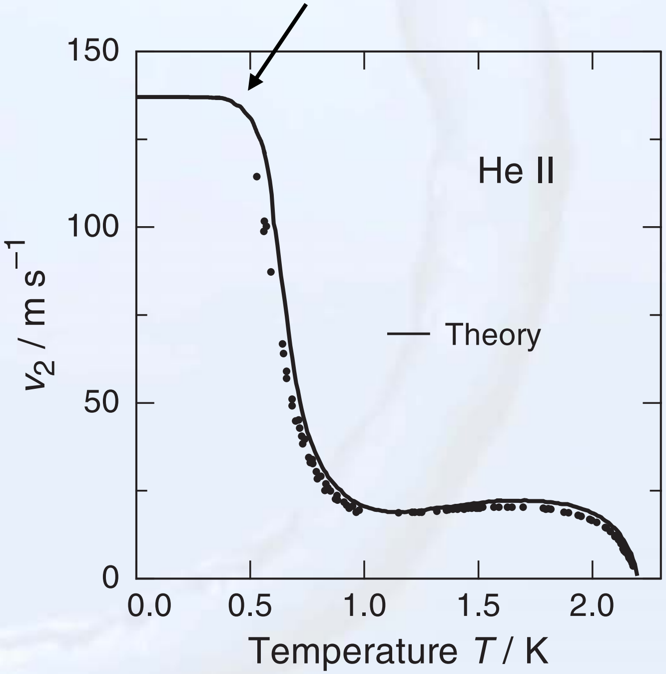
$$\varrho_\mathrm{s} \approx \varrho$$

$$\varrho_\mathrm{n} = A\varrho T^4/v_1^2$$

for $T \to 0$:

$$v_2 \to v_1/\sqrt{3} \approx 137\,\mathrm{m\,s^{-1}}$$

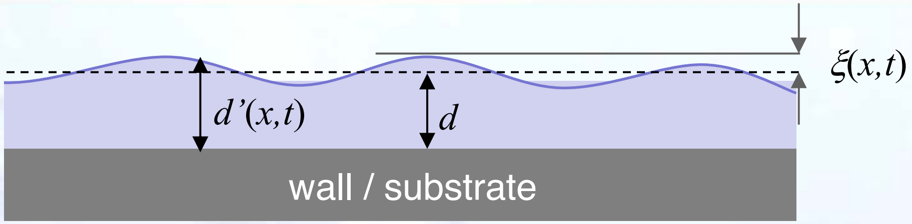
for $T \to 0$ second sound difficult determine since $\varrho_\mathrm{n} \to 0$



# 2.3 Properties of He-II described using the two-fluid model

## (iii) Third sound

sound propagating in thin films



$d'(x,t) = d + \xi(x,t)$

mean film thickness

assumptions: thin films $\boldsymbol{v}_\mathrm{n} = 0$

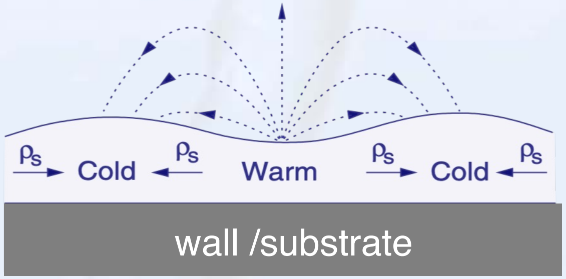
$\lambda \gg d$ ⟶ motion parallel to substrate in x-direction $(v_y = v_z = 0)$

$\mathrm{grad}\, T \approx 0$ (questionable ?)

