Statistical Methods in Particle Physics

8. Confidence Limits and Intervals

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Reminder: Hypothesis tests

Bayesian:

- *Posterior* $p(H_0 \,|\, \vec{d})$ gives probability for hypothesis to be true after considering the data
- Depends on prior $p(H_0)$, which may be hard to agree on precisely
- Likelihood ratio $p(H_0 \,|\, \vec{d})/p(H_1 \,|\, \vec{d})$ gives an objective value for how much data favours H_0 compared to H_1, but only if hypothesis is simple
- Easy to interpret, but not objective due to prior

Frequentist:

- No probability for hypothesis
- *p-value* gives probability of observing a test statistic at least this extreme assuming that H_0 is true with the alternative hypothesis H_1 defining the direction of “extremeness”
- Hard to interpret, but objective

Point estimates and limits

One often reports a point estimate and its standard deviation: $\hat{\theta}, \hat{\sigma}_{\hat{\theta}}$

In some situation this is not adequate, e.g. when

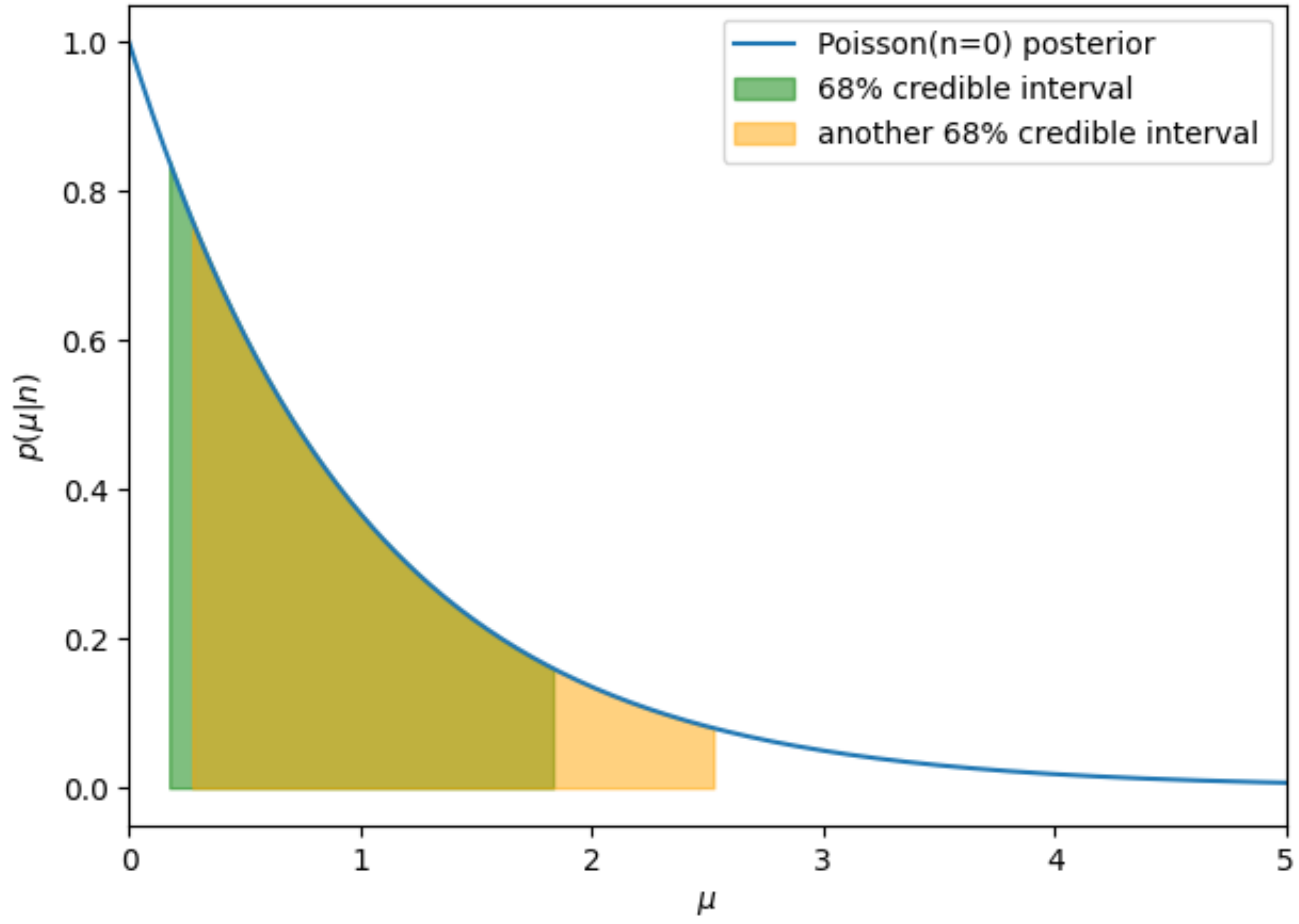
- the p.d.f. of the estimator is not Gaussian
- The standard deviation strongly depends on the true value
- one has physical boundaries on the possible values of the parameter

Example:

- Poisson distribution $p(n\,|\,\mu)$, we measure $n = 0$
- No good estimate for $\sigma_n = \sqrt{\mu}$
- But it feels like we should be able to exclude some ranges, like $\mu > 100$
- Can we make some statement about what a reasonable range for μ should be?
  - We can in the Bayesian framework
  - For the frequentist approach we again have to modify the question

