Standard Model of Particle Physics

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2. Relativistic Quantum Fields

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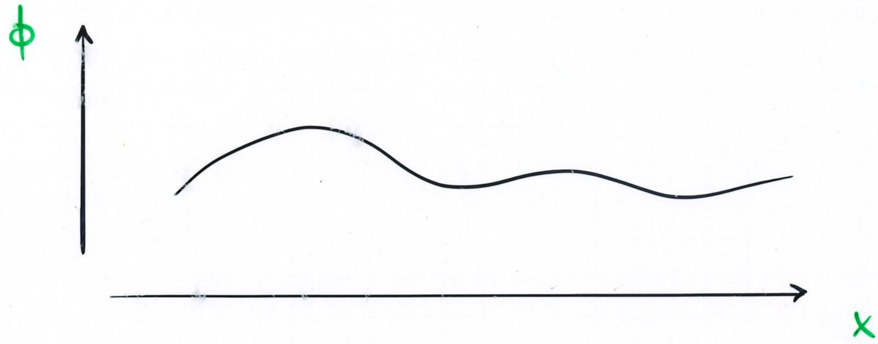
Classical Field Theory



* Consider waves on string, density ρ, tension T, displacement $\phi(x,t)$.

K.E. $$T = \int \frac{1}{2}\rho \left(\frac{\partial\phi}{\partial t}\right)^2 dx$$

P.E. $$V = \int \frac{1}{2} T \left(\frac{\partial\phi}{\partial x}\right)^2 dx = \int \frac{1}{2}\rho c^2 \left(\frac{\partial\phi}{\partial x}\right)^2 dx$$

wave velocity $c = \sqrt{T/\rho}$

* Lagrangian $L = T - V = \int \mathcal{L}\, dx$

with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\rho\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - c^2\left(\frac{\partial\phi}{\partial x}\right)^2\right].$$

For brevity, write

$$\frac{\partial\phi}{\partial t} = \dot\phi \quad , \quad \frac{\partial\phi}{\partial x} = \phi'$$

$$\Longrightarrow \quad \mathcal{L} = \frac{1}{2}\rho\left(\dot\phi^2 - c^2\phi'^2\right)$$

* Equations of motion given by least action

$$\delta S = 0 \quad \text{where} \quad S = \int L \, dt$$

$$\rightarrow \delta S = \int \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \phi'} \delta\phi' + \frac{\partial \mathcal{L}}{\partial \dot\phi} \delta\dot\phi \right) dx \, dt$$

But
$$\int \frac{\partial \mathcal{L}}{\partial \phi'} \delta\phi' \, dx = \int \frac{\partial \mathcal{L}}{\partial \phi'} \frac{\partial}{\partial x} \delta\phi \, dx$$
$$= \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \phi'} \delta\phi \right]_{-\infty}^{+\infty}}_{=0} - \int \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'} \right) \delta\phi \, dx$$

Similarly,
$$\int \frac{\partial \mathcal{L}}{\partial \dot\phi} \delta\dot\phi \, dt = - \int \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi} \right) \delta\phi \, dt$$

$$\Rightarrow \delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'} \right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi} \right) \right] \delta\phi \, dx \, dt$$

This has to vanish for any $\delta\phi(x,t)$, so we obtain Euler - Lagrange equation of motion for the field $\phi(x,t)$

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'} \right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi} \right) = 0}$$

* For the string $\mathcal{L} = \frac{1}{2}\rho\left(\dot\phi^2 - c^2 \phi'^2\right)$

$$\rightarrow \rho c^2 \frac{\partial \phi'}{\partial x} - \rho \frac{\partial \dot\phi}{\partial t} = 0$$

$$\rightarrow c^2 \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = 0 \quad \leftarrow \text{wave equation}$$

* We shall also need Hamiltonian

$$H = \int \mathcal{H} \, dx$$

Hamiltonian density

Recall that for a single coordinate q we have $L = L(\dot{q}, q)$ and

$$H = p\dot{q} - L$$

where p is the generalized momentum

$$p = \frac{\partial L}{\partial \dot{q}} .$$

* Similarly, for field $\phi(x,t)$ we define momentum density $\pi(x,t) = \partial \mathcal{L} / \partial \dot{\phi}$.

Then

$$\mathcal{H}(\pi, \phi) = \pi \dot{\phi} - \mathcal{L} .$$

* For the string $\pi = \rho \dot{\phi}$ (as expected)

$$\Rightarrow \quad \mathcal{H} = \pi \left(\frac{\pi}{\rho}\right) - \frac{1}{2}\rho\left(\frac{\pi^2}{\rho}\right) + \frac{1}{2}\rho c^2 \phi'^2$$

$$= \frac{1}{2\rho}\pi^2 + \frac{1}{2}\rho c^2 \left(\frac{\partial \phi}{\partial x}\right)^2$$

for string

Klein-Gordon Field

We choose the Lagrangian density $(\hbar = c = 1)$

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 - \frac{1}{2} m^2 \phi^2$$

to obtain the Klein-Gordon equation of motion:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0$$

$$\Rightarrow \quad -m^2 \phi + \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = 0$$

The momentum density is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \phi}{\partial t}$$

and so the Klein-Gordon Hamiltonian density is

$$\mathcal{H} = \pi^2 - \mathcal{L}$$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2} m^2 \phi^2$$

* In 3 spatial dimensions $\partial \phi/\partial x \longrightarrow \underline{\nabla} \phi$,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial \phi/\partial x)}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial(\partial \phi/\partial y)}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \mathcal{L}}{\partial(\partial \phi/\partial z)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0$$

$$\Rightarrow \quad -m^2 \phi + \underline{\nabla}^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = 0$$

* Covariant notation:

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} (\partial^\mu \phi)(\partial^\nu \phi) - \frac{1}{2} m^2 \phi^2 ,$$

Euler - Lagrange equation:

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi)} \right) = 0}$$

$$\Rightarrow \quad - m^2 \phi - \partial^\mu (g_{\mu\nu} \partial^\nu \phi) = 0$$

i.e. $\quad \partial^\mu \partial_\mu \phi + m^2 \phi = 0 \quad$ (KG equation)

* Note that the Lagrangian density $\mathcal{L}$ and the action $S = \int \mathcal{L} \, d^3\underline{r} \, dt = \int \mathcal{L} \, d^4x$ are <u>scalars</u> (invariant functions), like ϕ.

* On the other hand the momentum density $\pi = \partial\phi/\partial t$ and the Hamiltonian density

$$\mathcal{H} = \frac{1}{2} \pi^2 + \frac{1}{2} (\underline{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2$$

are not.