$$\Rightarrow \quad \frac{\partial^2 \xi}{\partial t^2} = f d \frac{\varrho_\mathrm{s}}{\varrho} \frac{\partial^2 \xi}{\partial x^2} \quad \Rightarrow \quad \boxed{v_3^2 = \frac{\varrho_\mathrm{s}}{\varrho} 3 g z}$$

v. d. Waals force

height over liquid level



problem: evaporation and condensation

⟹ increases the amplitude and changes velocity

can be taken into account

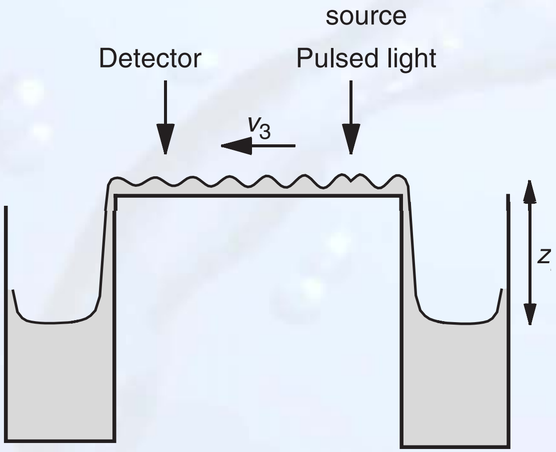
$$v_3^2 \approx \frac{\varrho_\mathrm{s}}{\varrho} 3 g z \left(1 + \frac{TS}{L}\right)$$

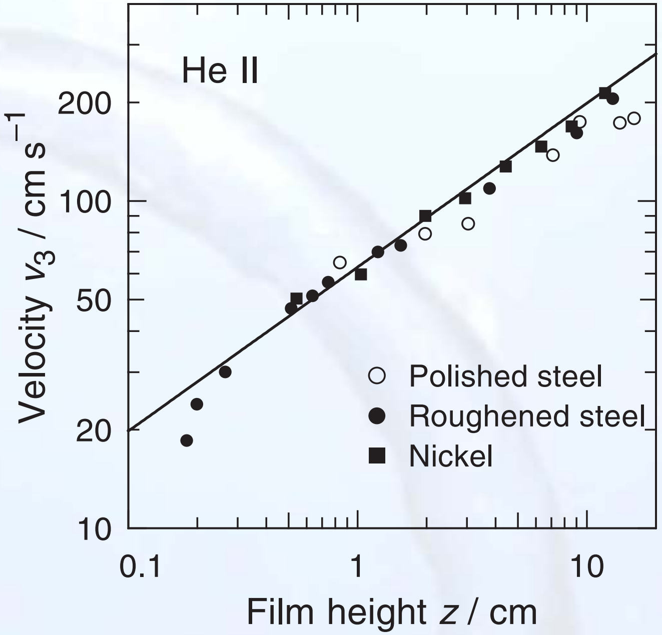
$TS/L$ = 0.01 at 1 K
$TS/L$ = 0.15 at $T_\lambda$

not really a problem

# 2.3 Properties of He-II described using the two-fluid model

## 3rd sound experiment





## Procedure

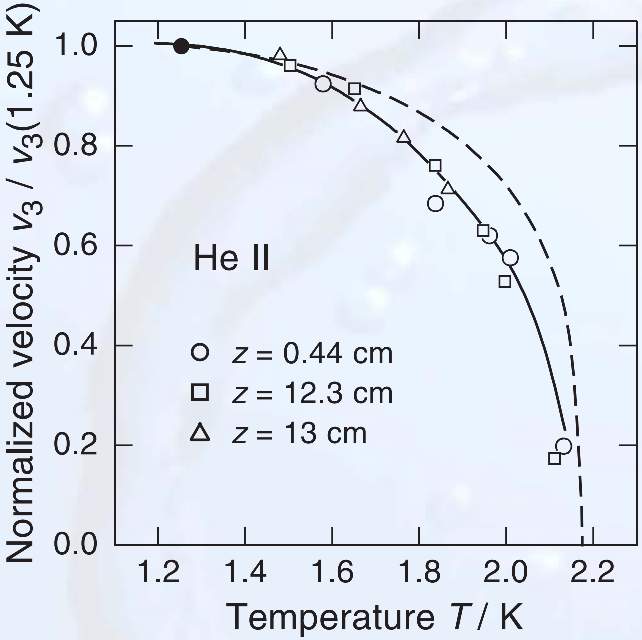
- periodic local heating
- $\varrho_s$ flows to warm location ⟶ thickness changes
- surface wave ≙ 3rd sound
- optical detection of thickness