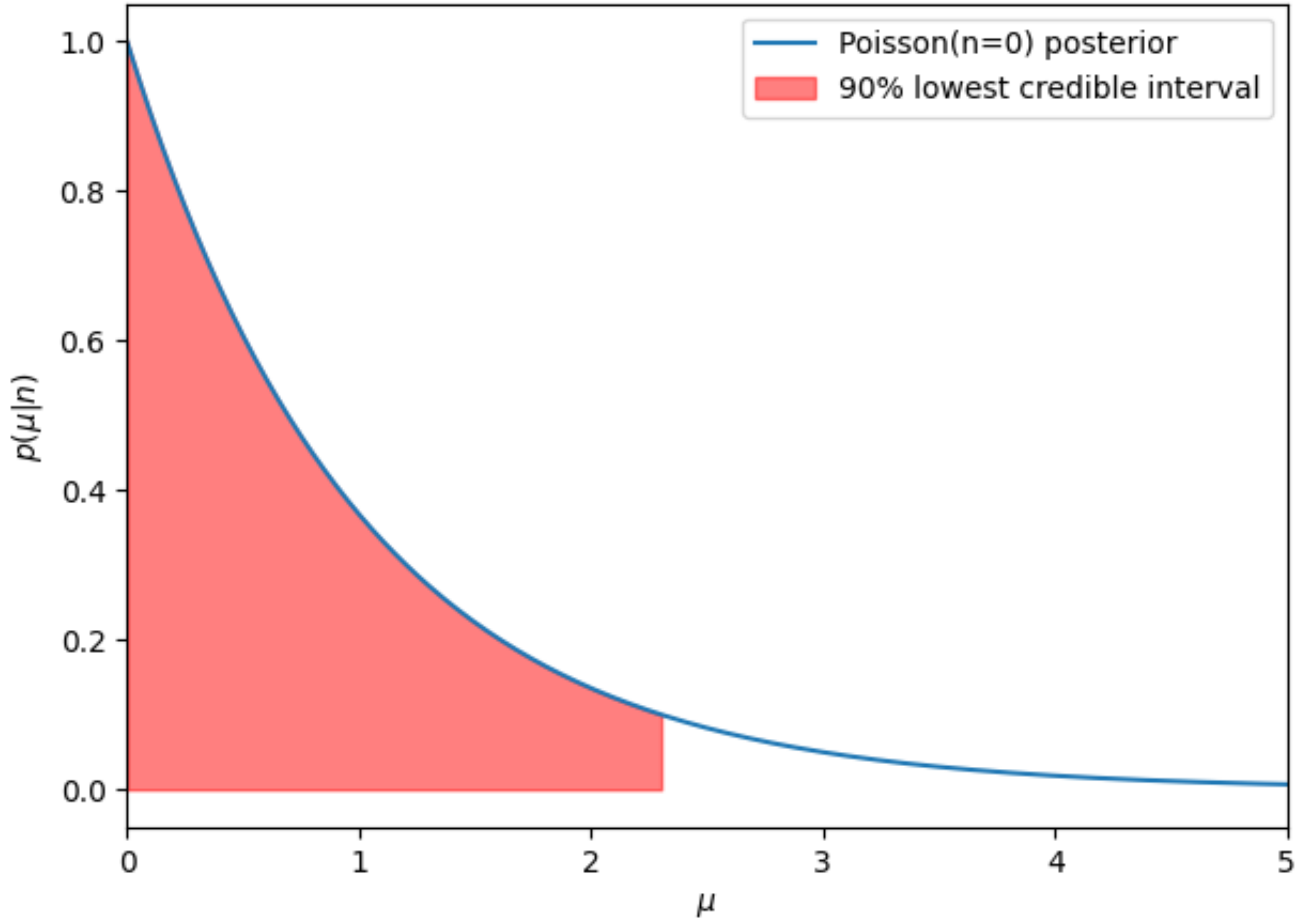
Bayesian credible intervals

- Assume uniform prior
- From posterior: Find region with a given total probability
- E.g. a region corresponding to a total probability of 68% is called the “68% *credible interval*”
- There is some freedom in choosing the interval
- Meaning: The true parameter is within this particular interval with a probability of 68%.



Bayesian limits

- For upper (or lower) limits, simply put one edge of the interval to 0 (or ∞)
- Interpretation: There is a 90% probability that the true parameter is below 2.3.



Example:
Bayesian upper limits for a poisson variable *n* (2)

Special case: $b = 0$

```python
import numpy as np
from scipy.stats import gamma
```

```python
def ul(alpha, n, b):
    """
    Baysian Poisson upper limits
    1 - alpha: confidence level
    n: observed counts
    b: background
    """
    return gamma.ppf(1. - alpha, n + 1) - b
```

```python
print("n   s_up")
print("-------")
for n in range(10):
    print(f"{n}  {ul(0.1, n, 0):.2f}")
```

```
n   s_up
-------
0   2.30
1   3.89
2   5.32
3   6.68
4   7.99
5   9.27
6   10.53
7   11.77
8   12.99
9   14.21
```

Can write this also in terms of the χ^2 distribution:

$$s_{\mathrm{up}} = \frac{1}{2} F_{\chi^2}^{-1}[1-\alpha, 2(n+1)] \quad (b=0)$$

Frequentist confidence intervals

Reminder: frequentist parameter estimation

- Make a rule to calculate a value $\hat{\theta}(\vec{d})$ of the dimension of the parameter of interest from the data
- Require e.g. consistency, unbiasedness etc. and study the properties

Now: interval estimation

- Make a rule to calculate an interval $(\hat{\theta}_1, \hat{\theta}_2)$ of the dimension of the parameter of interest from the data
- We cannot give a probability that the parameter is in any particular interval
- Instead: Require that when the measurement is repeated a large number of times, a particular fraction α should contain the true (fixed) parameter
- If we find a rule such that this probability is the same independent of the true parameter, then the intervals are called *confidence intervals* and α is called the confidence level