(Time development $\Rightarrow$ frame dependence)

Fourier Analysis

* We can express any real field $\phi(x,t)$ as a Fourier integral

$$\phi(x,t) = \int dk \; N(k) \left[a(k) e^{ikx - i\omega t} + a^*(k) e^{-ikx + i\omega t} \right]$$

where $N(k)$ is a convenient normalizing factor for the Fourier transform $a(k)$. The frequency $\omega(k) \geq 0$ is obtained by solving the equation of motion

* KG equation $\Rightarrow$ $-m^2 - k^2 + \omega^2 = 0$

$\Rightarrow$ $\omega = + \sqrt{k^2 + m^2}$

* The Hamiltonian

$$H = \int \left(\frac{1}{2} \pi^2 + \frac{1}{2} \phi'^2 + \frac{1}{2} m^2 \phi^2 \right) dx$$

takes a simpler form in terms of the Fourier amplitudes $a(k)$.

* Write e.g.

$$\phi^2 = \int dk \; N(k) [\dots] \int dk' \; N(k') [\dots]$$

and use

$$\int dx \; e^{i(k \pm k')x} = 2\pi \, \delta(k \pm k')$$

$$\rightarrow \int \phi^2 dx = 2\pi \int dk\, dk'\, N(k)\, N(k') \,\times$$

$$\times \Big[a(k)\, a(k')\, \delta(k+k')\, e^{-i(\omega+\omega')t} + a^*(k)\, a^*(k')\, \delta(k+k')\, e^{+i(\omega+\omega')t}$$

$$+ a(k)\, a^*(k')\, \delta(k-k')\, e^{-i(\omega-\omega')t} + a^*(k)\, a(k')\, \delta(k-k')\, e^{+i(\omega-\omega')t} \Big]$$

* Noting that $\omega(-k) = \omega(k)$ and choosing $N(k)$ such that $N(k) = N(-k)$, this gives

$$\int \phi^2 dx = 2\pi \int dk\, [N(k)]^2 \cdot \Big[a(k)\, a(-k)\, e^{-2i\omega t} +$$

$$+ a^*(k)\, a^*(-k)\, e^{+2i\omega t} + a(k)\, a^*(k) + a^*(k)\, a(k) \Big]$$

* Similarly,

$$\int \phi'^2 dx = 2\pi \int dk\, [k\, N(k)]^2 \,[\,—\; " \;—\,]$$

while

$$\int \dot{\phi}^2 dx = 2\pi \int dk\, [\omega(k)\, N(k)]^2 \cdot \Big[-a(k)\, a(-k)\, e^{-2i\omega t} +$$

$$- a^*(k)\, a^*(-k)\, e^{+2i\omega t} + a(k)\, a^*(k) + a^*(k)\, a(k) \Big]$$

and hence (using $k^2 = \omega^2 - m^2$)

$$H = 2\pi \int dk\, [N(k)\, \omega(k)]^2 \,[a(k)\, a^*(k) + a^*(k)\, a(k)]$$

or, choosing

$$N(k) = \frac{1}{2\pi \cdot 2\omega(k)}$$

$$\boxed{H = \int dk\, N(k)\, \tfrac{1}{2}\, \omega(k)\, [a(k)\, a^*(k) + a^*(k)\, a(k)]}$$

$$H = \int dk \, N(k) \, \underbrace{\omega(k) \, |a(k)|^2}_{\text{mode energy}}$$

$N(k)$: density of modes

* Each normal mode of the system behaves like an independent harmonic oscillator with amplitude $a(k)$.

* In 3 spatial dimensions we write

$$\phi(\underline{r},t) = \int d^3\underline{k} \, N(\underline{k}) \left[a(\underline{k}) e^{i\underline{k}.\underline{r} - i\omega t} + a^*(\underline{k}) e^{-i\underline{k}.\underline{r} + i\omega t} \right]$$

and use

$$\int d^3\underline{r} \, e^{i(k \pm k').\underline{r}} = (2\pi)^3 \delta^3(\underline{k} \pm \underline{k}') .$$

Therefore we should choose

$$N(\underline{k}) = \frac{1}{(2\pi)^3 \, 2\omega(\underline{k})}$$

$$\Rightarrow \quad H = \int \underbrace{\frac{d^3\underline{k}}{(2\pi)^3 \, 2\omega(\underline{k})}}_{\substack{\text{usual relativistic}\\ \text{phase space}\\ \text{(density of states)}}} \cdot \omega(\underline{k}) \, |a(\underline{k})|^2$$

Second Quantization

* First quantization was the procedure of replacing classical dynamical variables q and p for a particle by quantum operators $\hat{q}$ and $\hat{p}$ such that

$$[\hat{q}, \hat{p}] = i \qquad (\hbar = 1)$$

* Second quantization is replacing the field variable $\phi(x,t)$ and its conjugate momentum density $\pi(x,t)$ by operators such that

$$[\hat{\phi}(x,t), \hat{\pi}(y,t)] = i\,\delta(x-y)$$

NB: x and y are not dynamical variables but labels for the field values at different points. Compare (and contrast)

$$[\hat{q}_j, \hat{p}_k] = i\,\delta_{jk} \qquad (j,k = 1,2,3)$$

* The wave function ϕ satisfying the Klein-Gordon equation is replaced by the field operator $\hat{\phi}$, satisfying the same equation.

* The Fourier representation becomes

$$\hat{\phi}(x,t) = \int dk \; N(k) \left[\hat{a}(k) e^{ikx - i\omega t} + \hat{a}^{+}(k) e^{-ikx + i\omega t} \right]$$

i.e. $\hat{\phi}$ is hermitian but the Fourier conjugate operator $\hat{a}$ is not.

* Keeping track of the order of operators, the Hamiltonian operator is

$$\hat{H} = \int dk \; N(k) \cdot \tfrac{1}{2} \omega(k) \left[\hat{a}(k) \hat{a}^{+}(k) + \hat{a}^{+}(k) \hat{a}(k) \right]$$

* Compare this with the simple harmonic oscillator:

$$\hat{H}_{SHO} = \tfrac{1}{2} \omega \left(\hat{a} \hat{a}^{+} + \hat{a}^{+} a \right)$$

→ $\hat{a}^{+}(k)$ and $\hat{a}(k)$ are ladder operators for the mode of wave number k:

They add / remove one quantum of excitation of the mode.

These quanta are the particles corresponding to that field,

$\hat{a}^{+}(k)$ = <u>creation</u> operator

$\hat{a}(k)$ = <u>annihilation</u> operator

for Klein-Gordon particles.

