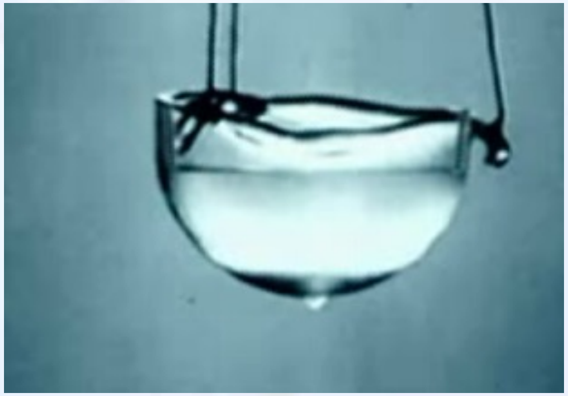
2.3 Properties of He-II described using the two-fluid model

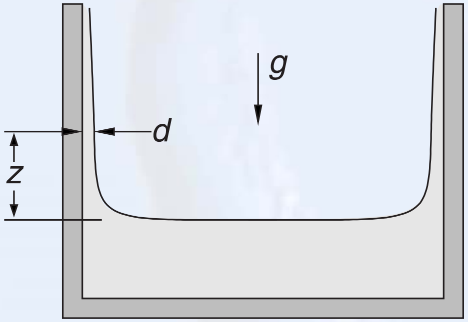
b) Beaker experiments

films are formed with a thickness of ~ 200 Å in saturated vapor pressure also against gravity



let us understand how
comment: the film formation is a “classical” phenomenon

(i) Film formation in saturated vapor



In thermal equilibrium

$$\mu_{\mathrm{f}} = \mu_{\mathrm{g}} = \mu_{\ell}$$

chemical potential for film (gas and liquid)

gravitational force is compensated by v. Waals forces

$$\mu_{\mathrm{f}} = \mu_{\ell} + \underbrace{\mu_{\mathrm{grav}} + \mu_{\mathrm{vdW}}}_{=0} = \mu_{\ell}$$

2.3 Properties of He-II described using the two-fluid model

film thinkness:

$$\left.\begin{aligned}\mu_{\mathrm{grav}} &= gz \\ \mu_{\mathrm{vdW}} &= -\alpha/d^3\end{aligned}\right\} \quad gz - \alpha/d^3 = 0 \quad \curvearrowright \quad \boxed{d = \sqrt[3]{\frac{\alpha}{gz}}}$$

depends on film thickness: $\mu_{\mathrm{vdW}} = -\alpha/d^3$ for $d < 30\,\mathrm{nm}$

$\mu_{\mathrm{vdW}} = -\alpha/d^4$ for $d > 80\,\mathrm{nm}$

atomic polarisability of helium + wall
(Hamaker constant)

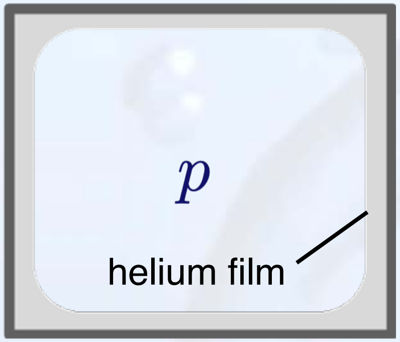
retardation of potential

typical value: $d \sim 20$ nm at $z = 10$ cm

comment: property of superfluidity is unimportant for the film formation and thickness, but for the film flow

2.3 Properties of He-II described using the two-fluid model

(ii) film formation in unsaturated vapor



How does d depend on p ?

