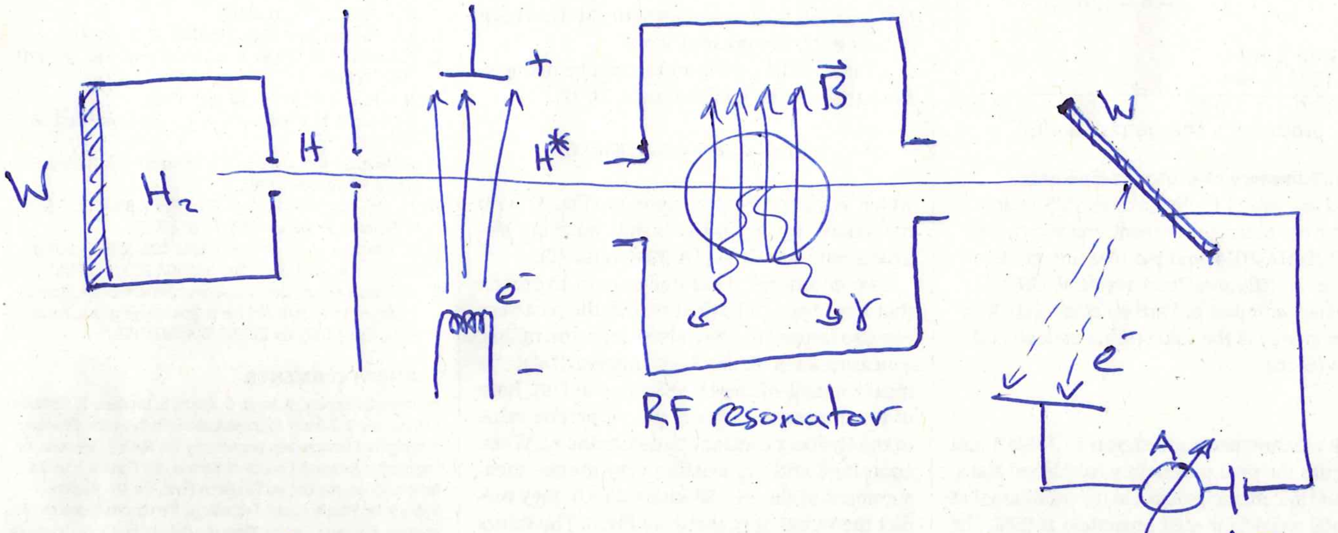
Lamb Shift and QED/bound states

Lamb-Retherford (1947) provided stimulus for a theory of renormalization. Handling of divergences... This helped to drive the development of modern QED.



- Excitation to <u>metastable</u> state, $\tau \sim \frac{1}{7}$ s
- Deflect from main beam, pass μwave/RF with tunable $\vec{B}$
- Hit metal target; <u>excited</u> atoms drop to 1S, emit e^-
    - detect via current, ~10 fA
- If the $\vec{B}$ field-induced energy difference $= \hbar\omega_{RF}$
    - ⇒ 2S H^* atoms absorb a photon
    - ⇒ transition to short-lived 2P ($\tau \sim 1$ ns)
    - ⇒ atoms de-excite rapidly to 1S
    - ⇒ current in detector <u>disappears</u>
- Determined the $2S_{1/2} - 2P_{1/2}$ energy difference ⇒ <u>not zero!</u>

Shelter Island: discussions on self-energy  interaction of a charge w/ its own field

formally: renormalization, corrects the classical energy and depends on distance

P electron: spends much less time near the atomic nucleus, as compared to S electrons

$\Rightarrow$ different (much smaller) correction

Leading effect is 1-loop QED

- virtual photons emitted and re-absorbed
- "zero-point" fluctuations $\rightarrow$ electron response "smeared" over some range