Ladder operators of simple harmonic oscillator satisfy

$$[\hat{a}, \hat{a}^{+}] = 1 .$$

The analogous commutation relation for the creation and annihilation operators is

$$N(k)\,[\hat{a}(k), \hat{a}^{+}(k')] = \delta(k-k')$$

i.e. $$[\hat{a}(k), \hat{a}^{+}(k')] = 2\pi \cdot 2\omega(k)\, \delta(k-k')$$

or in 3 dimensions

$$\boxed{[\hat{a}(\underline{k}), \hat{a}^{+}(\underline{k}')] = (2\pi)^3 \cdot 2\omega(k)\, \delta^3(\underline{k}-\underline{k}')}$$

On the other hand

$$[\hat{a}(\underline{k}), \hat{a}(\underline{k}')] = [\hat{a}^{+}(\underline{k}), \hat{a}^{+}(\underline{k}')] = 0$$

* These correspond to the field commutation relation

$$[\hat{\phi}(\underline{x},t), \hat{\pi}(\underline{y},t)] = \int d^3\underline{k}\, d^3\underline{k}'\, N(\underline{k})\, N(\underline{k}') \,\times$$

$$\times \left[\hat{a}(\underline{k})e^{i\underline{k}.\underline{x}-i\omega t} + \hat{a}^{+}(\underline{k})e^{-i\underline{k}.\underline{x}+i\omega t}, -i\omega'\hat{a}(\underline{k}')e^{i\underline{k}'.\underline{y}-i\omega' t} + i\omega'\hat{a}^{+}(\underline{k}')e^{-i\underline{k}'.\underline{y}+i\omega' t}\right]$$

$$= i\int d^3\underline{k}\, N(\underline{k})\, \omega(\underline{k}) \left(e^{i\underline{k}.(\underline{x}-\underline{y})} + e^{-i\underline{k}.(\underline{x}-\underline{y})}\right)$$

$$= i\,\delta^3(\underline{x}-\underline{y}) \qquad \text{as expected}$$

<u>N.B.</u> On the other hand

$$[\hat{\phi}(\underline{x},t), \hat{\phi}(\underline{y},t)] = 0$$

* The fact that the field operator has positive- and negative-frequency parts now appears quite natural:

$$\hat{\phi}(x,t) = \int dk\, N(k) \Big[\underbrace{\hat{a}(k)\, e^{ikx - i\omega t}}_{\substack{\text{positive-frequency part} \\ \text{annihilates particles}}} + \underbrace{\hat{a}^{\dagger}(k)\, e^{-ikx + i\omega t}}_{\substack{\text{negative-frequency part} \\ \text{creates particles}}} \Big]$$

* $\pm \hbar\omega$ is the energy released/absorbed in the annihilation/creation process.

  <u>NB</u>: The Hermitian field describes particles that are identical to their antiparticles, e.g. π^0 mesons.

* More generally (as we shall see shortly) the <u>negative-frequency</u> part of $\hat{\phi}$ <u>creates antiparticles</u>

Single-Particle States

If $|0\rangle$ represents the state with no particles present (the vacuum), then $\hat{a}^+(\underline{k})\,|0\rangle$ is a state containing a particle of wave vector $\underline{k}$ (i.e. momentum $\hbar\underline{k}$).

* More generally, to make a state with wave function $\phi(\underline{r},t)$ where

$$\phi(\underline{r},t) = \int d^3\underline{k}\ \tilde{\phi}(\underline{k})\, e^{i\underline{k}.\underline{r} - i\omega t}$$

we should operate on the vacuum with

$$\int d^3\underline{k}\ \tilde{\phi}(\underline{k})\ \hat{a}^+(\underline{k}) .$$

Writing this state as

$$|\phi\rangle = \int d^3\underline{k}\ \tilde{\phi}(\underline{k})\,\hat{a}^+(\underline{k})\,|0\rangle$$

we can find the wave function using the relation

$$\boxed{\langle 0|\,\hat{\phi}(\underline{r},t)\,|\phi\rangle = \phi(\underline{r},t)}$$

i.e. the field operator is an operator for "finding out the wave function" (for single-particle states).

Two-Particle States

Similarly we can make a state $|\phi_{12}\rangle$ of two particles:

$$|\phi_{12}\rangle = \int d^3\underline{k}_1\, d^3\underline{k}_2\, \tilde{\phi}(\underline{k}_1, \underline{k}_2)\, \hat{a}^+(\underline{k}_1)\, \hat{a}^+(\underline{k}_2)\, |0\rangle .$$

Since the two particles are identical bosons, this is symmetric in the labels 1 and 2 (even if $\tilde{\phi}$ isn't) because $\hat{a}^+(\underline{k}_1)$ and $\hat{a}^+(\underline{k}_2)$ commute

$$\begin{aligned} |\phi_{12}\rangle &= \int d^3\underline{k}_1\, d^3\underline{k}_2\, \tilde{\phi}(\underline{k}_1, \underline{k}_2)\, \hat{a}^+(\underline{k}_1)\, \hat{a}^+(\underline{k}_2)\, |0\rangle \\ &= \int d^3\underline{k}_1\, d^3\underline{k}_2\, \tilde{\phi}(\underline{k}_2, \underline{k}_1)\, \hat{a}^+(\underline{k}_2)\, \hat{a}^+(\underline{k}_1)\, |0\rangle \\ &= |\phi_{21}\rangle \end{aligned}$$

* The 2-particle wave function is

$$\begin{aligned} \phi(\underline{r}_1, r_2, t) &= \langle 0|\, \hat{\phi}(\underline{r}_1, t)\, \hat{\phi}(r_2, t)\, |\phi_{12}\rangle \\ &= \phi(\underline{r}_2, r_1, t) . \end{aligned}$$

Thus quantum field theory is also a good way of dealing with systems of many identical particles.

Number Operator

* Recall that for the simple harmonic oscillator the operator $\hat{a}^+\hat{a}$ tells us the number of quanta of excitation:

$$\hat{a}^+ \hat{a} \, |\phi_n\rangle = n \, |\phi_n\rangle$$

n^{th} excited state

* Similarly, in field theory the operator

$$\hat{N} = \int dk \, N(k) \, \hat{a}^+(k) \, \hat{a}(k)$$

counts the number of particles in a state.

E.g.

$$\hat{N} \, |\phi_{12}\rangle = \int dk \, dk_1 \, dk_2 \, N(k) \, \tilde{\phi}(k_1,k_2) \, \hat{a}^+(k) \hat{a}(k) \, \hat{a}^+(k_1) \hat{a}^+(k_2) \, |0\rangle$$

$$N(k) \, \hat{a}^+(k) \, \underline{\hat{a}(k) \, \hat{a}^+(k_1)} \, \hat{a}^+(k_2) =$$

$$= N(k) \, \hat{a}^+(k) \, \hat{a}^+(k_1) \, \underline{a(k) \, \hat{a}^+(k_2)} + \hat{a}^+(k) \, \hat{a}^+(k_2) \, \delta(k-k_1)$$

$$= N(k) \, \hat{a}^+(k) \, \hat{a}^+(k_1) \, \hat{a}^+(k_2) \, a(k) +$$

$$+ \hat{a}^+(k) \, \hat{a}^+(k_1) \, \delta(k-k_2) + \hat{a}^+(k) \, \hat{a}^+(k_2) \, \delta(k-k_1)$$

$$\Rightarrow \quad \hat{N} \, |\phi_{12}\rangle = 0 + |\phi_{12}\rangle + |\phi_{12}\rangle = \underline{\underline{2}} \, |\phi_{12}\rangle$$

because $\boxed{\hat{a}(k) \, |0\rangle = 0}$ $\longleftarrow$ definition of $|0\rangle$

<u>NB</u>: In general, states do not have to be eigenstates of $\hat{N}$ — they need not contain a definite number of particles.