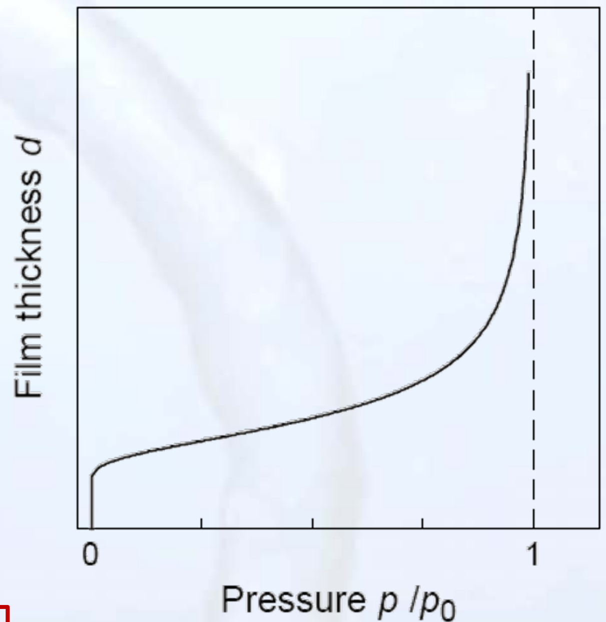
➡ barometric formula

$$\frac{p}{p_0} = \mathrm{e}^{-mgh/k_\mathrm{B}T}$$

$$mgh = -\,k_\mathrm{B}T\ln\left(\frac{p}{p_0}\right)$$

➡ $$\frac{\alpha}{d^3} = \frac{k_\mathrm{B}T}{m_4}\ln\left(\frac{p_0}{p}\right)$$

➡ $$\boxed{d = \sqrt[3]{\frac{\alpha\, m_4}{k_\mathrm{B}T(\ln p_0 - \ln p)}}}$$



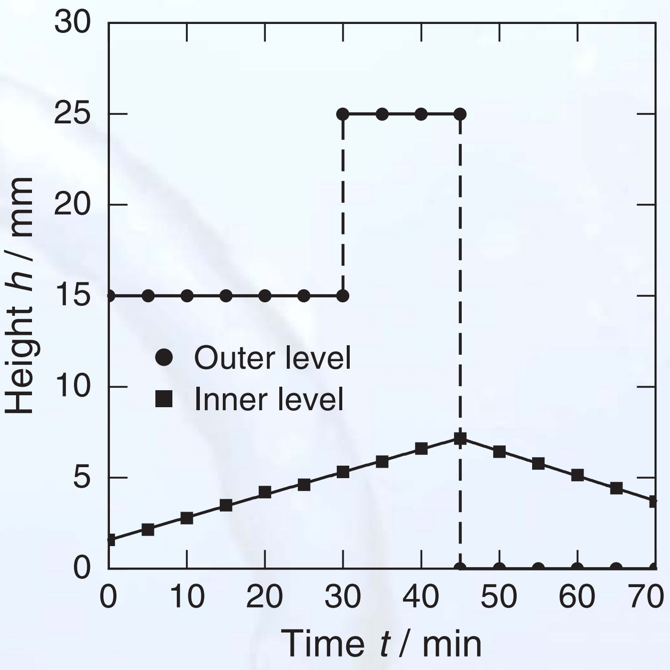
- decrease of pressure ➡ decrease of film thickness
- in practice: thicknesses of sub-mono layers are possible and realized

investigation of superfluidity with third sound: onset of superfluidity at $n > 2.1$ layers

2.3 Properties of He-II described using the two-fluid model

now back to the film flow:

- films are formed
- ϱ_{s} is moving without friction
- equalizing the chemical potential is driving force
- const. rate $\triangleq$ critical velocity

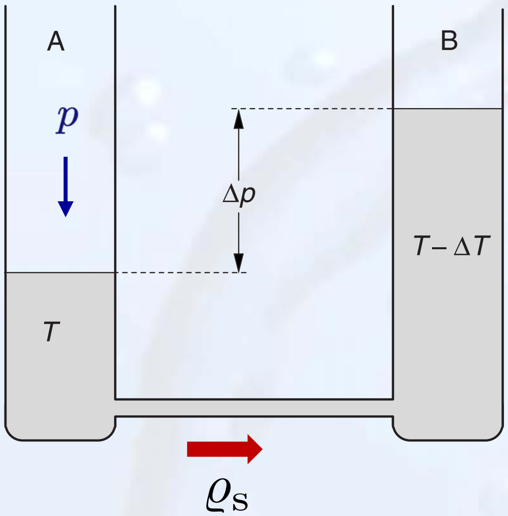


Interesting question: ϱ_{s} flows with $S = 0$! ➡ rest should warm up and helium flowing into a vessel should have $T = 0$!