$\Rightarrow$ modification to Coulomb potential, eigen-energies:

$$V(r) \rightarrow \left\langle -\frac{Z\alpha\hbar c}{r+\delta r} \right\rangle$$

$$\delta E_{nlj} = \frac{4\alpha}{3\pi} \frac{(Z\alpha)^5}{n^3} m_e \times \begin{cases} \ln \dfrac{m_e}{2\langle E_{n0}\rangle} + \dfrac{19}{30} & (l=0) \\ \ln \dfrac{Z^2 R_\infty}{\langle E_{nl}\rangle} + \dfrac{3}{8} C_{lj} & (l \neq 0) \end{cases}$$

$$C_{lj}^{-1} = \begin{cases} l+1 & (j = l+\frac{1}{2}) \\ -l & (j = l-\frac{1}{2},\ l \geq 1) \end{cases}$$

Bethe used a cut-off on the order of $m_e c^2$ to argue that there should be contributions from RF modes only up to some energy scale $\Rightarrow$ EFT


Vacuum polarization lowers the 2S state, -27 MHz

Self-energy  over-compensates this $\rightarrow$ needed to get agreement with experiment

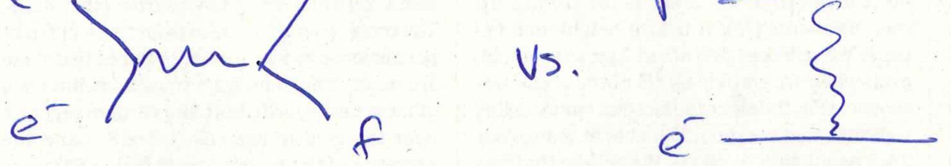
overall: ~ 1057 MHz shift

Static charge at the origin:

$$\frac{Z\alpha\hbar c}{r} \rightarrow \frac{Z\alpha\hbar c}{r} Q(r)$$

$$Q(r) = 1 + \begin{cases} \frac{\alpha}{3\pi} \ln \frac{1}{(mr)^2} + \cdots & (mr \ll 1) \\ \frac{\alpha}{4(\pi m^3 r^3)^{1/2}} e^{-2mr} + \cdots & (mr \gg 1) \end{cases}$$

So we need to bridge scales:



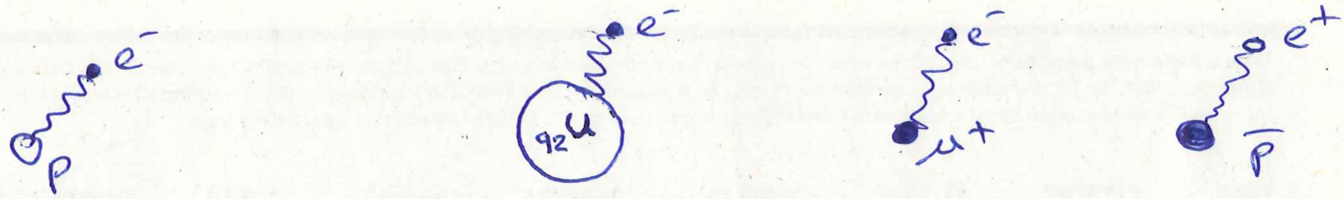
Different test systems:

hadronic structure finite size

$\boxed{\text{strong field}}$

only leptons finite size

antimatter



$m_p \sim 1836 m_e$
$Z\alpha \sim 0.007$
$r_c \sim 0.8\,\text{fm}$

$m_U \sim 4\times10^5 m_e$
$\boxed{Z\alpha \sim \frac{2}{3}}$
$r_c \sim 6\,\text{fm}$

$m_\mu \sim 207 m_e$
$Z\alpha \sim 0.007$

NRSE (Non-Relativistic Schrödinger Equation)

solutions are wavefunctions built from spherical harmonics, Laguerre polynomials

$$H = \frac{p^2}{2\mu} - \frac{Z\alpha}{r}\hbar c$$

$$E_n = -\frac{\mu c^2}{2n^2} Z^2 \alpha^2$$

- no dependence on the e^- angular momentum
- cf. $\Delta E_{\text{Lamb}} \sim (Z\alpha)^4$