Electromagnetic Field

* Each component of A^μ (in Lorenz gauge) is quantized like a massless Klein-Gordon field (i.e. with $\omega = |\underline{k}|$):

$$\hat{A}^\mu(\underline{r},t) = \sum_{p=L,R} \int \frac{d^3\underline{k}}{(2\pi)^3\, 2\omega} \cdot \Big[\varepsilon_p^\mu(\underline{k})\, \hat{a}_p(\underline{k})\, e^{i\underline{k}.\underline{r} - i\omega t} + \varepsilon_p^{*\mu}(\underline{k})\, \hat{a}_p^\dagger(\underline{k})\, e^{-i\underline{k}.\underline{r} + i\omega t} \Big]$$

where $\hat{a}_p(\underline{k})$ annihilates a photon with momentum $\underline{k}$ and polarization p (= L or R) and $\hat{a}_p^\dagger(\underline{k})$ creates one.

* $\varepsilon^\mu_{L,R}(\underline{k})$ are the left/right-handed circular polarization 4-vectors for wave vector $\underline{k}$,

[e.g. recall that for $\underline{k}$ along the z-axis

$$\varepsilon^\mu_{L,R} = \frac{1}{\sqrt{2}}(0, 1, \mp i, 0)$$]

and

$$\boxed{\left[\hat{a}_p(\underline{k}), \hat{a}_{p'}^\dagger(\underline{k}')\right] = (2\pi)^3\, 2\omega\, \delta_{pp'}\, \delta^3(\underline{k} - \underline{k}')}$$

different polarizations commute

* The Hamiltonian for the e.m. field is

$$\hat{H} = \frac{1}{2}\int d^3\underline{r}\,\left(\hat{\underline{E}}^2 + \hat{\underline{B}}^2\right)$$

where (in $A^0 = 0$ gauge) from

$$\hat{\underline{E}} = -\frac{\partial \hat{A}}{\partial t} \quad , \quad \hat{\underline{B}} = \underline{\nabla} \times \hat{\underline{A}}$$

we have

$$\hat{\underline{E}} = \sum_p \int \frac{d^3\underline{k}}{(2\pi)^3 2\omega}\, i\omega \left[\underline{\varepsilon}_p\, \hat{a}_p e^{-ik.x} - \underline{\varepsilon}_p^*\, \hat{a}_p^+ e^{+ik.x}\right]$$

$$\hat{\underline{B}} = \sum_p \int \frac{d^3\underline{k}}{(2\pi)^3 2\omega}\, i\underline{k} \times \left[\underline{\varepsilon}_p\, \hat{a}_p e^{-ik.x} - \underline{\varepsilon}_p^*\, \hat{a}_p^+ e^{+ik.x}\right]$$

Using

$$\underline{k} \times \underline{\varepsilon}_L = i\omega\, \underline{\varepsilon}_L$$

$$\underline{k} \times \underline{\varepsilon}_R = -i\omega\, \underline{\varepsilon}_R$$

we find, as expected,

$$\hat{H} = \sum_p \int \frac{d^3\underline{k}}{(2\pi)^3 2\omega} \cdot \frac{1}{2}\omega \left[\hat{a}_p(\underline{k})\, \hat{a}_p^+(\underline{k}) + \hat{a}_p^+(\underline{k})\, a_p(\underline{k})\right]$$

Vacuum Energy and Normal Ordering

* Using the above expression for the e.m. Hamiltonian and the commutation relation for the photon annihilation + creation operators, we find that the energy of the vacuum is

$$\hat{H}\,|0\rangle = \sum_p \int d^3\underline{k} \cdot \tfrac{1}{2}\omega \cdot \lim_{\underline{k}' \to \underline{k}} \delta^3(\underline{k}-\underline{k}')\,|0\rangle$$

* In a finite volume V, we should interpret

$$\delta^3(\underline{k}-\underline{k}') = \int_V \frac{d^3\underline{r}}{(2\pi)^3}\, e^{i(\underline{k}-\underline{k}')\cdot\underline{r}} \xrightarrow[\underline{k}\to\underline{k}']{} \frac{V}{(2\pi)^3}$$

Hence the vacuum energy density is

$$\underline{u_0 = \infty} \quad !$$

* This is due to the zero-point fluctuations of the e.m. field ($\frac{1}{2}\hbar\omega$ per mode). In reality, we don't understand physics at very short distances (very large k) and presumably the integral gets cut off at some large $|\underline{k}| \sim \Lambda$. We shall see that quantities we can measure do not depend much on Λ or the form of the cut-off.

* For most purposes we can throw away the vacuum energy term in $\hat{H}$, which corresponds to measuring all energies relative to the vacuum. This means writing

$$\hat{H} = \sum_p \int \frac{d^3k}{(2\pi)^3 2\omega} \cdot \omega \, \hat{a}_p^\dagger(\underline{k}) \, \hat{a}_p(\underline{k})$$

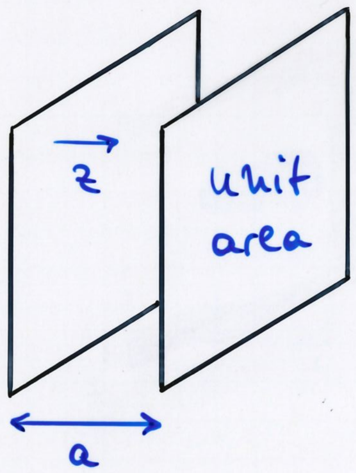
which is called the <u>normal-ordered</u> form of $\hat{H}$, i.e. a form with the annihilation operator $\hat{a}$ on the right. (Note that $\hat{H}$ is still hermitian.)

Clearly a normal-ordered operator has the vacuum as an eigenstate with eigenvalue zero.

* However, we should not think that this means vacuum fluctuations are not there. They give rise to real effects, such as spontaneous transitions and... :

The Casimir Effect

Suppose a region of vacuum is bounded by the plane surfaces of two semi-infinite conductors. Then some (low-frequency) vacuum fluctuations are forbidden by the boundary conditions ($E_{\parallel} = B_{\perp} = 0$). Thus the vacuum energy is reduced, corresponding to an attractive force between the conductors.