but thermal equilibrium via gas phase

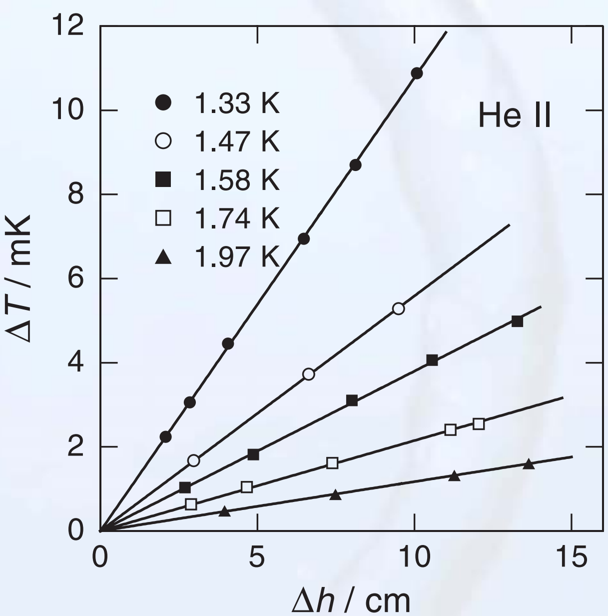
2.3 Properties of He-II described using the two-fluid model

c) Thermomechanical effect



$+\,\varrho_\mathrm{s}$ → cooling in B
$-\,\varrho_\mathrm{s}$ → warming in A

$\left.\right\}$ $T_\mathrm{B} < T_\mathrm{A}$



Using (6) in stationary state

$$\frac{\partial \boldsymbol{v}_\mathrm{s}}{\partial t} = S\,\mathrm{grad}\,T - \frac{1}{\varrho}\,\mathrm{grad}\,p = 0$$

in equilibrium nothing flows

$$\boxed{\frac{\Delta p}{\Delta T} = \varrho S}$$

London equation
(H. London 1939)

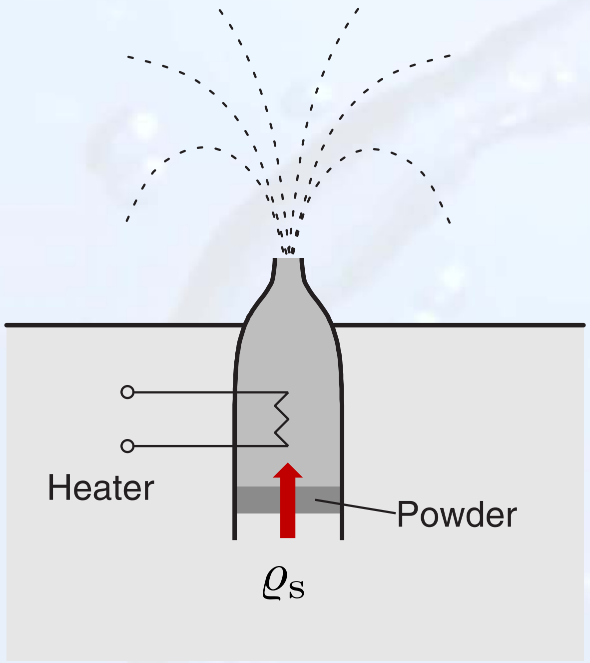
→ Linear relation between Δp and ΔT

$$\left.\begin{array}{l}\Delta h = 2\,\mathrm{cm}\\ T = 1.5\,\mathrm{K}\end{array}\right\}\ \Delta T = 1\ \mathrm{mK}$$