Relativistic Dirac Equation (solved by Gordon for hydrogen-like atoms)

$$E_{nj} = \mu c^2 \left[1 + \left(\frac{Z\alpha}{n - |j+1/2| + \sqrt{(j+1/2)^2 - Z^2\alpha^2}} \right)^2 \right]^{-1/2}$$

$$= \mu c^2 \left[1 - \frac{Z^2\alpha^2}{2n^2} + \frac{Z^4\alpha^4}{2n^3}\left(\frac{1}{|j+1/2|} - \frac{3}{4n} \right) + \ldots \right]$$

- depends on angular momentum
- spherical harmonics/Laguerre polynomials → bispinors (4 complex fns)
- nucleus nonmagnetic + pointlike

Radiative corrections: QED

Fluctuations of the fields associated w/ QED vacuum

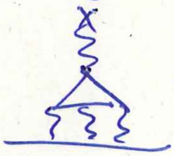
- high Z: small recoil → Lamb
- low Z: leading recoil, separated

overall, $E = E_{Dirac} + E_{R/N} + E_{Lamb}$

Vacuum polarization

$\sim (Z\alpha)^2 \cdot \alpha$

Light-by-light



$\sim \alpha^5 \cdot (Z/Z^2/Z^3)$

Nuclear recoil (finite mass) $\sim (Z\alpha)^4$

Nuclear size and structure $\sim r_c^2$

H: $\frac{\Delta E}{E} \sim 5\times10^{-7}$ for 1S

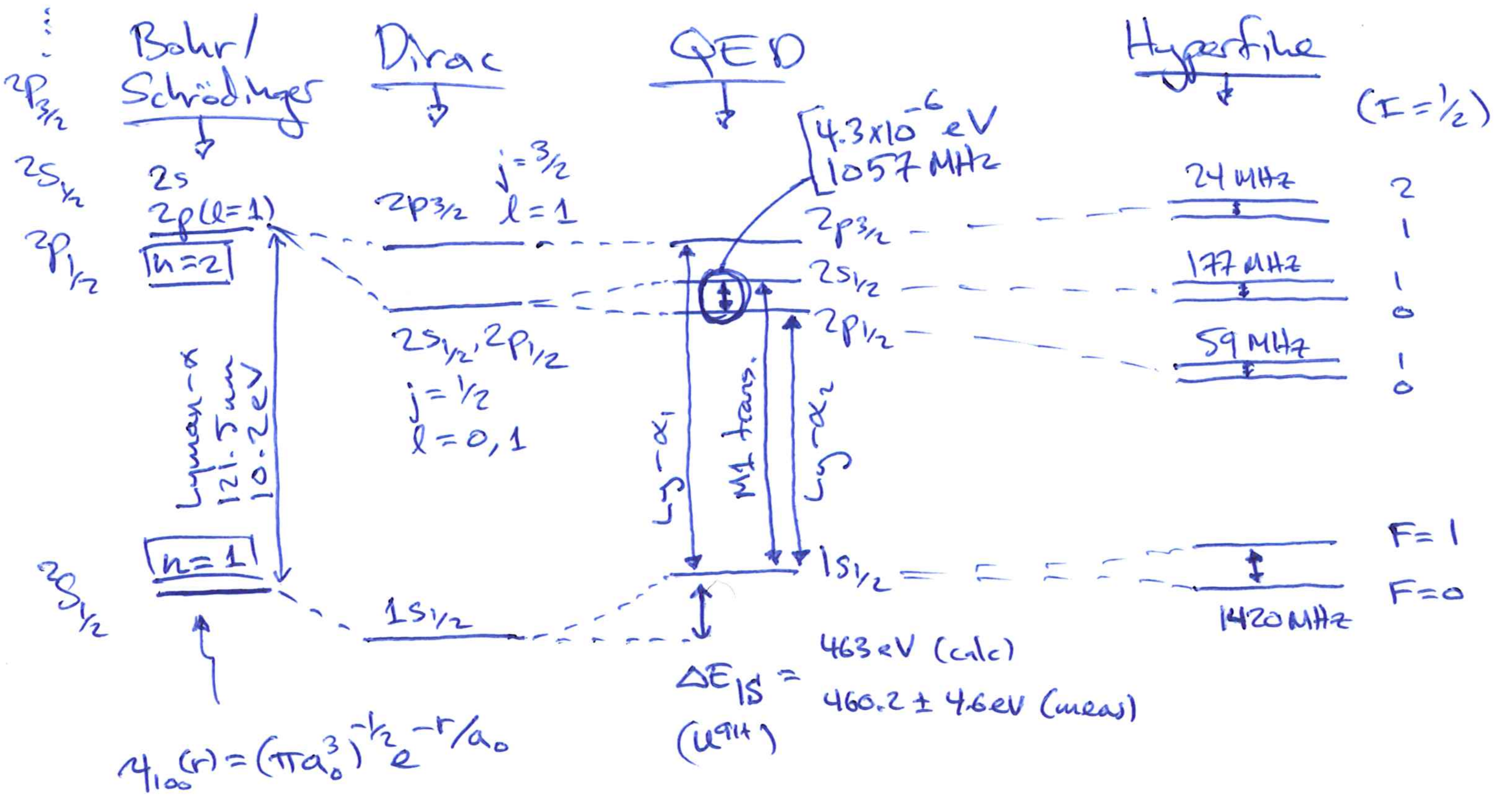
U^{91+}: $\frac{\Delta E}{E} \sim 3.5\times10^{-3}$