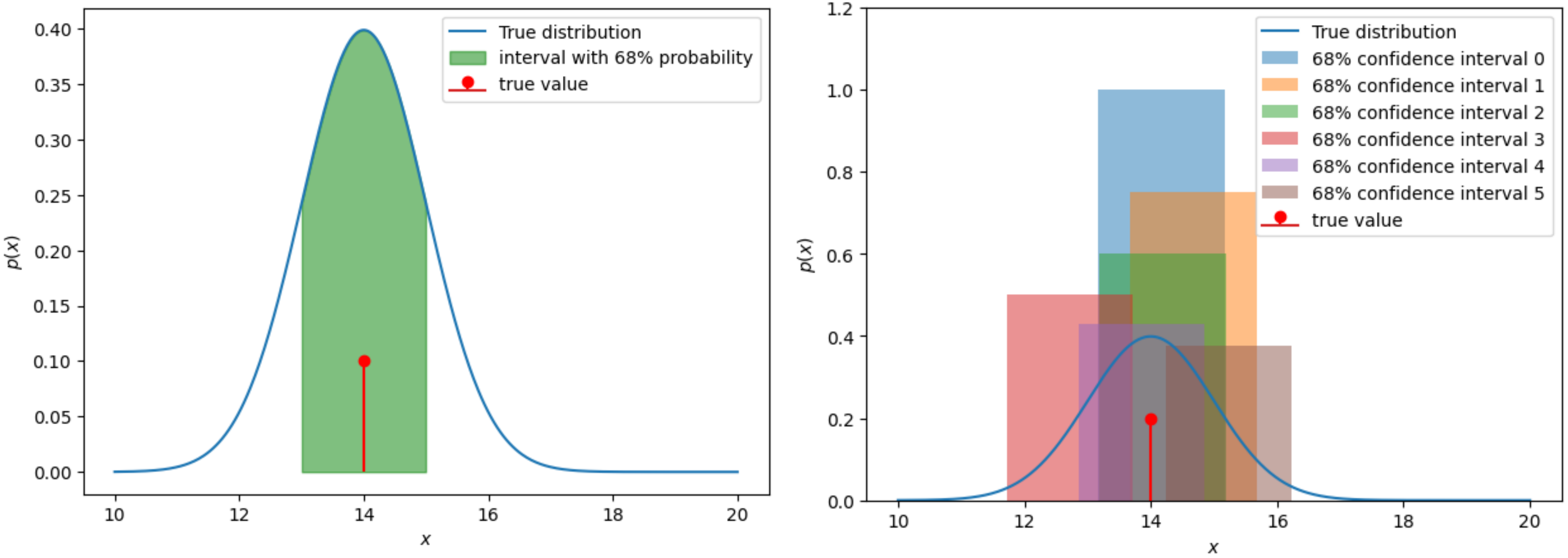
Understanding the meaning of confidence intervals

- One could reasonably write:
  - "A Bayesian 90% *credible interval* contains the true value with a probability of 90%."
  - "A frequentist 90% *confidence interval* contains the true value with a probability of 90%."

- This is not a good explanation but technically true.
- The important part is:
  - For credible intervals, this is a statement about the unknown parameter
  - For confidence intervals, this is a statement about the rule that makes the intervals
- A single confidence interval does not have a probability to contain the parameter!

- The "optimal" rule for confidence intervals produces the shortest possible intervals, but this is not always what we want

The simplest example: Gaussian with fixed variance



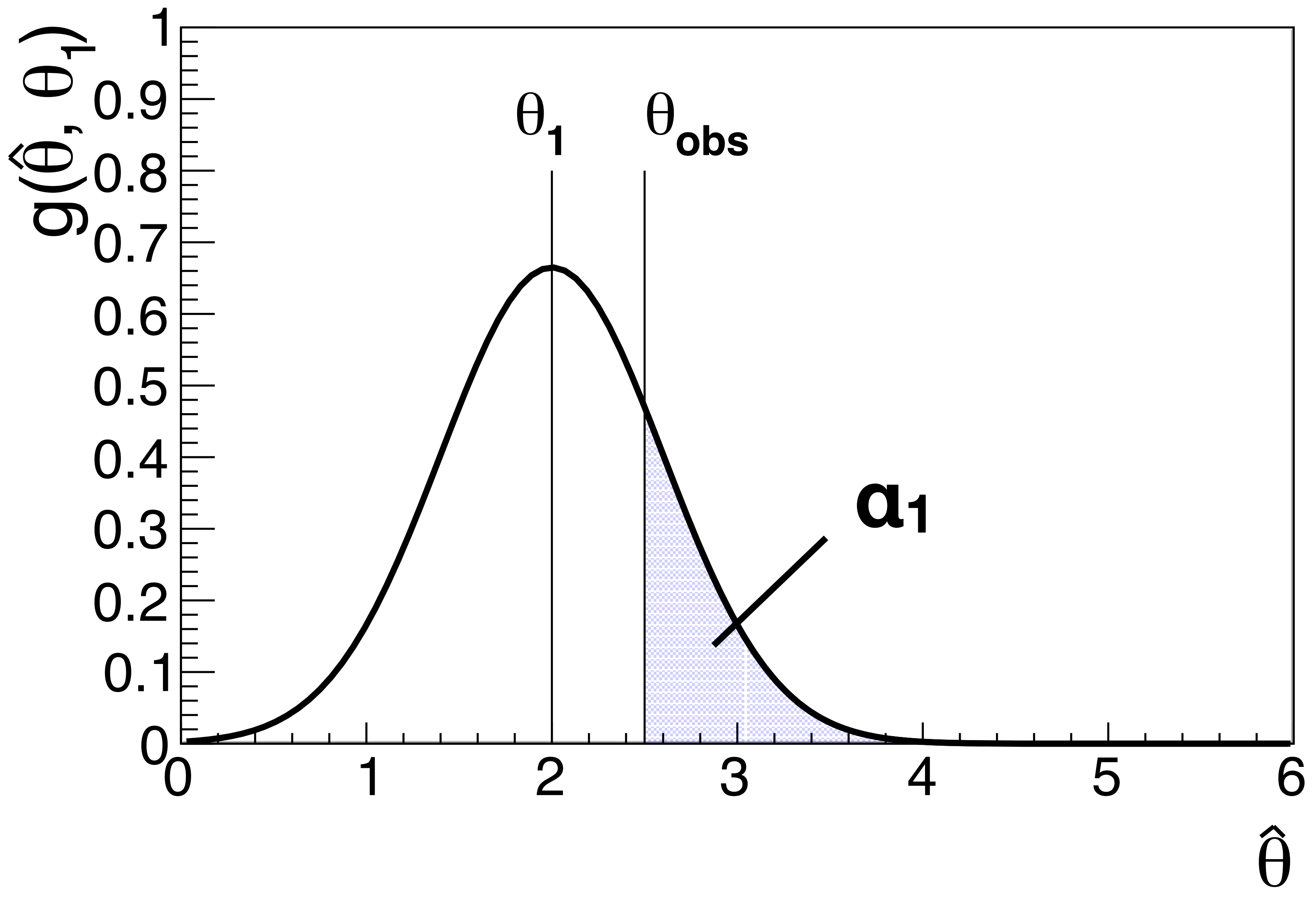
- The measurements are symmetric around the true value with fixed variance
- Choose symmetric interval around measured values
- Now 68% of these intervals will cover the true value
- Here we get the same result as before, where we asked how the estimate varies