## Measurement and results

- 3rd sound velocity vs. $z$ (log/log plot)
- different surfaces: $v_3$ almost independent
- line ≙ theory $v_3 \propto \sqrt{z}$
- good agreement except for very thick films

# 2.3 Properties of He-II described using the two-fluid model

3rd sound experiment: temperature dependence



Measurement and results

- 3rd sound velocity vs $T$
- points at $T$ = 1.25 K normalized to (●)
- $v_3$ is rising with decreasing $T$
- $T \to 0$: $v_3$ = 1.5 m/s (very slow)
- dashed line ≙ theory $v_3 \propto \sqrt{\varrho_\mathrm{s}}$
- systematic deviations: origin unknow, but likely due to generation process

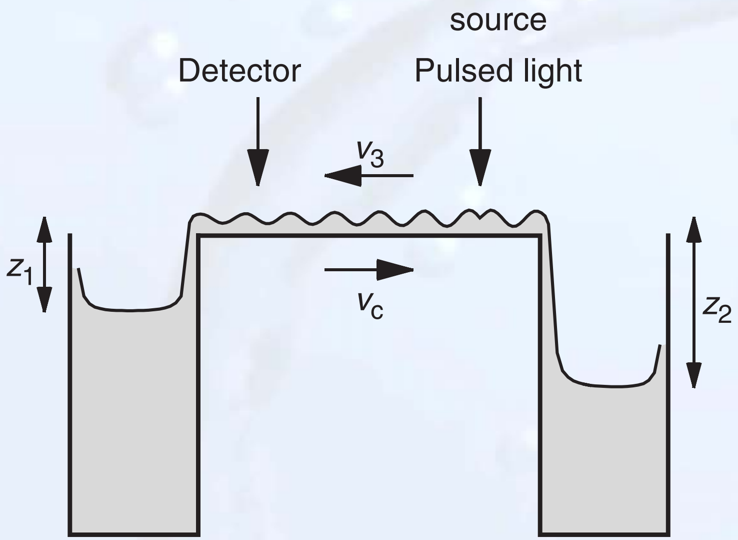
3rd sound in very thin films:

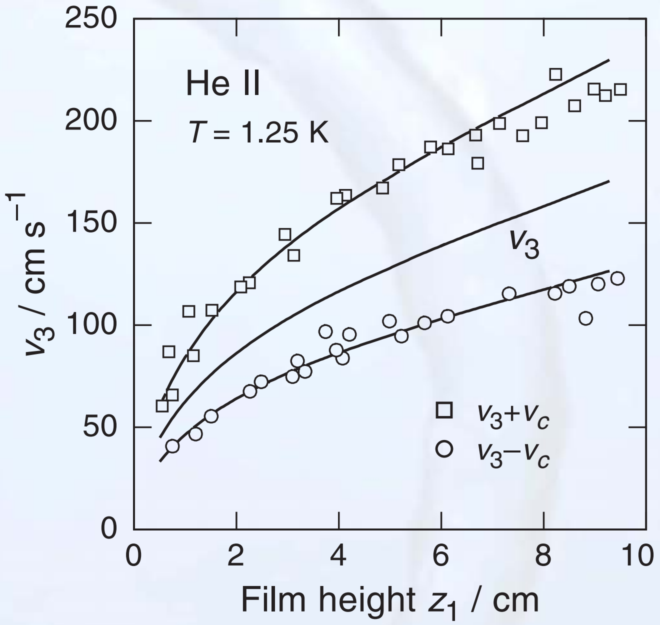
3rd sound propagation can be observed down to 2.1 monolayers

➡ onset of superfluidity

# 2.3 Properties of He-II described using the two-fluid model

3rd sound in moving films:



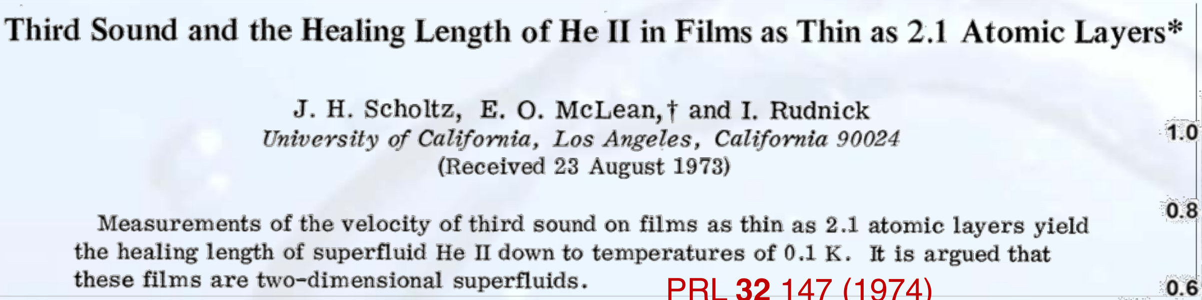


3rd sound propagation in moving films ➡ Doppler effect $v_3 \pm v_\mathrm{f}$

$v_c$ critical velocity

# 2.3 Properties of He-II described using the two-fluid model

## Detection of 3rd sound experiment in ultralow films:

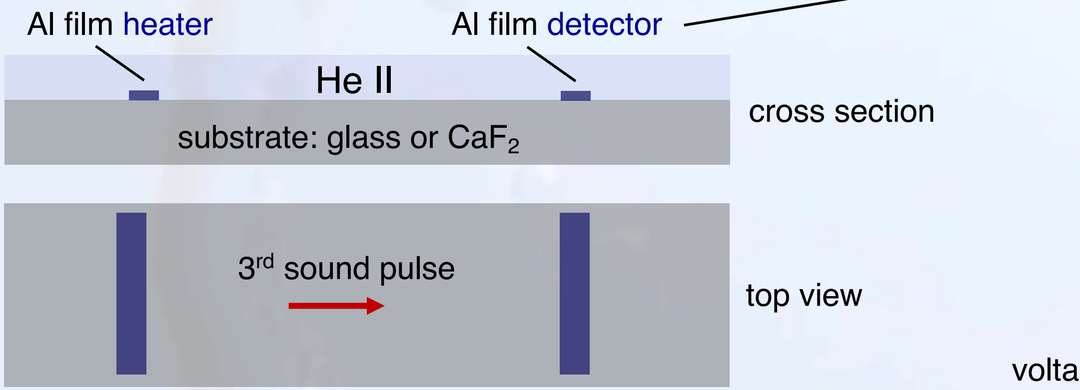


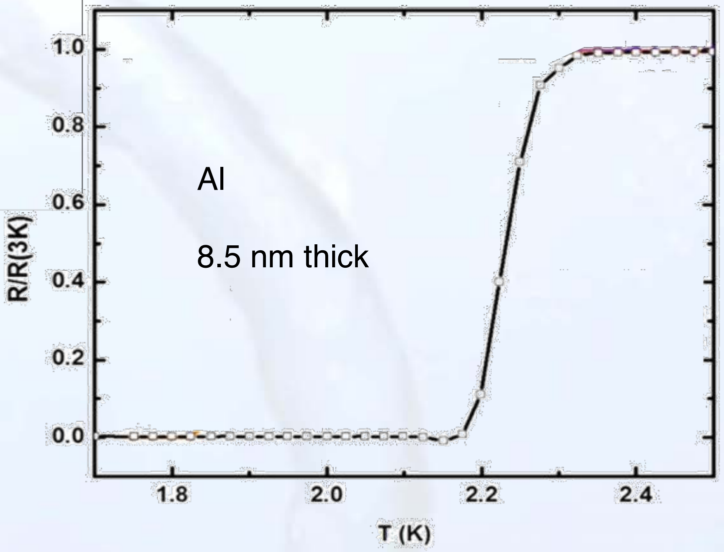
### Third Sound and the Healing Length of He II in Films as Thin as 2.1 Atomic Layers*