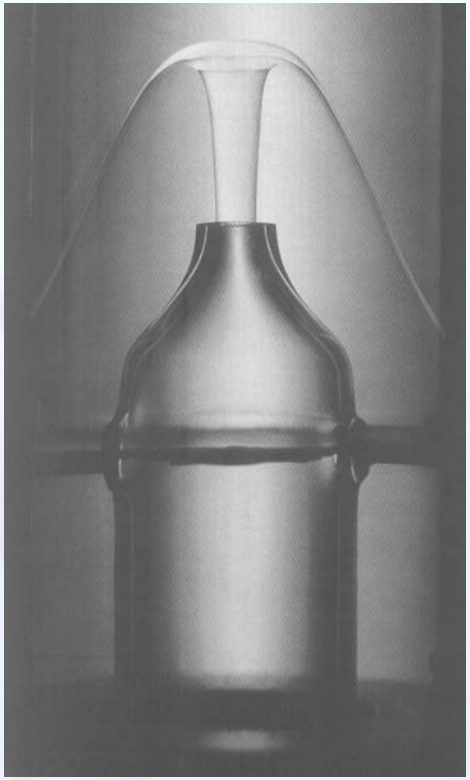
not very effective cooling

2.3 Properties of He-II described using the two-fluid model

Reverse thermomechanical effect: Fountain effect



$$\Delta T \curvearrowright \Delta p$$



- heating of helium inside vessel $\longrightarrow$ ratio of ϱ_n / ϱ_s increases inside the vessel
- the temperature inside is higher than outside
- to equalize the system ϱ_s flows through superleak (compressed powder)
- pressure rises and fountain starts to flow (and flows as long as heater is on)

2.3 Properties of He-II described using the two-fluid model

d) Heat Transport

- in not too small capillaries $v_n \neq 0$
- even in equilibrium ($\Delta p = \varrho S \Delta T$) there is a constant flow of ϱ_n from the warm end to the cold end and ϱ_s in the opposite direction by "convection"

$$\left.\begin{array}{l} \varrho_n \longrightarrow \text{cold end} \\ \varrho_s \longrightarrow \text{warm end} \end{array}\right\} \text{entropy transport} \triangleq \text{heat transport}$$

heat transport maximum at 1.8 K where $\varrho_n \approx \varrho_s$

- limited only by the mobility of ϱ_n and therefore η_n
- viscos mass flow of ϱ_n :

$$\dot{V}_n = \frac{\beta}{\eta_n} \frac{\Delta p}{L} \qquad (*)$$

volume rate

$\beta \propto r^4$ for capillaries

$\beta \propto d^3$ for slits

- entropy flow $\dot{V}_n \varrho S \longrightarrow$ heat flow $\dot{Q} = T \dot{V}_n \varrho S \quad (**)$

$\delta Q = T \delta S$

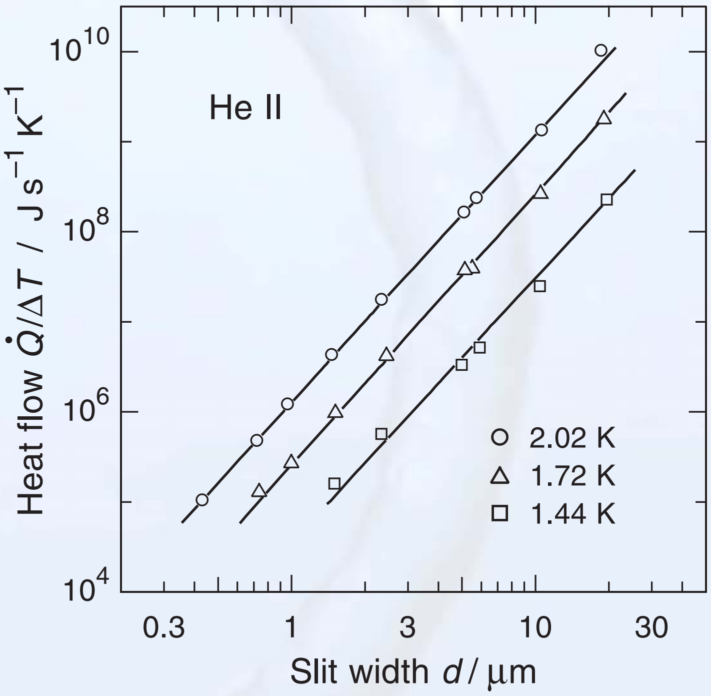
2.3 Properties of He-II described using the two-fluid model

