Muon decay and G_F

From the propagator $\frac{-i g_{\mu\nu}}{q^2 - m_W^2}$ we would expect $G \sim \frac{g_W^2}{m_W^2}$

In fact G_F was originally written down for a pure vector coupling, so for the Standard Model (V-A) we have a factor of $\sqrt{2}$. Including factors of 2 for the chiral projection $\frac{1}{2}(1-\gamma_5)$, the relation is:

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8 m_W^2} = \frac{1}{2v^2}$$

where $v \approx 246$ GeV is the Higgs vev.

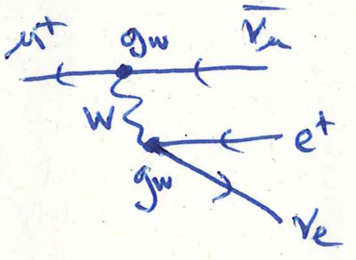
The relation can also be seen from transition matrix elements: ($l \sim e, \mu, \tau$)

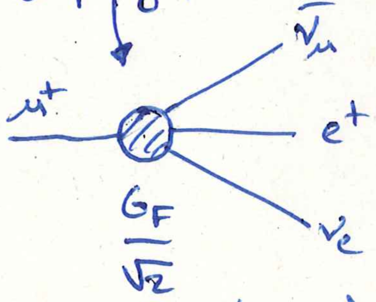
$$M_F = \frac{G_F}{\sqrt{2}} \left[(\bar{l} \gamma_\mu (1-\gamma_5) \nu_l)(\bar{\nu}_l \gamma^\mu (1-\gamma_5) l) \right]$$

$$M_{SM} = \frac{g_W^2}{8} \left[(\bar{l} \gamma_\mu (1-\gamma_5) \nu_l) \frac{1}{q^2 - m_W^2} (\bar{\nu}_l \gamma^\mu (1-\gamma_5) l) \right]$$

In the $q^2 \to 0$ limit, $M_{SM} \to M_F$ and the current-current coupling of the Fermi theory is recovered as a low-energy *effective theory*.

non-local current-current coupling
pointlike 4-fermion





Convenient effective theory for charged-current weak interactions, and for the definition of G_F.

Reminder: helicity/chirality

$$\left.\begin{array}{l} P_R = \frac{1}{2}(1+\gamma_5) \\ P_L = \frac{1}{2}(1-\gamma_5) \end{array}\right\} P_R + P_L = 1$$

→ chiral eigenstates are Lorentz invariant but not stationary states under time evolution

Helicity operator:

$$\vec{\Sigma} \cdot \hat{p} = \begin{pmatrix} \vec{\sigma} \cdot \hat{p} & \vec{0} \\ \vec{0} & \vec{\sigma} \cdot \hat{p} \end{pmatrix} = 2 \vec{S} \cdot \hat{p}$$

negative: $\vec{\sigma} \cdot \hat{p} = -1$

positive: $\vec{\sigma} \cdot \hat{p} = +1$

(often say RH for +1 and LH for −1, but this is only strictly connected to chiral states in the massless/ultrarelativistic limit)

→ helicity states are not Lorentz invariant, but are stationary states of the motion

- Weak interactions (unlike strong/EM) affect all particles.
- Every observed process appears consistent with a universal dimensionless coupling; compare $\frac{g_W^2}{\hbar c}$ similar to $\alpha = \frac{e^2}{\hbar c}$

While the Fermi theory is not renormalizable,

- QED corrections to the leading Fermi interaction are finite to all orders ($\Rightarrow$ separate parameterization of weak interaction vs. EM corrections)
- G_F includes all weak interaction effects in the low energy EFT

The muon lifetime is a powerful means to determine G_F

$\mu^\pm \to e^\pm \nu_e \nu_\mu$

$\tau_\mu \approx 2.2\,\mu s$

- no contribution from hadronic corrections until sub-ppm level (via 2-loop QED) since no hadrons lighter than μ
- τ_μ is well suited to precision measurement
- clear theory interpretation

$$\tau_\mu^{-1} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left(1 + \sum_i \Delta q_i\right)$$

$\Delta q_0 \sim$ phase space

$\Delta q_1 \sim 1^{st}$ order QED

$\Delta q_2 \sim 2^{nd}$ order QED

etc.