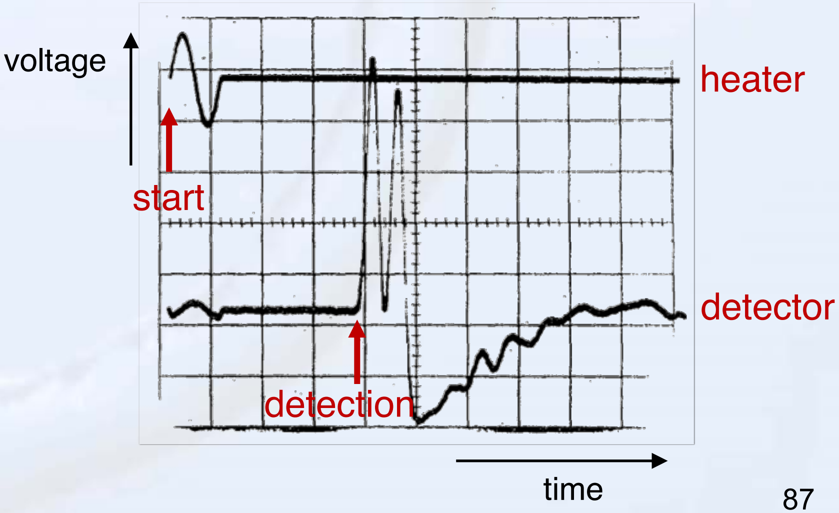
J. H. Scholtz, E. O. McLean,† and I. Rudnick
*University of California, Los Angeles, California 90024*
(Received 23 August 1973)

Measurements of the velocity of third sound on films as thin as 2.1 atomic layers yield the healing length of superfluid He II down to temperatures of 0.1 K. It is argued that these films are two-dimensional superfluids.

PRL **32** 147 (1974)