$(*)$ insert in $(**)$ and London equation $(\Delta p = \varrho S \Delta T)$

$$\dot{Q} = \frac{\beta T (\varrho S)^2}{\eta_{\mathrm{n}} L} \Delta T$$

linear regime

heat flow log $\dot{Q}/\Delta T$ vs log d



experimental results:

$\dot{Q} \propto \beta \propto d^3$ (as expected)

$\dot{Q}$ rises with T (as expected)

2.3 Properties of He-II described using the two-fluid model

Momentum of heat flow

Heat flow in He-II $\longrightarrow$ momentum flow $\varrho\,\boldsymbol{v}\cdot\boldsymbol{v}$

momentum flow / volume

resulting pressure acting on a heat source

$$p = \varrho_\mathrm{n} v_\mathrm{n}^2 + \varrho_\mathrm{s} v_\mathrm{s}^2 \qquad (*)$$

no net mass transport (closed vessel)

$$\varrho_\mathrm{n} v_\mathrm{n} + \varrho_\mathrm{s} v_\mathrm{s} = 0 \longrightarrow v_\mathrm{s} = -\frac{\varrho_\mathrm{n}}{\varrho_\mathrm{s}} v_\mathrm{n}$$

insert in $(*)$ $\longrightarrow$ $p = \dfrac{\varrho_\mathrm{n}\varrho}{\varrho_\mathrm{s}} v_\mathrm{n}^2$

with heat flow / per area $\dfrac{\dot Q}{A} = \varrho S T v_\mathrm{n}$ $\quad \dfrac{\dot V_\mathrm{n}}{A}$ $\longrightarrow$ $v_\mathrm{n} = \dfrac{\dot Q}{A \varrho S T}$

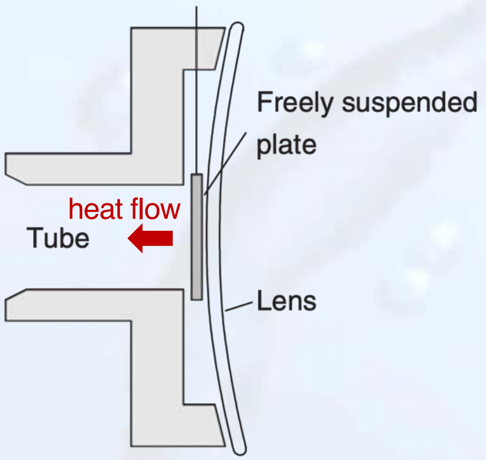
$$\boxed{p = \frac{\varrho_\mathrm{n}}{\varrho_\mathrm{s}\varrho}\left(\frac{\dot Q}{AST}\right)^2}$$

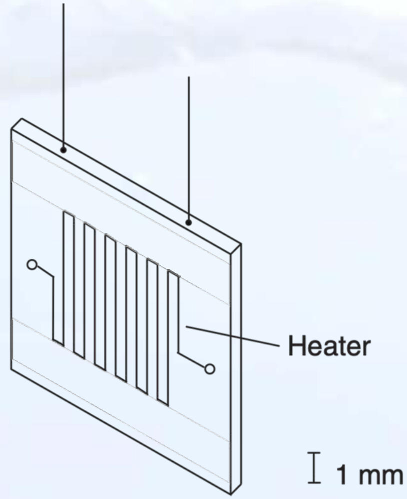
pressure associated with uni-directional heat flow

$$p = \varrho_n v_n^2 + \varrho_s v_s^2$$
$$v_s = -\frac{\varrho_n}{\varrho_s} v_n$$
$$= \varrho_n v_n^2 + \varrho_s \left(-\frac{\varrho_n}{\varrho_s} v_n\right)^2$$
$$= \varrho_n v_n^2 \left(1 + \frac{\varrho_n}{\varrho_s}\right)$$
$$= \varrho_n v_n^2 \left(\frac{\varrho_s + \varrho_n}{\varrho_s}\right)$$
$$= \frac{\varrho_n \varrho}{\varrho_s} v_n^2$$