Determine G_F from $\tau_\mu \Rightarrow$ relate any fundamental parameters at high energy cf. EW SM interaction

$$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8 m_W^2}\Big(1 + \sum_i \underbrace{\Delta r_i}_{\text{all higher EW corrections}}\Big)$$

can also be rewritten in terms of only observables and higher-order corrections:

Important prediction of SM: $m_W^2\left(1 - \frac{m_W^2}{m_Z^2}\right) = \frac{\pi\alpha}{\sqrt{2}G_F}\left(1 + \sum_i \Delta r_i\right)$

Known decay modes:
$$\begin{cases} \mu^+ \to e^+ \nu_e \bar{\nu}_\mu & \\ \mu^+ \to e^+ \nu_e \bar{\nu}_\mu \gamma & (1.4 \pm 0.4)\times 10^{-2} \\ \mu^+ \to e^+ e^+ e^- \nu_e \bar{\nu}_\mu & (3.4 \pm 0.4)\times 10^{-5} \end{cases}$$

- no neutrinos detected
- photons barely detectable
- positrons overwhelmingly from the ordinary decay branch

For many years the theory uncertainties on the relation of τ_μ to G_F were below ppm, while experimentally τ_μ was known only with 18 ppm uncertainty (PDG 1998) with a resulting 9 ppm uncertainty for G_F:

$$\frac{\Delta G_F}{G_F} = -\frac{5}{2}\underbrace{\frac{\Delta m_\mu}{m_\mu}}_{2.2\times 10^{-8}} - \frac{1}{2}\underbrace{\frac{\Delta \tau_\mu}{\tau_\mu}}_{\boxed{1.0\times 10^{-6}}} + 4\frac{m_{\nu_\mu}^2}{m_\mu^2} + \underbrace{\text{theory uncertainty}}_{1.4\times 10^{-7}}$$

(formerly 1.8×10^{-5})

1.4×10^{-7} (3-loop QED corrections dominate)

Thus there was a strong motivation to improve experimental uncertainty for τ_μ. This was achieved by the MuLan experiment at PSI, with the result:

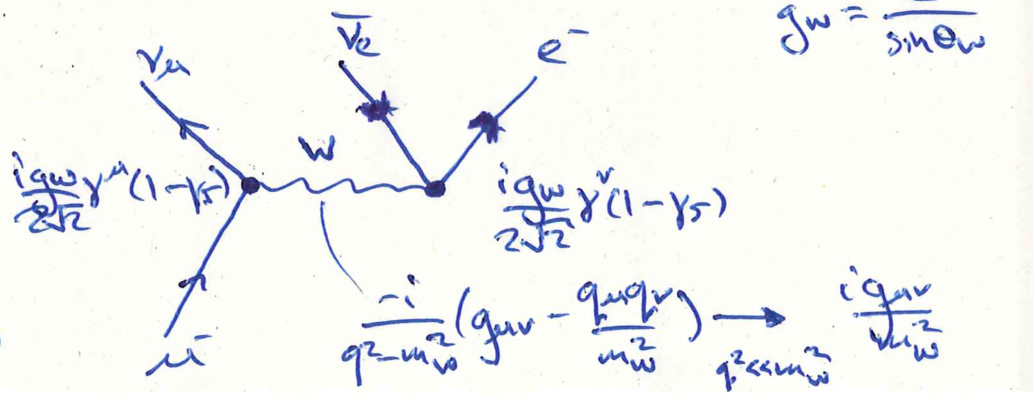
$$\tau_\mu = 2.1969803(22)\ \mu s \qquad (1.0\ \text{ppm})$$

$$G_F = 1.1663787(6)\times 10^{-5}\ \text{GeV}^{-2} \qquad (0.5\ \text{ppm})$$

Note that experimental uncertainty on τ_μ still limits the determination of G_F ...

$$g_W = \frac{e}{\sin\theta_W}$$

Basic process in SM, diagram for $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ at tree-level. (Note that MuLan measured μ^+)



Differential transition rate

$$d\Gamma = W_{fi} \cdot \rho_f$$

$\hookrightarrow$ phase space: $\prod_f \left(\frac{d^3 p_f}{(2\pi)^3} \cdot V \right)$

transition probability per unit time: $\dfrac{(2\pi)^4 \delta^4(\sum_f p_f - \sum_i p_i) V |M_{fi}|^2}{\prod_i (2E_i V) \prod_f (2E_f V)}$

transition amplitude: M_{fi} from theory, Feynman rules

$$= (2\pi)^4 \delta^4(p_\mu - \sum_f p_f) \frac{|M_{fi}|^2}{2m_\mu} \underbrace{\prod_f \frac{d^3 p_f}{(2\pi)^3 2E_f}}_{dLIPS_n}$$ for $\mu \to n$ particles

$dLIPS_n$ ~ Lorentz-Invariant Phase Space for n-particle final state

Now from the diagram and Feynman rules,

$$M_{fi} = \frac{g_W^2}{8m_W^2} \underbrace{\left[\bar{u}(\nu_\mu) \gamma^\mu (1-\gamma_5) u(\mu) \right]}_{j^\mu(\mu, \nu_\mu)} g_{\mu\nu} \underbrace{\left[\bar{u}(e) \gamma^\nu (1-\gamma_5) v(\bar{\nu}_e) \right]}_{j^\nu(e, \bar{\nu}_e)}$$

$$\Rightarrow |M_{fi}|^2 = \left(\frac{g_W^2}{8m_W^2} \right)^2 |j(\mu, \nu_\mu)|^2 |j(e, \nu_e)|^2$$ (with Lorentz indices contracted appropriately!)