* Since the vacuum energy density is proportional to $\hbar c$, by dimensions this Casimir force per unit area must be

$$F_c \propto \frac{\hbar c}{a^4}$$

* To find the constant of proportionality, note that $k_z = \frac{n\pi}{a}$, $n = 0, 1, 2, \ldots$, so

$$\omega = c|\underline{k}| = c\sqrt{\underline{k}_{\parallel}^2 + \left(\frac{n\pi}{a}\right)^2}$$

* For $n = 1, 2, 3 \ldots$ there are 2 polarizations, for $n = 0$, only one polarization is allowed ($E_{\parallel} = 0$)

Hence the energy per unit area is

$$E = \frac{1}{2}\hbar c \int \frac{d^2\underline{k}_\parallel}{(2\pi)^2} \left\{ |k_\parallel| + 2\sum_{n=1}^{\infty} \sqrt{\underline{k}_\parallel^2 + \left(\frac{n\pi}{a}\right)^2} \right\}$$

This is assumed to be cut off by new physics at wave numbers $|\underline{k}| > \Lambda$. Thus

$$E = \frac{\hbar c}{2\pi} \left[\frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n) \right]$$

where

$$F(n) = \int k_\parallel \, dk_\parallel \sqrt{k_\parallel^2 + \left(\frac{n\pi}{a}\right)^2} \, f\left(\sqrt{k_\parallel^2 + \left(\frac{n\pi}{a}\right)^2}\right)$$

with

$$f(k) = \begin{cases} 1 & \text{for } k \ll \Lambda \\ 0 & \text{for } k \gg \Lambda \end{cases}$$

Changing variable $k_\parallel \longrightarrow k = \sqrt{k_\parallel^2 + (n\pi/a)^2}$

$$F(n) = \int_{\frac{n\pi}{a}}^{\infty} k^2 \, dk \, f(k)$$

Removing the boundary conditions would allow n to be a continuous variable:

$$E_0 = \frac{\hbar c}{2\pi} \int_0^\infty dn \, F(n)$$

Hence the change in the vacuum energy is

$$\delta E = E - E_0 = \frac{\hbar c}{2\pi} \left[\frac{1}{2}F(0) + \sum_{n=1}^{\infty} F(n) - \int_0^\infty dn \, F(n) \right]$$

* The Euler - MacLaurin formula is

$$\int_0^\infty dn\, F(n) = \frac{1}{2} F(0) + \sum_{n=1}^{\infty} \left\{ F(n) + \frac{1}{(2n)!} B_{2n} F^{(2n-1)}(0) \right\}$$

where B_{2n} are Bernoulli numbers

$$B_2 = \frac{1}{6} \ , \qquad B_4 = -\frac{1}{30} \ , \quad \ldots$$

$$\Rightarrow \quad \delta E = \frac{\hbar c}{2\pi} \left[-\frac{1}{12} F'(0) + \frac{1}{720} F'''(0) + \cdots \right]$$

Now
$$F(n) = \int_{n\pi/a}^{\infty} k^2 f(k)\, dk$$

$$\Rightarrow \quad F'(n) = -\frac{\pi}{a} \left(\frac{n\pi}{a}\right)^2 f\left(\frac{n\pi}{a}\right)$$

Hence $F'(0) = 0$ and $F'''(0) = -2\left(\frac{\pi}{a}\right)^3$

$$\Rightarrow \quad \delta E = \frac{-\pi^2}{720} \frac{\hbar c}{a^3}$$

$$\boxed{F_c = \frac{d}{da} \delta E = \frac{\pi^2}{240} \frac{\hbar c}{a^4}}$$

* This is very small,

$$F_c = \frac{1.3 \times 10^{-27}}{a^4} \ \mathrm{Pa\ m^4}$$

but it has been measured (Sparnay, 1957)

Complex Fields

Suppose $\hat{\phi}$ is the second-quantized version of a complex field, i.e. $\hat{\phi}^+ \neq \hat{\phi}$.
We can always decompose it into

$$\hat{\phi} = \frac{1}{\sqrt{2}}\left(\hat{\phi}_1 + i\,\hat{\phi}_2\right)$$

where $\hat{\phi}_1$ and $\hat{\phi}_2$ are hermitian. Then

$$\hat{\phi}(x,t) = \int dk\; N(k)\left[\hat{a}(k)\,e^{ikx-i\omega t} + \hat{b}^+(k)\,e^{-ikx+i\omega t}\right]$$

where $\hat{a} = \frac{1}{\sqrt{2}}(\hat{a}_1 + i\,\hat{a}_2)$, $\hat{b}^+ = \frac{1}{\sqrt{2}}(\hat{a}_1^+ + i\,\hat{a}_2^+) \neq \hat{a}^+$.

* From the canonical commutation relations

$$\left[\hat{\phi}_j(x,t),\, \hat{\pi}_l(y,t)\right] = i\,\delta_{jl}\,\delta(x-y) \qquad (j,l = 1,2)$$

we can deduce

$$N(k)\left[\hat{a}_j(k),\, \hat{a}_l^+(k')\right] = \delta_{jl}\,\delta(k-k')$$

and hence

$$N(k)\left[\hat{a}(k),\, \hat{a}^+(k')\right] = N(k)\left[\hat{b}(k),\, \hat{b}^+(k')\right] = \delta(k-k')$$

$$\left[\hat{a}(k),\, \hat{b}^+(k')\right] = \left[\hat{a}(k),\, \hat{b}(k')\right] = \left[\hat{a}^+(k),\, \hat{b}^+(k')\right] = 0$$

* Hence $\hat{a}^+$ and $\hat{b}^+$ are creation operators for <u>different</u> particles.

* The Lagrangian density

$$\mathcal{L} = \mathcal{L}[\phi_1] + \mathcal{L}[\phi_2]$$

can be written as

$$\mathcal{L} = \frac{\partial \hat{\phi}^+}{\partial t}\frac{\partial \hat{\phi}}{\partial t} - \frac{\partial \hat{\phi}^+}{\partial x}\frac{\partial \hat{\phi}}{\partial x} - m^2 \hat{\phi}^+ \hat{\phi} .$$

The canonical momentum density is thus

$$\hat{\pi} \equiv \partial\mathcal{L}/\partial\dot{\hat{\phi}} = \partial\hat{\phi}^+/\partial t$$

and the Hamiltonian density is

$$\hat{\mathcal{H}} = \hat{\pi}\,\dot{\hat{\phi}} + \hat{\pi}^+\,\dot{\hat{\phi}}^+ - \mathcal{L}$$

$$= \hat{\pi}^+\hat{\pi} + \frac{\partial \hat{\phi}^+}{\partial x}\frac{\partial \hat{\phi}}{\partial x} + m^2 \hat{\phi}^+ \hat{\phi}$$

* Using the Fourier expansion of $\hat{\phi}$ and integrating over all space, we find

$$\hat{H} = \int dx\, \hat{\mathcal{H}} = \frac{1}{2}\int dk\, N(k)\,\omega(k)\cdot$$

$$\cdot \left[\hat{a}(k)\hat{a}^+(k) + \hat{a}^+(k)\hat{a}(k) + \hat{b}(k)\hat{b}^+(k) + \hat{b}^+(k)\hat{b}(k)\right]$$

or after normal ordering

$$\hat{H} = \int dk\, N(k)\,\omega(k)\left[\hat{a}^+(k)\hat{a}(k) + \hat{b}^+(k)\hat{b}(k)\right]$$

i.e. both $\hat{a}^+$ and $\hat{b}^+$ create particles with <u>positive energy</u> $\hbar\omega(k)$.

Symmetries and Conservation Laws

We want to find a current and a density that satisfy the continuity equation for the complex Klein-Gordon field.

We use an important general result called Noether's Theorem (Emmy Noether, 1918), which tells us that there is a conserved current associated with every continuous symmetry of the Lagrangian, i.e. with symmetry under transformations of the form

$$\phi \longrightarrow \phi + \delta\phi$$

where $\delta\phi$ is infinitesimal. Symmetry means

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi'}\,\delta\phi' + \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\,\delta\dot{\phi} = 0$$

where

$$\delta\phi' = \delta\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}\,\delta\phi$$

$$\delta\dot{\phi} = \delta\left(\frac{\partial\phi}{\partial t}\right) = \frac{\partial}{\partial t}\,\delta\phi\,,$$

which is easily generalized to 3 spatial dimensions.