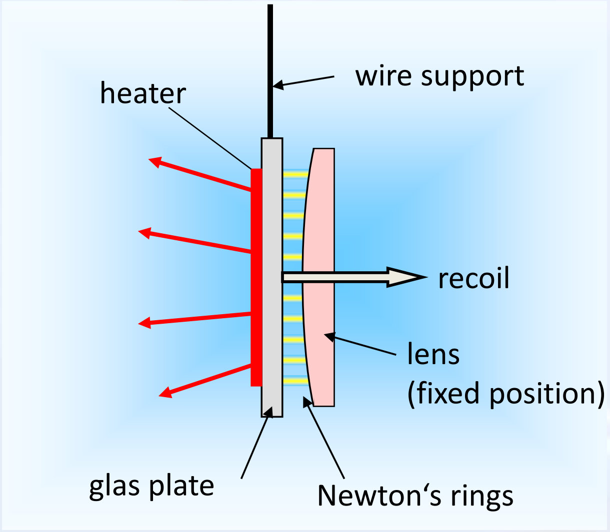
2.3 Properties of He-II described using the two-fluid model

Momentum of heat flow: Measurement

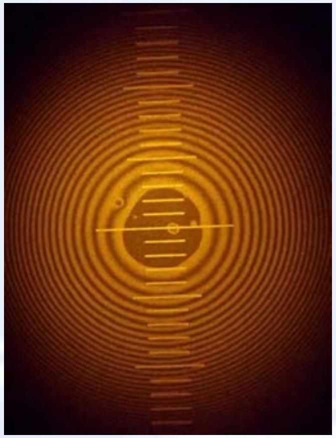




change of distance between glass plate and lens measured by Newton rings ⟶ force



Newton rings



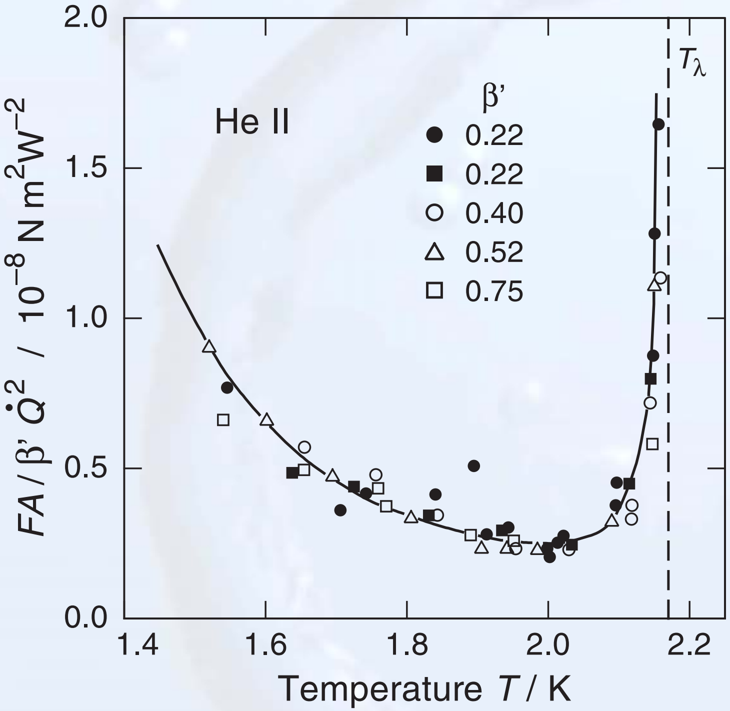
Expected force

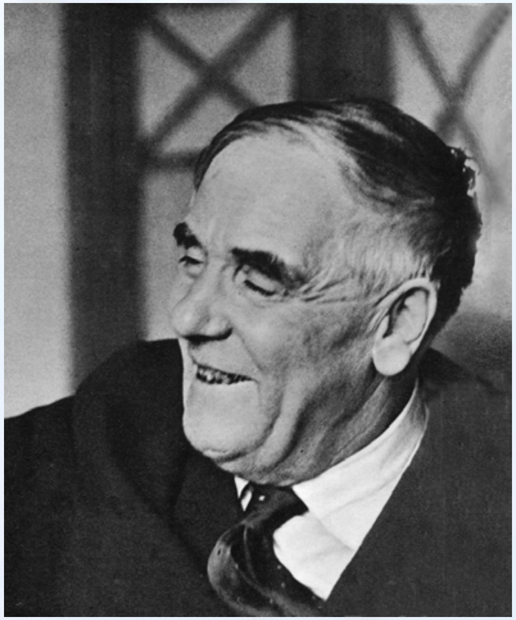
$$F = pA = \beta' \frac{\varrho_\mathrm{n}}{\varrho_\mathrm{s}\varrho A}\left(\frac{\dot{Q}}{ST}\right)^2$$

geometry dependent factor of the order of one

2.3 Properties of He-II described using the two-fluid model

Momentum of heat flow: results plotted as $\dfrac{FA}{\beta'\dot{Q}^2} = \dfrac{\varrho_\mathrm{n}}{\varrho_\mathrm{s}\varrho}\dfrac{1}{T^2S^2}$





Pyotr Leonidovich Kapitsa (1894 – 1984)

- results are independent of geometry
- because of $\varrho_\mathrm{n}v_\mathrm{n} + \varrho_\mathrm{s}v_\mathrm{s} = 0$ ⟶ rise at low and high T
- line: two-fluid model (without free parameter)

Two-Fluid Hydrodynamics

density $$\varrho = \varrho_\mathrm{n} + \varrho_\mathrm{s} \tag{1}$$

mass flow $$\boldsymbol{j} = \varrho_\mathrm{n}\boldsymbol{v}_\mathrm{n} + \varrho_\mathrm{s}\boldsymbol{v}_\mathrm{s} \tag{2}$$

mass conservation
continuity eqn. $$\frac{\partial \varrho}{\partial t} = -\mathrm{div}\,\boldsymbol{j} \tag{3}$$

ideal fluid $$\frac{\partial \boldsymbol{j}}{\partial t} = -\mathrm{grad}\,p \tag{4}$$