The squared fermion currents can be simplified with the aid of gamma-matrix identities and spinor definitions,

$s^\mu = (0, \hat{s})$ relativistic 4-spin in <u>rest frame</u>,
massive
$\hat{s}$ = unit vector in direction of 3-spin

$\bar{u}^\alpha u^\beta = 2m \, \delta_{\alpha\beta}$ $\quad \alpha, \beta$ ~ polarization index
$\bar{v}^\alpha v^\beta = -2m \, \delta_{\alpha\beta}$

using $s_\mu p^\mu = 0$, the projectors over the polarization state associated to s^μ are: $\dfrac{\not{p} \pm m}{2}(1 + \gamma_5 \not{s})$ $\begin{cases} \text{upper sign } (+) \sim u\bar{u} \\ \text{lower sign } (-) \sim v\bar{v} \end{cases}$

For the neutrinos in the massless limit, we have $u\bar{u} = v\bar{v} = \not{p}$ and the spin 4-vector is lightlike ($s^2=0$), with helicity indicated by the sign of the spatial component: $s^\mu = (1, \pm\hat{p})$ $\pm\hat{p} = \pm\vec{p}/|\vec{p}|$ takes the upper sign for anti-neutrinos: $\nu \sim (-)$, $\bar{\nu} \sim (+)$

Now we can express the factors $|j(\mu,\nu_\mu)|^2$ and $|j(e,\bar{\nu}_e)|^2$ in terms of traces over spinor indices:

$$|j(\mu,\nu_\mu)|^2 = \overbrace{\bar{u}_a(\nu_\mu)[\gamma^\rho(1-\gamma_5)]_{ab}}^{j^\rho(\mu,\nu_\mu)}\cdot\overbrace{[u(\mu)\bar{u}(\mu)]_{bc}\cdot[\gamma^\sigma(1-\gamma_5)]_{cd}\cdot u_d(\nu_\mu)}^{(j^\sigma(\mu,\nu_\mu))^*}$$

$$= \mathrm{Tr}\Big[\underbrace{u(\nu_\mu)\bar{u}(\nu_\mu)}_{\not{p}(\nu_\mu)}\cdot\gamma^\rho(1-\gamma_5)\cdot\underbrace{(u(\mu)\bar{u}(\mu))}_{\frac{1}{2}(\not{p}(\mu)+m_\mu)(1+\gamma_5\not{s}(\mu))}\cdot\gamma^\sigma(1-\gamma_5)\Big]$$

$$= \mathrm{Tr}\big[\not{p}(\nu_\mu)\gamma^\rho(\not{p}(\mu)+m_\mu\gamma_5\not{s}(\mu))\cdot\gamma^\sigma(1-\gamma_5)\big] \quad \begin{cases} (1-\gamma_5)^2 = 2(1-\gamma_5) \\ \text{trace is cyclic} \\ \mathrm{Tr}[\underbrace{\gamma^\alpha\cdots\gamma^\kappa}_{\text{if odd }\#}] = 0 \end{cases}$$

$$= \mathrm{Tr}\big[\not{p}(\nu_\mu)\gamma^\rho(\not{p}(\mu)-m_\mu\not{s}(\mu))\cdot\gamma^\sigma(1-\gamma_5)\big]$$

$$= \mathrm{Tr}\big[\gamma^\alpha p_\alpha(\nu_\mu)\gamma^\rho\gamma^\beta(p_\beta(\mu)-m_\mu s_\beta(\mu))\cdot\gamma^\sigma(1-\gamma_5)\big] \quad \begin{cases} \gamma_5\gamma^\sigma = -\gamma^\sigma\gamma_5 \\ \gamma_5(1-\gamma_5) = \gamma_5 - 1 \end{cases}$$

which is of the form:

$$\mathrm{Tr}[\gamma^\alpha\gamma^\rho\gamma^\beta\gamma^\sigma(1-\gamma_5)] = 4\Big(\overbrace{g^{\alpha\rho}g^{\beta\sigma} - g^{\alpha\beta}g^{\rho\sigma} + g^{\alpha\sigma}g^{\rho\beta}}^{\text{from the }\gamma^\alpha\gamma^\rho\gamma^\beta\gamma^\sigma\text{ term}} + \underbrace{i\,\varepsilon^{\alpha\rho\beta\sigma}}_{\text{from the }-\gamma_5\text{ term}}\Big)$$