Confidence interval for a Gaussian distributed estimator

Consider a parameter θ whose estimator is distributed as

$$g(\hat{\theta};\theta) = \frac{1}{\sqrt{2\pi}\sigma_{\hat{\theta}}} \exp\left(-\frac{1}{2}\frac{(\hat{\theta}-\theta)^2}{\sigma_{\hat{\theta}}^2}\right)$$

"sampling distribution"



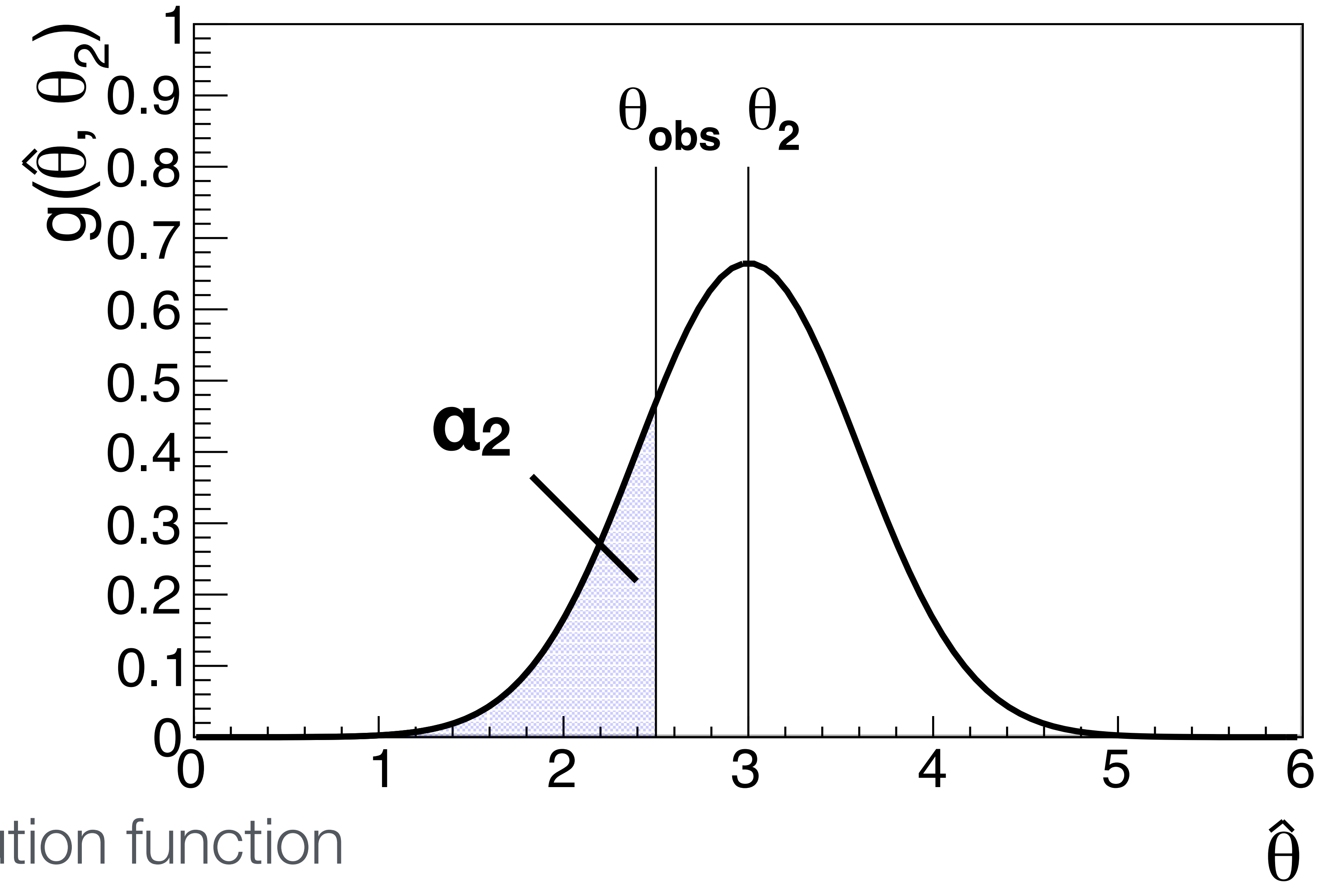
Determine lower bound θ_1 of the confidence interval for θ by solving

$$\alpha_1 = \int_{\hat{\theta}_{obs}}^{\infty} g(\hat{\theta};\theta_1)\,\mathrm{d}\hat{\theta} \equiv 1 - G(\hat{\theta}_{obs},\theta_1)$$