entropy conservation $$\frac{\partial(\varrho S)}{\partial t} = -\mathrm{div}(\varrho S \boldsymbol{v}_\mathrm{n}) \tag{5}$$

an equation of motion for
superfluid component $$\frac{\partial \boldsymbol{v}_\mathrm{s}}{\partial t} = S\,\mathrm{grad}\,T - \frac{1}{\varrho}\,\mathrm{grad}\,p \tag{6}$$

2.3 Properties of He-II described using the two-fluid model

d) Sound propagation (precision test of two-fluid model)

differentiation of (3) in respect to time and insert in (4)

$$\frac{\partial^2 \varrho}{\partial t^2} = \nabla^2 p \qquad (*)$$

eliminate $\boldsymbol{v}_{\mathrm{s}}$ and $\boldsymbol{v}_{\mathrm{n}}$ in (5) and (6) with (2)

since not observable

neglect terms of 2nd order

$$\frac{\partial^2 S}{\partial t^2} = \frac{\varrho_{\mathrm{s}} S^2}{\varrho_{\mathrm{n}}} \nabla^2 T \qquad (**)$$

with $(*)$ and $(**)$ one can fully describe the sound propagation in He-II
(under the assumption we made)

$$\frac{\partial \varrho}{\partial t} = -\operatorname{div} \vec{J} \qquad (3)$$

$$\frac{\partial^2 \varrho}{\partial t^2} = -\operatorname{div}\left(\frac{\partial \vec{J}}{\partial t}\right)$$

$$\frac{\partial \vec{J}}{\partial t} = -\operatorname{grad} p \qquad (4)$$

$$\frac{\partial^2 \varrho}{\partial t^2} = -\operatorname{div}(-\operatorname{grad} p)$$

$$\frac{\partial^2 \varrho}{\partial t^2} = \nabla^2 p$$

2.3 Properties of He-II described using the two-fluid model

we have 2 equations, but 4 variables $(\varrho,\ S,\ p,\ T)$ however, only 2 independent variables