Similarly, now with lowered Lorentz indices ρ, σ

$$|j(e,\bar{\nu}_e)|^2 = \mathrm{Tr}\Big[[u(e)\bar{u}(e)]\cdot[\gamma_\rho(1-\gamma_5)]\cdot[v(\bar{\nu}_e)\bar{v}(\bar{\nu}_e)]\cdot\gamma_\sigma(1-\gamma_5)\Big]$$

$$= \mathrm{Tr}\big[(\not{p}(e)-m_e\not{s}(e))\cdot\gamma_\rho\not{p}(\bar{\nu}_e)\gamma_\sigma(1-\gamma_5)\big]$$

$$= \mathrm{Tr}\big[\gamma_{\alpha'}(p^{\alpha'}(e)-m_e s^{\alpha'}(e))\gamma_\rho\gamma_{\beta'}p^{\beta'}(\bar{\nu}_e)\gamma_\sigma(1-\gamma_5)\big],$$

which is of the same form. It can be shown that the product of two such traces is

$$\mathrm{Tr}[\gamma^\alpha\gamma^\rho\gamma^\beta\gamma^\sigma(1-\gamma_5)]\,\mathrm{Tr}[\gamma_{\alpha'}\gamma_\rho\gamma_{\beta'}\gamma_\sigma(1-\gamma_5)] = 64\, g^\alpha{}_{\alpha'}\, g^\beta{}_{\beta'}$$

Thus we can simplify the squared matrix element,

$$|M_{fi}|^2 = \left(\frac{g_w^2}{8m_w^2}\right)^2 \cdot 64 \cdot (p(e) - m_e s(e))^\alpha p_\alpha(\nu_\mu) \cdot (p(\mu) - m_\mu s(\mu))_\beta \, p^\beta(\nu_e)$$

Now the neutrinos cannot realistically be detected, so we must integrate over their momenta to get the differential decay rate in terms of the electron spectrum alone:

$$d\Gamma = \frac{(2\pi)^4}{2m_\mu} \delta^4(p(\mu) - (p(e) + p(\bar{\nu}_e) + p(\nu_\mu))) \, |M_{fi}|^2 \, \frac{d^3p(e)\, d^3p(\bar{\nu}_e)\, d^3p(\nu_\mu)}{(2\pi)^9 \cdot 2E_e \cdot 2E_{\bar{\nu}_e} \cdot 2E_{\nu_\mu}}$$

$$\Rightarrow d\Gamma_{(e\text{ only})} = \left(\frac{g_w^2}{8m_w^2}\right)^2 \cdot \frac{4\, d^3p(e)}{(2\pi)^5 m_\mu E_e} (p(e) - m_e s(e))^\alpha (p(\mu) - m_\mu s(\mu))^\beta$$

$$\times \int \frac{d^3p(\bar{\nu}_e)}{E_{\bar{\nu}_e}} \int \frac{d^3p(\nu_\mu)}{E_{\nu_\mu}} p_\alpha(\nu_\mu) p_\beta(\bar{\nu}_e) \, \delta^4(q - (p(\nu_\mu) + p(\bar{\nu}_e)))$$

where we have defined $q = p(\mu) - p(e)$ as the 4-momentum transferred from muon to electron. The integral can be decomposed in Lorentz invariants (by contracting with $g^{\alpha\beta}$ and $q^\alpha q^\beta$) and then evaluated in a specific frame.

The result is: $\frac{\pi}{6}(q^2 g_{\alpha\beta} + 2 q_\alpha q_\beta)$

We can most simply express $d\Gamma$ in the muon rest frame, where $q = (m_\mu - E_e, -\vec{p}_e)$ and $s(\mu) = (0, \hat{s}(\mu))$. The explicit form for $s(e)$ is obtained by Lorentz transformation of the respective electron-rest-frame ~~polarisation~~ covariant spin $s(e) = (0, \hat{s}(e$ into a frame where the electron has three-momentum $\vec{p}_e$.

The differential decay rate, now writing p_e for $p(e)$ etc,

$$d\Gamma_{(e\text{ only})} = \left(\frac{g_w^2}{8m_w^2}\right) \cdot \frac{d^3 p_e}{3 \cdot (2\pi)^4 m_\mu E_e} (p_e - m_e s_e)^\rho (p_\mu - m_\mu s_\mu)^\sigma (q^2 g_{\rho\sigma} + 2 q_\rho q_\sigma)$$

can now be used to deduce the electron energy and angular spectra.