* The Euler - Lagrange equation of motion

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = 0$$

then implies that

$$\delta \mathcal{L} = \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right)\delta\phi + \frac{\partial \mathcal{L}}{\partial \phi'}\frac{\partial}{\partial x}(\delta\phi) + \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right)\delta\phi + \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\frac{\partial}{\partial t}(\delta\phi) = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\delta\phi\right) + \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\delta\phi\right) = 0$$

* Comparing with the conservation equation

$$\frac{\partial}{\partial x}(J_x) + \frac{\partial \rho}{\partial t} = 0 \qquad \text{(in 1 dim.)}$$

shows that the conserved density and current are (proportional to)

$$\boxed{\begin{aligned} \rho &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\,\delta\phi \\ J_x &= \frac{\partial \mathcal{L}}{\partial \phi'}\,\delta\phi \end{aligned}}$$

* In 3 spatial dimensions

$$J_x = \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x)}\,\delta\phi \;, \quad J_y = \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial y)}\,\delta\phi \;, \;\dots$$

and hence in covariant notation

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta\phi$$

* If the Lagrangian involves several fields $\phi_1, \phi_2, \dots$, the symmetry involves changing them all in general:

invariance w.r.t. $\phi_j \longrightarrow \phi_j + \delta\phi_j$

$\Rightarrow$ conserved Noether current

$$\boxed{J^\mu = \sum_j \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_j)}\,\delta\phi_j}$$

* After second quantization, the same procedure (being careful about the order of operators!) can be used to define conserved current and density operators.

Phase (Gauge) Invariance

* The Klein-Gordon Lagrangian density

$$\mathcal{L} = \frac{\partial \hat{\phi}^+}{\partial t} \cdot \frac{\partial \hat{\phi}}{\partial t} - \frac{\partial \hat{\phi}^+}{\partial x} \cdot \frac{\partial \hat{\phi}}{\partial x} - m^2 \hat{\phi}^+ \hat{\phi}$$

or in 3 spatial dimensions

$$\mathcal{L} = \frac{\partial \hat{\phi}^+}{\partial t} \cdot \frac{\partial \hat{\phi}}{\partial t} - \underline{\nabla} \hat{\phi}^+ \cdot \underline{\nabla} \hat{\phi} - m^2 \hat{\phi}^+ \hat{\phi}$$

$$= \partial_\mu \hat{\phi}^+ \cdot \partial^\mu \hat{\phi} - m^2 \hat{\phi}^+ \phi$$

is invariant w.r.t. a <u>global phase change</u> in $\hat{\phi}$:

$$\begin{cases} \hat{\phi} \longrightarrow e^{-i\varepsilon} \hat{\phi} \simeq \hat{\phi} \underbrace{- i\varepsilon \hat{\phi}}_{\delta \hat{\phi}} \\ \hat{\phi}^+ \longrightarrow e^{+i\varepsilon} \hat{\phi}^+ \simeq \hat{\phi}^+ \underbrace{+ i\varepsilon \hat{\phi}^+}_{\delta \hat{\phi}^+} \end{cases} \qquad (\varepsilon \text{ const.})$$

* The corresponding conserved Noether current is just that derived earlier:

$$\hat{j}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \hat{\phi})} \delta \hat{\phi} + \delta \hat{\phi}^+ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \hat{\phi}^+)}$$

$$= -i \left(\partial^\mu \hat{\phi}^+\right) \hat{\phi} + i \hat{\phi}^+ \left(\partial^\mu \hat{\phi}\right)$$

* We can define an associated **conserved charge**, which is the integral of $\hat{\rho}$ over all space:

$$\hat{Q} = \int \hat{\rho}\, d^3\underline{r}$$

$$\left[\Rightarrow \frac{d\hat{Q}}{dt} = \int \frac{\partial \hat{\rho}}{\partial t} d^3\underline{r} = -\int \underline{\nabla} \cdot \underline{J}\, d^3\underline{r} = \int_{\infty\ \text{sphere}} \underline{J} \cdot d\underline{s} = 0 \right]$$

* In this case

$$\hat{Q} = -i \int \left(\frac{\partial \hat{\phi}^+}{\partial t} \hat{\phi} - \hat{\phi}^+ \frac{\partial \hat{\phi}}{\partial t} \right) d^3\underline{r} .$$

Inserting the Fourier decomposition of the field,

$$\hat{\phi} = \int d^3\underline{k}\ N(\underline{k}) \left[\hat{a}(\underline{k})\, e^{-ik.x} + \hat{b}^+(\underline{k})\, e^{+ik.x} \right]$$

where $N(\underline{k}) = \left[(2\pi)^3\, 2\omega(\underline{k}) \right]^{-1}$, we find

$$\hat{Q} = \int d^3\underline{k}\ N(\underline{k}) \left[\hat{a}^+(\underline{k})\, \hat{a}(\underline{k}) - \hat{b}(\underline{k})\, \hat{b}^+(\underline{k}) \right]$$

i.e. after normal ordering

$$\boxed{\hat{Q} = \int d^3\underline{k}\ N(\underline{k}) \left[\hat{a}^+(\underline{k})\, \hat{a}(\underline{k}) - \hat{b}^+(\underline{k})\, \hat{b}(\underline{k}) \right]}$$

!

* Comparing with energy

$$\hat{H} = \int d^3\underline{k}\ N(\underline{k})\, \omega(\underline{k}) \left[\hat{a}^+(\underline{k})\, \hat{a}(\underline{k}) + \hat{b}^+(\underline{k})\, \hat{b}(\underline{k}) \right]$$

we see that the particles created by $\hat{a}^+$ and $\hat{b}^+$ have **opposite charge**.

* Thus we see that in quantum field theory all the 'problems' with the negative-energy solutions of the Klein-Gordon equation are resolved, as follows:

> The object ϕ that satisfies the K.-G. equation is in fact the field operator $\hat{\phi}$.

The Fourier decomposition of $\hat{\phi}$,

$$\hat{\phi} = \int d^3\underline{k}\; N(\underline{k}) \left[\hat{a}(\underline{k})\, e^{i\underline{k}.\underline{r} - i\omega t} + \hat{b}^{+}(\underline{k})\, e^{-i\underline{k}.\underline{r} + i\omega t} \right],$$

has a positive-frequency part that annihilates a particle (with energy $\hbar\omega$ momentum $\hbar\underline{k}$ and charge $+1$), and a negative-frequency part that creates an antiparticle (with energy $\hbar\omega$, momentum $\hbar\underline{k}$ and charge -1).

* Similarly $\hat{\phi}^{+}$ creates a particle or annihilates an antiparticle.