ordinary hydrogen

$$E_{nl} = -\frac{R_\infty}{n^2} + \delta_{l,0}\frac{L_{1S} + \alpha r_p^2}{n^3}$$

$$hcR_\infty = \frac{e^2}{4\pi\varepsilon_0 a_0} = \frac{1}{2} m_e c^2 \alpha^2$$

→ need two measurements



To get a feeling for finite nuclear-size effects, consider

$$4\pi\int_0^R r^2|\psi_{100}(r)|^2\,dr = 1-e^{-2r/a_0}\left(1+2\frac{R}{a_0}+2\frac{R^2}{a_0^2}\right)$$

$$= \frac{4}{3}\left(\frac{R}{a_0}\right)^3 - 2\left(\frac{R}{a_0}\right)^4 + \frac{8}{5}\left(\frac{R}{a_0}\right)^5 + \cdots$$

cf. $\frac{4}{3}\pi R^3 \cdot \frac{1}{\pi a_0^3} = \frac{4}{3}\left(\frac{R}{a_0}\right)^3$ take value at origin ~ const.

$$|\psi(0)|^2 = \frac{1}{\pi}\left(\frac{Z}{n a_0}\right)^2$$

Consider how this looks for some alternate systems:

H: $\frac{R}{a_0} \sim 2\times10^{-5}$ $\qquad a_0 \sim 0.5$ Å

proton size contributes $\sim 10^{-4}$ of the 2S-2P Lamb Shift
⇒ fractional shift of transition $\sim 10^{-10}$
⇒ use 10^{-15} precision of 1S-2S measurements (linewidth) + $R_\infty \to r_p$

$\boxed{\mu H: \frac{R}{a_0} \sim 3\times10^{-3}}$ → 10 ppm measurement improved r_p by 10x

U^{91+}: $\frac{R}{a_0} \sim 0.01$

M: $a_0^M \sim a_0^H$

$e^- \to \mu^-$, $a_0 \to a_0/207$
⇒ proton size ~ 2% of 2S-2P Lamb shift!