Analogously for the upper bound θ_2:

$$\alpha_2 = \int_{-\infty}^{\hat{\theta}_{obs}} g(\hat{\theta};\theta_2)\,\mathrm{d}\hat{\theta} \equiv G(\hat{\theta}_{obs},\theta_2)$$

cumulative distribution function



Confidence interval for a Gaussian distributed estimator

With the aid of the CDF of the standard Gaussian **Φ** we can write this as:

$$\alpha_1 = 1 - G(\hat{\theta}_{\mathrm{obs}}, \theta_1) = 1 - \Phi\left(\frac{\hat{\theta}_{\mathrm{obs}} - \theta_1}{\sigma_{\hat{\theta}}}\right)$$

$$\alpha_2 = G(\hat{\theta}_{\mathrm{obs}}, \theta_2) = \Phi\left(\frac{\theta_{\mathrm{obs}} - \theta_2}{\sigma_{\hat{\theta}}}\right)$$

This gives:

$$\theta_1 = \theta_{\mathrm{obs}} - \sigma_{\hat{\theta}} \Phi^{-1}(1 - \alpha_1)$$

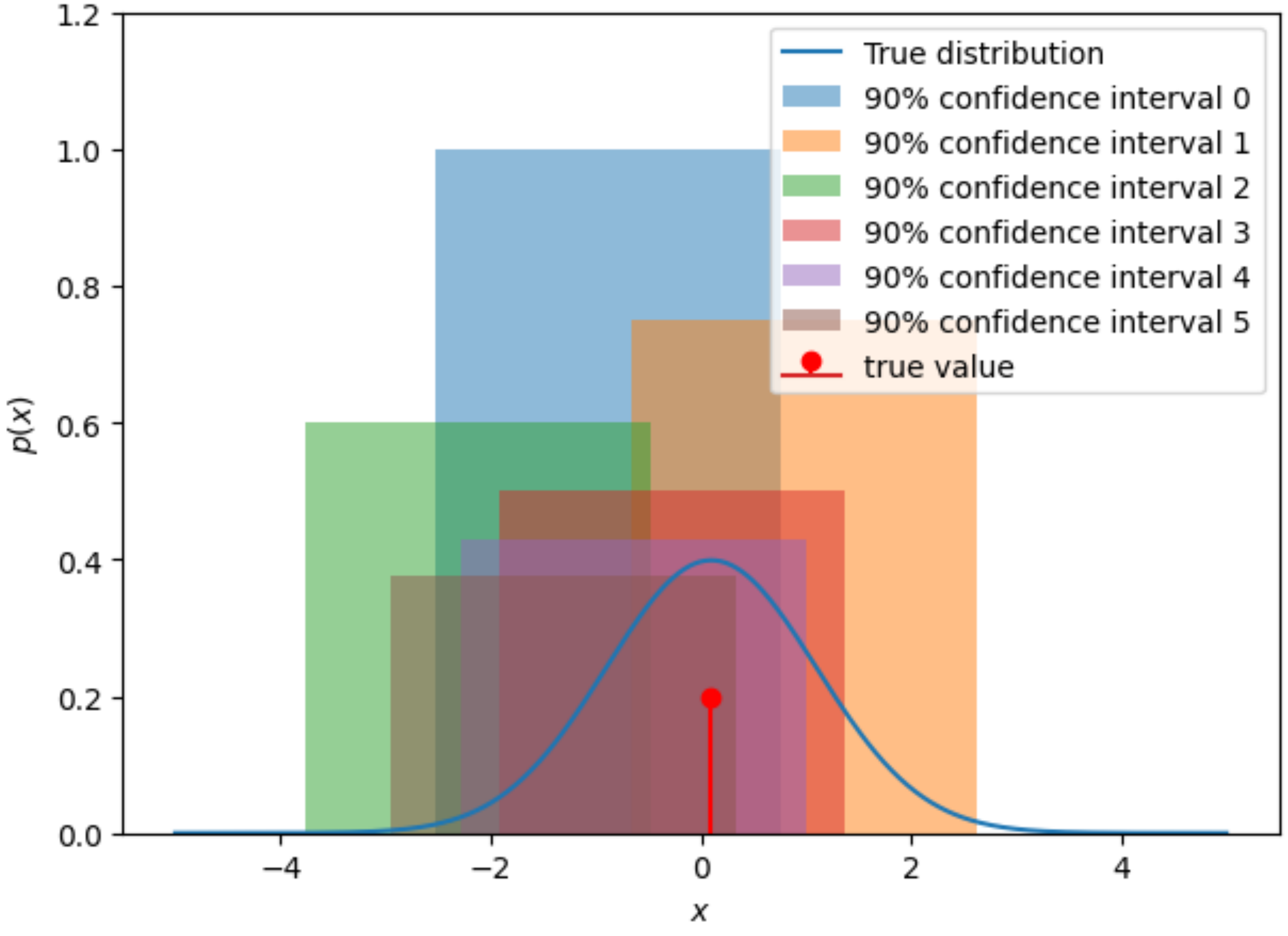
$$\theta_2 = \theta_{\mathrm{obs}} + \sigma_{\hat{\theta}} \Phi^{-1}(1 - \alpha_2)$$

$$-\Phi^{-1}(y) = \Phi^{-1}(1 - y)$$

Here Φ^{-1} is the inverse function of Φ, i.e., the quantile function of the standard Gaussian.