The Dirac Field

The Lagrangian density that gives the Dirac equation of motion,

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \quad ,$$

is

$$\mathcal{L}_D = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \quad .$$

As in the Klein-Gordon case we should treat ψ and $\bar{\psi} = \psi^+\gamma^0$ as independent fields. The Euler-Lagrange equation for $\bar{\psi}$ gives the Dirac equation for ψ:

$$\frac{\partial \mathcal{L}_D}{\partial \bar{\psi}} - \partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}_D}{\partial(\partial_\mu \bar{\psi})} \right)}_{=0} = 0$$

$$\Rightarrow \qquad i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

while that for ψ gives the Dirac equation for $\bar{\psi}$:

$$\frac{\partial \mathcal{L}_D}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}_D}{\partial(\partial_\mu \psi)} \right)$$

$$= -m\bar{\psi} - i\partial_\mu (\bar{\psi}\gamma^\mu) = 0$$

Check: $-i\partial_\mu \psi^+ \gamma^{\mu +} - m\psi^+ = 0 \quad , \qquad \bar{\psi} = \psi^+\gamma^0 ,$

$$\gamma^{\mu +}\gamma^0 = \gamma^0\gamma^\mu \quad \Rightarrow \quad -i\partial_\mu \bar{\psi}\gamma^\mu = m\bar{\psi}$$

* The canonical momentum densities are

$$\pi = \partial \mathcal{L}_D / \partial \dot{\psi} = \bar{\psi} i \gamma^0 = i \psi^+$$

$$\bar{\pi} = \partial \mathcal{L}_D / \partial \dot{\bar{\psi}} = 0$$

* Hence the Hamiltonian density is

$$\mathcal{H}_D = \pi \dot{\psi} - \mathcal{L}_D = \bar{\psi} i \gamma^0 \frac{\partial \psi}{\partial t} - \mathcal{L}_D$$
$$= - \bar{\psi} i \underline{\gamma} \cdot \underline{\nabla} \psi + m \bar{\psi} \psi$$

and the total energy is given by

$$H = \int d^3\underline{r}\, \mathcal{H}_D = \int d^3\underline{r}\, \bar{\psi} (-i \underline{\gamma} \cdot \underline{\nabla} + m) \psi$$
$$= \int d^3\underline{r}\, \psi^+ \underbrace{(-i \underline{\alpha} \cdot \underline{\nabla} + \beta m)}_{\text{original Dirac Hamiltonian}} \psi$$

* Now we second quantize by expressing the field operator $\hat{\psi}$ as a Fourier integral over plane-wave spinors with operator coefficients:

$$\hat{\psi}(\underline{r}, t) = \int d^3\underline{k}\, N(\underline{k}) \sum_s \left[\hat{c}_s(\underline{k})\, u_s(\underline{k})\, e^{-ik\cdot x} + \hat{d}_s^+(\underline{k})\, v_s(\underline{k})\, e^{+ik\cdot x} \right]$$

where $s = 1, 2$ labels spin up / down, i.e. $u_1 = u^\uparrow$ free particle spinor, etc.

* We expect that the operator $\hat{c}_s(\underline{k})$ annihilates a particle (e.g. an electron) of momentum $\underline{k}$, spin orientation s, while $\hat{d}_s^+(\underline{k})$ creates an antiparticle (positron) of the same momentum and spin.

However, the Hamiltonian operator is

$$\hat{H} = \int d^3\underline{r}\ \hat{\psi}^+ (-i\underline{\alpha}.\underline{\nabla} + \beta m)\, \hat{\psi}$$

and using the Dirac equation,

$$(-i\underline{\alpha}.\underline{\nabla} + \beta m)\, u_s = E u_s = \omega u_s$$
$$(-i\underline{\alpha}.\underline{\nabla} + \beta m)\, v_s = -E v_s = -\omega v_s \quad ,$$

we find that

$$\hat{H} = \int d^3\underline{k}\ N(\underline{k})\, \omega(\underline{k}) \sum_s \left[\hat{c}_s^+(\underline{k})\, \hat{c}_s(\underline{k}) - \hat{d}_s(\underline{k})\, \hat{d}_s^+(\underline{k}) \right]$$

!

* Thus there appears to be a problem: unlike the Klein-Gordon case we get a <u>negative</u> contribution to the energy from the antiparticles.

* The Dirac Lagrangian also has phase (gauge) symmetry

$$\hat{\psi} \longrightarrow e^{-i\varepsilon}\hat{\psi} \simeq \hat{\psi} - i\varepsilon\hat{\psi}$$

$$\hat{\psi}^{+} \longrightarrow e^{+i\varepsilon}\hat{\psi}^{+} \simeq \hat{\psi}^{+} + i\varepsilon\hat{\psi}^{+}$$

(ε const.)

with corresponding Noether current

$$\hat{j}^{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \hat{\psi})}\,\delta\hat{\psi} + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \hat{\psi}^{+})}}_{=0}\,\delta\hat{\psi}^{+}$$

$$= \bar{\hat{\psi}}\, i\gamma^{\mu}(-i\hat{\psi}) = \bar{\hat{\psi}}\gamma^{\mu}\hat{\psi}$$

and conserved charge

$$\hat{Q} = \int d^3\underline{r}\; \hat{j}^{0} = \int d^3\underline{r}\; \hat{\psi}^{+}\hat{\psi}$$

$$= \int d^3\underline{k}\; N(\underline{k}) \sum_{s} \left[\hat{c}_s^{+}(\underline{k})\,\hat{c}_s(\underline{k}) + \hat{d}_s(\underline{k})\,\hat{d}_s^{+}(\underline{k}) \right]$$

which also seems to have the wrong sign for the antiparticle contribution.

* However, we actually need the normal-ordered operators, which involve $\hat{d}_s^{+}\hat{d}_s$, not $\hat{d}_s\hat{d}_s^{+}$. Hence if

$$\hat{d}_s^{+}\,\hat{d}_s = -\,\hat{d}_s\,\hat{d}_s^{+} + \text{const.}$$

?

then all signs are correct.

* Thus we are forced to give the creation and annihilation operators for spin-½ particles anticommutation relations :

$$\{\hat{c}_s(\underline{k}), \hat{c}^+_{s'}(\underline{k}')\} \equiv \hat{c}_s(\underline{k})\,\hat{c}^+_{s'}(\underline{k}') + \hat{c}^+_{s'}(\underline{k}')\,\hat{c}_s(\underline{k})$$

$$= (2\pi)^3\,\delta_{ss'}\,\delta^3(\underline{k}-\underline{k}')$$

$$= \{\hat{d}_s(\underline{k}), \hat{d}^+_{s'}(\underline{k}')\} \equiv \hat{d}_s(\underline{k})\,\hat{d}^+_{s'}(\underline{k}') + \hat{d}^+_{s'}(\underline{k}')\,\hat{d}_s(\underline{k})$$

$$\{\hat{c}_s(\underline{k}), \hat{c}_{s'}(\underline{k}')\} = \{\hat{d}_s(\underline{k}), \hat{d}_{s'}(\underline{k}')\}$$

$$= \{\hat{c}^+_s(\underline{k}), \hat{c}^+_{s'}(\underline{k}')\} = \{\hat{d}^+_s(\underline{k}), \hat{d}^+_{s'}(\underline{k}')\} = 0$$