We choose ϱ, S as independent and express p, T with ϱ and S (for small changes)

$$\left.\begin{aligned}\delta p &= \left(\frac{\partial p}{\partial \varrho}\right)_S \delta\varrho + \left(\frac{\partial p}{\partial S}\right)_\varrho \delta S\,,\\ \delta T &= \left(\frac{\partial T}{\partial \varrho}\right)_S \delta\varrho + \left(\frac{\partial T}{\partial S}\right)_\varrho \delta S\end{aligned}\right\}$$

insert in $(*)$ and $(**)$

$$\boxed{\begin{aligned}\frac{\partial^2 \varrho}{\partial t^2} &= \left(\frac{\partial p}{\partial \varrho}\right)_S \nabla^2\varrho + \left(\frac{\partial p}{\partial S}\right)_\varrho \nabla^2 S\\ \frac{\partial^2 S}{\partial t^2} &= \frac{\varrho_\mathrm{s}}{\varrho_\mathrm{n}} S^2 \left[\left(\frac{\partial T}{\partial \varrho}\right)_S \nabla^2\varrho + \left(\frac{\partial T}{\partial S}\right)_\varrho \nabla^2 S\right]\end{aligned}}$$

2 partial differential equations of 2nd order

2.3 Properties of He-II described using the two-fluid model

Ansatz: $$\varrho = \varrho_0 + \varrho' \, \mathrm{e}^{\mathrm{i}\omega(t-x/v)} \,,$$

$$S = S_0 + S' \, \mathrm{e}^{\mathrm{i}\omega(t-x/v)}$$

frequency of wave

velocity in x direction

Insertion and differentiation leads to 2 linear equations in ϱ' and S'

$$\left[\left(\frac{v}{v_1}\right)^2 - 1\right] \varrho' + \left(\frac{\partial p}{\partial S}\right)_\varrho \left(\frac{\partial \varrho}{\partial p}\right)_S S' = 0 \,, \qquad \text{(i)}$$

$$\left(\frac{\partial T}{\partial \varrho}\right)_S \left(\frac{\partial S}{\partial T}\right)_\varrho \varrho' + \left[\left(\frac{v}{v_2}\right)^2 - 1\right] S' = 0 \qquad \text{(ii)}$$

with

$$v_1^2 = \left(\frac{\partial p}{\partial \varrho}\right)_S \qquad \text{and} \qquad v_2^2 = \frac{\varrho_\mathrm{s}}{\varrho_\mathrm{n}} S^2 \left(\frac{\partial T}{\partial S}\right)_\varrho$$

2.3 Properties of He-II described using the two-fluid model

the constrains equation for the coefficients is

$$\left[\left(\frac{v}{v_1}\right)^2 - 1\right]\left[\left(\frac{v}{v_2}\right)^2 - 1\right] = \underbrace{\left(\frac{\partial p}{\partial S}\right)_\varrho \left(\frac{\partial \varrho}{\partial p}\right)_S \left(\frac{\partial T}{\partial \varrho}\right)_S \left(\frac{\partial S}{\partial T}\right)_\varrho}$$

here standard thermodynamic relations are used

$$\boxed{\left[\left(\frac{v}{v_1}\right)^2 - 1\right]\left[\left(\frac{v}{v_2}\right)^2 - 1\right] = \frac{C_p - C_V}{C_p}}$$

for liquid helium $C_p \approx C_V$

$$\left[\left(\frac{v}{v_1}\right)^2 - 1\right]\left[\left(\frac{v}{v_2}\right)^2 - 1\right] \approx 0 \qquad \text{(iii)}$$

interpretation: two wave $\begin{cases} v_1 \\ v_2 \end{cases}$ weakly coupled via $\dfrac{C_p - C_V}{C_p}$