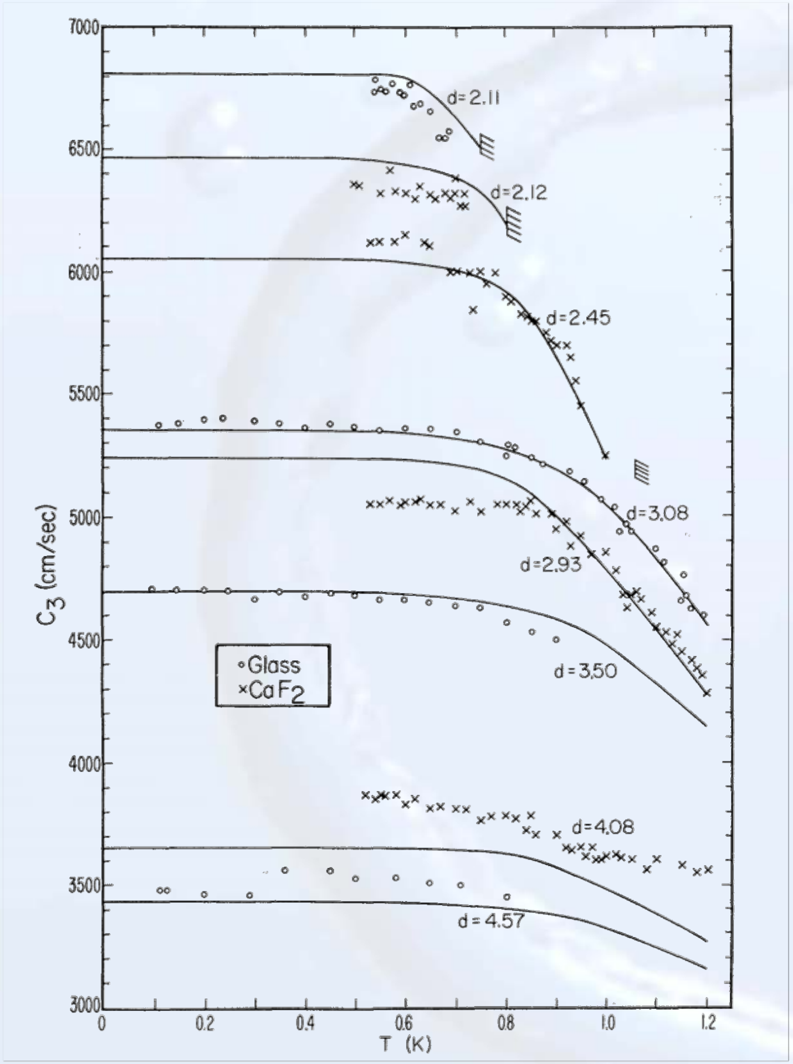


- ► time of flight detection of 3rd sound pulse
- ► thin film Al heater and detector
- ► detector operated at the transition temperature
- ► detection of heat pulse associated with 3rd sound pulse
- ► very sensitive because of steep transition curve

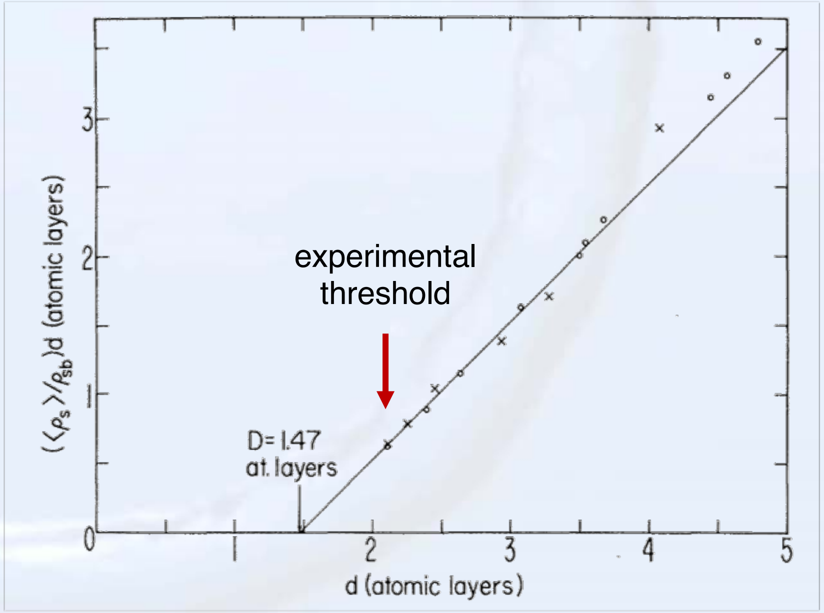
# 2.3 Properties of He-II described using the two-fluid model

**Experimental results:**

for ultrathin films: $$v_3^2 = \frac{\overline{\rho_\mathrm{s}}}{\rho_\mathrm{s,bulk}} \frac{3RT}{m} \ln \frac{p_0}{p}$$



- ► experimental threshold of 2.1 monolayers independent of substrate
- ► film thickness determine by amount of helium and surface area
- ► extrapolation suggests that 1.47 monolayers might be the onset threshold



# 2.3 Properties of He-II described using the two-fluid model

## (iv) Fourth sound

sound propagation in fine powders / slits $\boldsymbol{v}_\mathrm{n} \approx 0$

→ oscillations in total density, in ratio of superfluid to normalfluid density, in pressure, in temperature, in entropy

$$v_4^2 = \frac{\varrho_\mathrm{s}}{\varrho} v_1^2 \left[ 1 + \underbrace{\frac{2ST}{\varrho C_p} \left( \frac{\partial \varrho}{\partial T} \right)_p}_{\ll 1} \right] + \frac{\varrho_\mathrm{n}}{\varrho} v_2^2$$

$$\boxed{v_4 \approx \sqrt{\frac{\varrho_\mathrm{s}}{\varrho} v_1^2 + \frac{\varrho_\mathrm{n}}{\varrho} v_2^2}} \approx \sqrt{\frac{\varrho_\mathrm{s}}{\varrho} v_1^2}$$

5th sound

4th sound generation like for 1st sound, but $\boldsymbol{v}_\mathrm{n} \approx 0$