* This means that 2-particle states are antisymmetric :

$$|\phi_{12}\rangle = \int d^3\underline{k}_1\, d^3\underline{k}_2 \sum_{s_1,s_2} \tilde{\phi}_{s_1s_2}(\underline{k}_1,\underline{k}_2)\,\hat{c}^+_{s_1}(\underline{k}_1)\,\hat{c}^+_{s_2}(\underline{k}_2)\,|0\rangle$$

$$= -\int d^3\underline{k}_1\, d^3\underline{k}_2 \sum_{s_1,s_2} \tilde{\phi}_{s_1s_2}(\underline{k}_1,\underline{k}_2)\,\hat{c}^+_{s_2}(\underline{k}_2)\,\hat{c}^+_{s_1}(\underline{k}_1)\,|0\rangle$$

$$= -\int d^3\underline{k}_1\, d^3\underline{k}_2 \sum_{s_1,s_2} \tilde{\phi}_{s_2s_1}(\underline{k}_2,\underline{k}_1)\,\hat{c}^+_{s_1}(\underline{k}_1)\,\hat{c}^+_{s_2}(\underline{k}_2)\,|0\rangle$$

$$= -|\phi_{21}\rangle$$

* Thus spin-½ particles must be fermions. This is the spin-statistics theorem.

Interacting Fields

We introduce e.m. interactions into the Dirac Lagrangian density by the usual minimal substitution, $\partial^\mu \rightarrow \partial^\mu + ieA^\mu$:

$$\mathcal{L}_D = \bar{\psi} i \gamma^\mu (\partial_\mu + ieA_\mu)\psi - m\bar{\psi}\psi$$
$$= \mathcal{L}_0 - eA_\mu \bar{\psi}\gamma^\mu\psi$$

where $\mathcal{L}_0$ is the free particle Lagrangian.

* Notice that the canonical momentum $\pi = \partial\mathcal{L}_D / \partial\dot{\psi}$ is unchanged, and so the Hamiltonian density is

$$\mathcal{H}_D = \pi\dot{\psi} - \mathcal{L}_D = \mathcal{H}_0 + \mathcal{H}_I$$

where the interaction Hamiltonian density is

$$\mathcal{H}_I = eA_\mu \bar{\psi}\gamma^\mu\psi .$$

* In the second-quantized theory, $\mathcal{H}_I$ becomes an operator,

$$\hat{\mathcal{H}}_I = e\hat{A}_\mu \hat{\bar{\psi}}\gamma^\mu\hat{\psi}$$

where all the field operators are capable of creating or annihilating particles.

$$\hat{A}_\mu = \int d^3\underline{k}\ N(\underline{k}) \sum_{p} \left[\hat{a}_p(\underline{k})\, \varepsilon_{p\mu}(\underline{k})\, e^{-ik\cdot x} + \hat{a}_p^{+}(\underline{k})\, \varepsilon_{p\mu}^{*}\, e^{+ik\cdot x} \right]$$

$$\hat{\bar{\psi}} = \int d^3\underline{p}'\ N(\underline{p}') \sum_{s'} \left[\hat{c}_{s'}^{+}(\underline{p}')\, \bar{u}_{s'}(\underline{p}')\, e^{+ip'\cdot x} + \hat{d}_{s'}(\underline{p}')\, \bar{v}_{s'}(\underline{p}')\, e^{-ip'\cdot x} \right]$$

$$\hat{\psi} = \int d^3\underline{p}\ N(\underline{p}) \sum_{s} \left[\hat{c}_{s}(\underline{p})\, u_{s}(\underline{p})\, e^{-ip\cdot x} + \hat{d}_{s}^{+}(\underline{p})\, v_{s}(\underline{p})\, e^{+ip\cdot x} \right]$$

* In first-order perturbation theory, the transition matrix element is

$$A_{fi} = -i \int d^4x \, \langle f | \hat{\mathcal{H}}_I | i \rangle$$

* Suppose for example that the initial state $|i\rangle$ contains an electron of momentum $\underline{p}_i$, spin orientation s_i, and a photon of momentum $\underline{q}$, polarization R:

$$|i\rangle = \hat{c}_{s_i}^{+}(\underline{p}_i)\, \hat{a}_R^{+}(\underline{q})\, |0\rangle$$

* Using the (anti) commutation relations for the $\hat{c}$'s and $\hat{a}$'s, the positive-frequency parts of $\hat{A}$ and $\hat{\psi}$ give

$$\hat{A}_\mu^{(+)}\, \hat{\psi}^{(+)} |i\rangle = \varepsilon_{R\mu}(\underline{q})\, u_{s_i}(\underline{p}_i)\, e^{-i(p_i+q)\cdot x}\, |0\rangle$$

* Similarly, if $|f\rangle$ contains an electron of momentum $\underline{p}_f$, spin s_f

$$|f\rangle = \hat{c}_{s_f}^{+}(\underline{p}_f)\, |0\rangle$$

Hence

$$\langle f| = \langle 0|\,\hat{c}_{s_f}(p_f)$$

and hence the negative-frequency part of $\hat{\bar{\psi}}$ gives

$$\langle f|\,\hat{\bar{\psi}}^{(-)} = \bar{u}_{s_f}(p_f)\, e^{+ip_f\cdot x}$$

* Putting everything together,

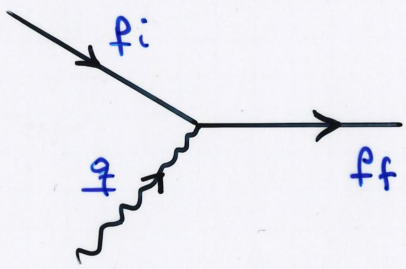
$$\langle f|\,\hat{A}_\mu^{(+)}\,\hat{\bar{\psi}}^{(-)}\,\gamma^\mu\,\hat{\psi}^{(+)}\,|i\rangle =$$

$$= \varepsilon_{R\mu}\,\bar{u}_{s_f}(p_f)\,\gamma^\mu\,u_{s_i}(p_i)\,e^{i(p_f-p_i-q)\cdot x}$$

which gives

$$A_{fi} = -ie\,(2\pi)^4\,\delta^4(p_f-p_i-q)\,\varepsilon_{R\mu}\,\bar{u}_{s_f}(p_f)\,\gamma^\mu\,u_{s_i}(p_i)$$

corresponding to the Feynman rule for the vertex



* Similarly, if the initial and final fermions are positrons, then the term

$$\langle f|\,\hat{A}_\mu^{(+)}\,\hat{\bar{\psi}}^{(+)}\,\gamma^\mu\,\hat{\psi}^{(-)}\,|i\rangle$$

gives the expected result

$$A_{fi} = -ie\,(2\pi)^4\,\delta^4(p_f-p_i-q)\,\varepsilon_{R\mu}\,\bar{v}_{s_i}(p_i)\,\gamma^\mu\,v_{s_f}(p_f)$$