# 2.3 Properties of He-II described using the two-fluid model

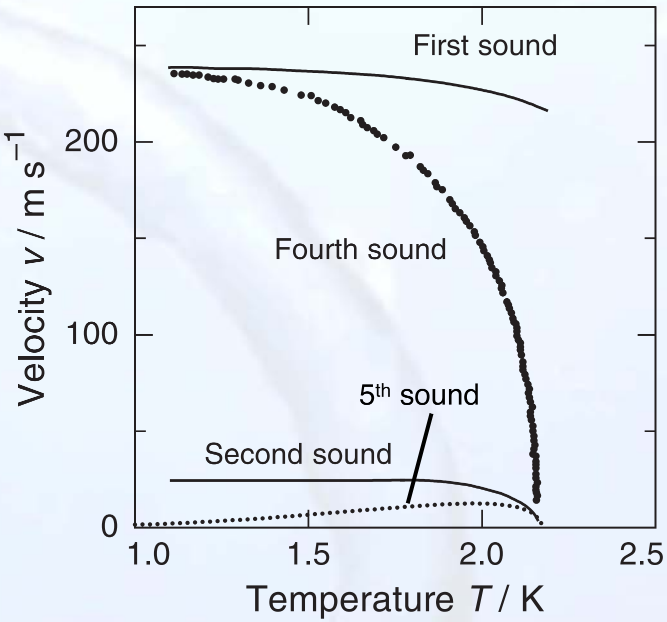
## 4th sound experiments

4th sound generation like for 1st sound, but $\boldsymbol{v}_\mathrm{n} \approx 0$

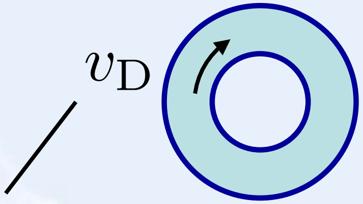
$T \to 0 \quad v_4 = v_1 \approx 238\,\mathrm{m/s}$, since $\varrho_\mathrm{s} = \varrho$

$T = T_\lambda \quad v_4 = 0$

$$v_4 \approx \sqrt{\frac{\varrho_\mathrm{s}}{\varrho}\, v_1^2 + \frac{\varrho_\mathrm{n}}{\varrho}\, v_2^2}$$



## Persistent flow and 4th sound



persistent flow velocity

$$v_4 \approx v_{4,0} \pm \frac{\varrho_\mathrm{s}}{\varrho}\, v_\mathrm{D}$$

coupling of a compression wave to second sound

# 2.4 Bose-Einstein Condensation

Einstein 1924
Bose 1925
London 1938

Basic idea of Fritz London:

dissipation-less motion ⟷ macroscopic wave function

**a) Ideal Bose gas**

non-interacting Bose gas (rough approximation for liquid He)

let's consider: 1 cm$^3$ cube of liquid $^4$He $\triangleq 10^{22}$ atoms with mass *m*

eigenstates for free particles in a cube:

$$E_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2 \qquad \text{with} \qquad n^2 = n_x^2 + n_y^2 + n_z^2$$

$T = 0$ $\longrightarrow$ all atoms are in the ground state $E_{111}$ trivial !

But at finite temperatures?

# 2.4 Bose-Einstein Condensation

consider energy difference between ground state and first excited state

$$\Delta E/k_{\mathrm{B}} = \left(E_{211} - E_{111}\right)/k_{\mathrm{B}} \approx 2 \times 10^{-14}\,\mathrm{K}$$

↷ if Boltzmann statistics would hold ➡ no condensate at 1 K!!!

however, Bose-Einstein distribution is relevant here

$$f(E,T) = \frac{1}{\mathrm{e}^{(E-\mu)/k_{\mathrm{B}}T} - 1}\,.$$

chemical potential $\mu = \dfrac{\partial F}{\partial N}$

what we know: $\mu < E_{111}$ ⟶ otherwise, negative occupation

$\mu \neq 0$ ⟶ since particle number conserved