The possibility of unphysical intervals

- Imagine measuring signal+background with large statistics
- Poisson can be approximated by normal distribution, with mean close to 0
- Draw 90% confidence intervals for different measurements
- Some intervals can be partially in the negative values
- The green CI is $(-3.75, -0.47)$ and thus completely unphysical
- Nothing went wrong here, the 90% simply does not refer to a particular interval
- Nevertheless, the single interval is what would be published, which can lead to confusion



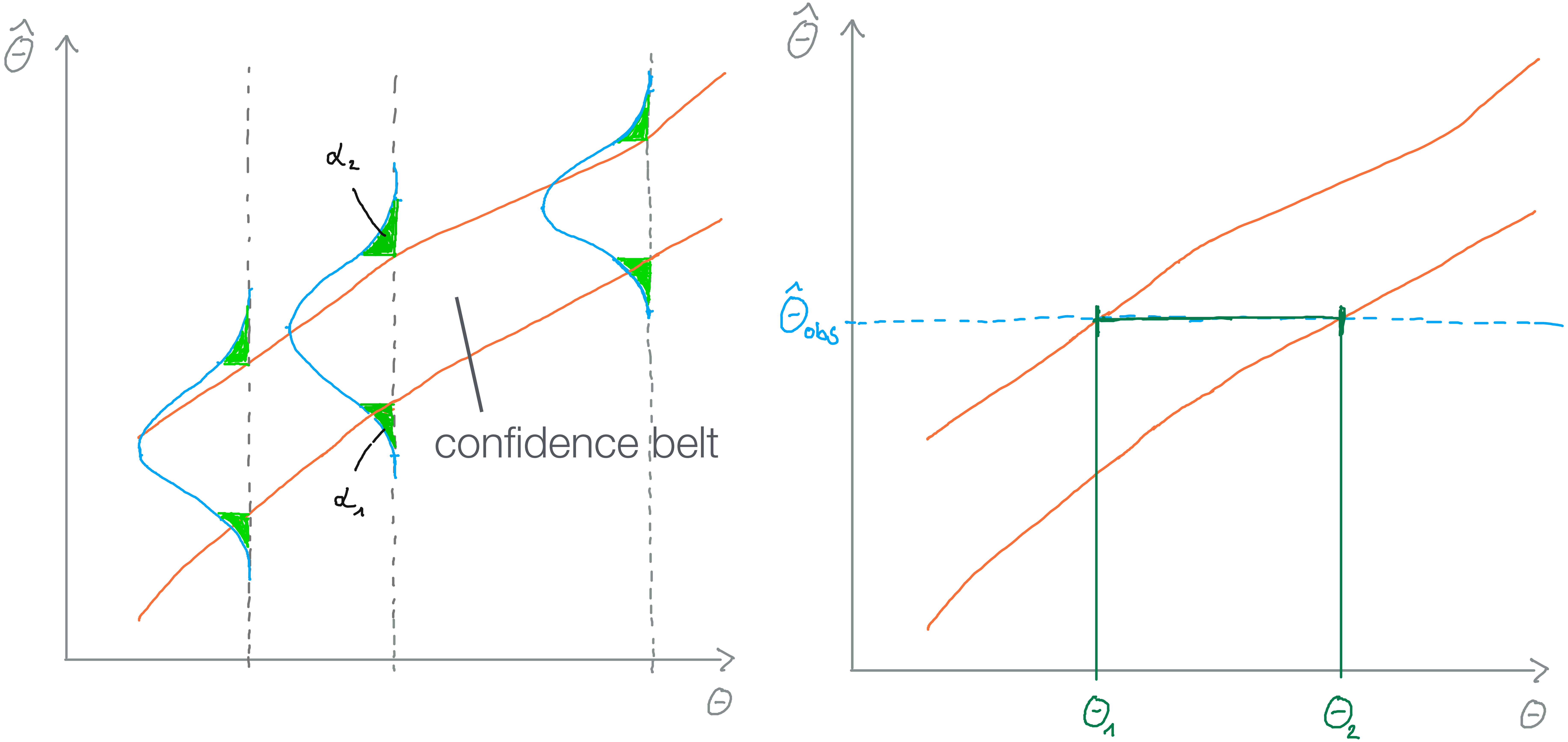
Neyman construction (1)

The Neyman construction for constructing frequentist confidence intervals involves the following steps:

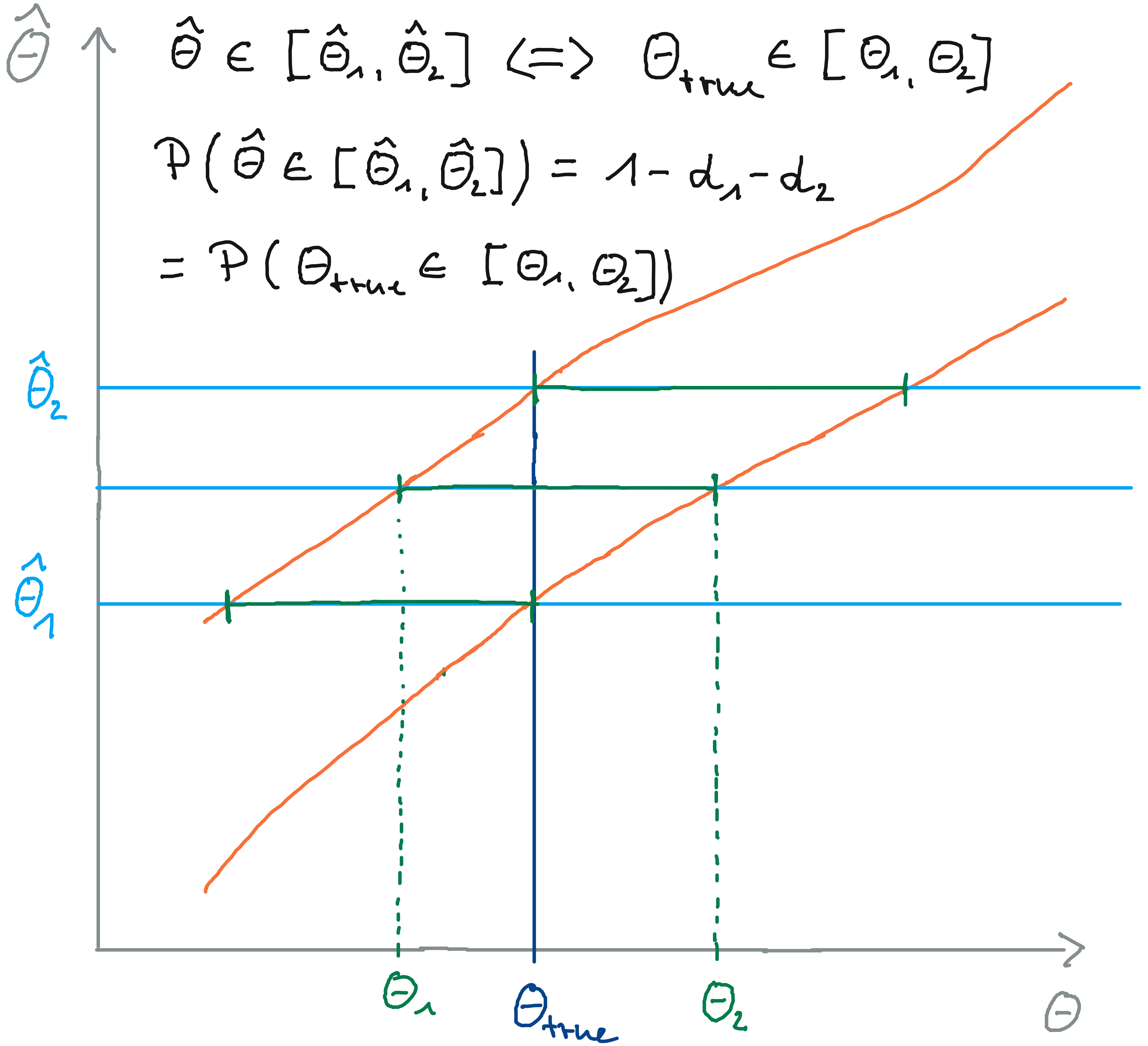
1. Given a true value of the parameter θ, determine a p.d.f. f(*x*; θ) for the outcome of the experiment. Often *x* is an estimator for the θ.
2. Using some procedure, define an interval in *x* that has a specified probability (say, 90%) of occurring
3. Do this for all possible true values of θ, and build a confidence belt of these intervals.

In practice, the p.d.f. of step 1 might come from Monte Carlo simulations.

Neyman Construction (2)



Coverage of the Neyman interval



Classical confidence intervals for the mean of the Poisson distribution (1)

$$f(n;\nu) = \frac{\nu^n}{n!} e^{-\nu}$$

Equations for the confidence interval limits θ_1 and θ_2:

$$\alpha_1 = P(n \geq n_{\mathrm{obs}}; \theta_1)$$
$$\alpha_2 = P(n \leq n_{\mathrm{obs}}; \theta_2)$$

As n is an integer there is often no exact solution. This leads to slight overcoverage.

This gives:

$$\alpha_1 = \sum_{n=n_{\mathrm{obs}}}^{\infty} f(n;\theta_1) = 1 - \sum_{n=0}^{n_{\mathrm{obs}}-1} f(n;\theta_1) = 1 - \sum_{n=0}^{n_{\mathrm{obs}}-1} \frac{\theta_1^n}{n!} e^{-\theta_1}$$

$$\alpha_2 = \sum_{n=0}^{n_{\mathrm{obs}}} f(n;\theta_2) = \sum_{n=0}^{n_{\mathrm{obs}}} \frac{\theta_2^n}{n!} e^{-\theta_2}$$

Classical confidence intervals for the mean of the Poisson distribution (2)

Using the the following relation between the Poisson distribution and the χ^2 distribution

$$\sum_{n=0}^{n_\mathrm{obs}} \frac{\nu^n}{n!} e^{-\nu} = \int_{2\nu}^{\infty} f_{\chi^2}(z; n_\mathrm{df} = 2(n_\mathrm{obs}+1))\,\mathrm{d}z$$
$$= 1 - F_{\chi^2}(2\nu; 2(n_\mathrm{obs}+1)))$$

F_{χ^2} : CFD of the χ^2 distribution

we obtain

$$\theta_1 = \frac{1}{2} F^{-1}_{\chi^2}[\alpha_1; 2n_\mathrm{obs}]$$
$$\theta_2 = \frac{1}{2} F^{-1}_{\chi^2}[1-\alpha_2; 2(n_\mathrm{obs}+1)]$$

[identical to Bayesian upper limits (b = 0)]

Classical confidence intervals for the mean of the Poisson distribution (3)

n_{obs}	lower limit θ_1			upper limit θ_2		
	$\alpha_1 = 0.1$	$\alpha_1 = 0.05$	$\alpha_1 = 0.01$	$\alpha_2 = 0.1$	$\alpha_2 = 0.05$	$\alpha_2 = 0.01$
0	–	–	–	2.30	3.00	4.61
1	0.105	0.051	0.010	3.89	4.74	6.64
2	0.532	0.355	0.149	5.32	6.30	8.41
3	1.10	0.818	0.436	6.68	7.75	10.04
4	1.74	1.37	0.823	7.99	9.15	11.60
5	2.43	1.97	1.28	9.27	10.51	13.11

cf. slide 6 (Bayesian upper limits, b = 0)

If e.g. $\theta_{true} = 0.54$, then probability of coverage $= p(0) + p(1) + p(2) \approx 0.98$ is higher than $90\,\%$

Classical Gaussian upper limits with physical limit

Suppose the estimator of a parameter θ follows a Gaussian with known standard deviation σ = 1:

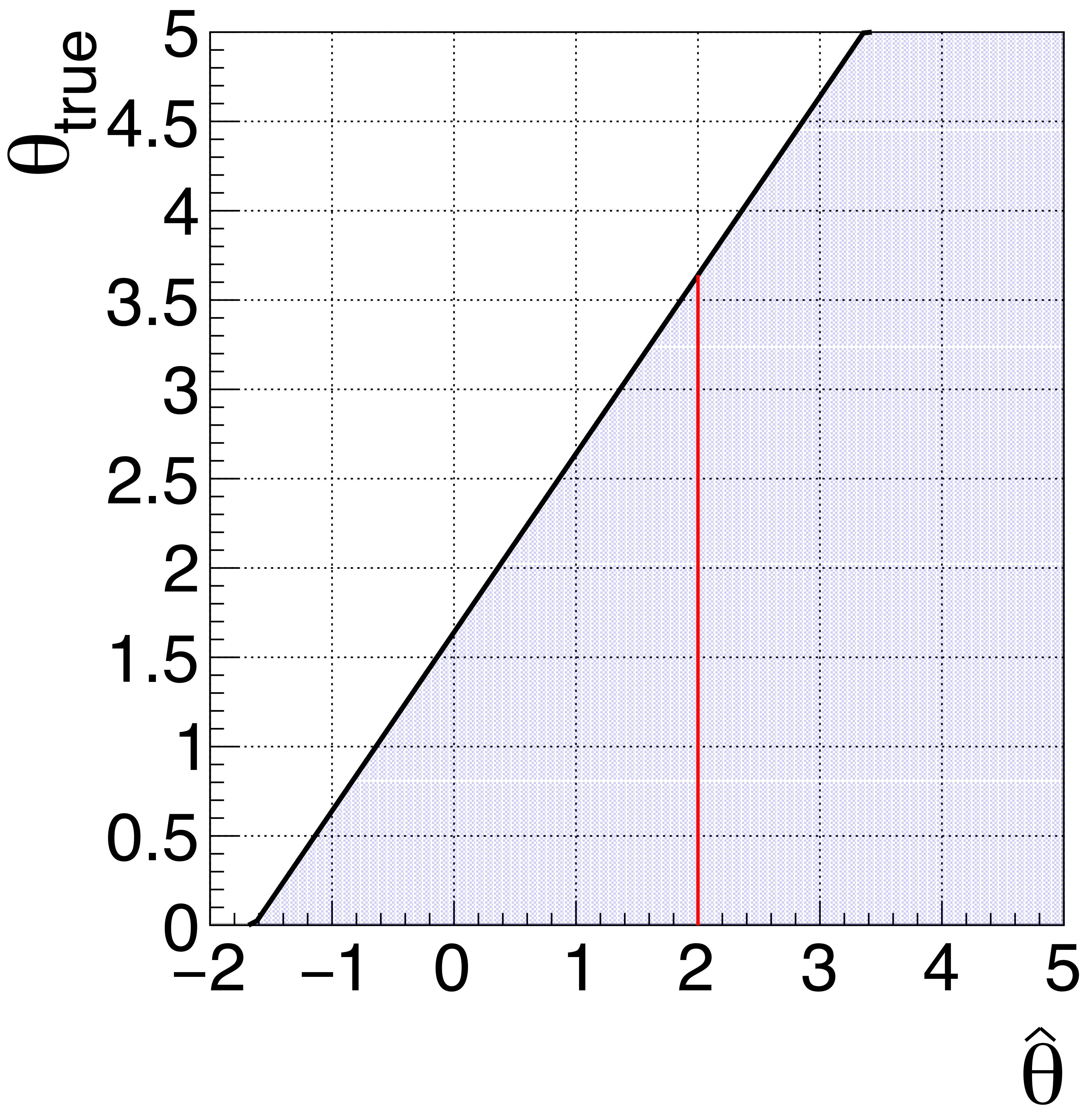
$$g(\hat{\theta};\theta) = \frac{1}{\sqrt{2\pi}} \exp\left(-(\hat{\theta}-\theta)^2/2\right)$$

Physically allowed region: $\theta \geq 0$

An example would be the measurement of the neutrino mass: $m \geq 0$

Let's construct the 95% CL upper limit confidence belt (1.64σ)

$$\hat{\theta} = 2 \rightsquigarrow s_{\text{up}} = 3.64 \text{ @ } 95\%\text{CL}$$



But what if we measured –2?: $\hat{\theta} = -2 \rightsquigarrow s_{\text{up}} = -0.36 \text{ @ } 95\%\text{CL}$

A negative upper limit? Has anything gone wrong?

Classical Gaussian Upper Limits with Physical Limit

$$\hat{\theta} = -2 \rightsquigarrow s_{\mathrm{up}} = -0.36 \text{ @ } 95\%\mathrm{CL}$$

We stipulated θ ≥ 0, i.e. the confidence interval is an empty set …

If we measured –1.63 the confidence interval would be [0, 0.01]. Does this really mean that in this case there is a 95% chance that the true value of θ is between 0 and 0.01?

No, it just means that we have observed a downward fluctuation

- Suppose the true value is zero (θ = 0) → acceptance region @ 95% CL is [–∞, 1.64]
- We expect a negative result in 50% of the cases
- We expect a measurement less than –1.64 in 5% of the cases
- We expect a measurement less than –2 in 2.3% of the cases

Sometimes a negative result is shifted to zero, i.e., 0 + 1.64 σ is reported as upper limit.

That's not helpful. Always report the observed value even if it is in the unphysical regime. Otherwise the result cannot be combined with other results in meta analyses.

Interpretation of Frequentist Confidence Intervals

So has anything gone wrong with the construction of the confidence interval?

Actually no, nothing has gone wrong.

- Even though one should not, there is a tendency to interpret frequentist confidence intervals as Bayesian objects. That is, if one constructs the confidence interval in our example one tends to think that the true value lies in this interval with 95% probability
- But that's not right. We have to think in terms of repeated experiments. The obtained interval covers the true value in 95% of the experiments.
- This does not mean that the interval obtained in a single experiment contains the true value with 95% probability.

The "flip-flop" problem

Feldman, Cousins, physics/9711021v2

Let us suppose that physicist X takes the following attitude in an experiment designed to measure a small quantity:

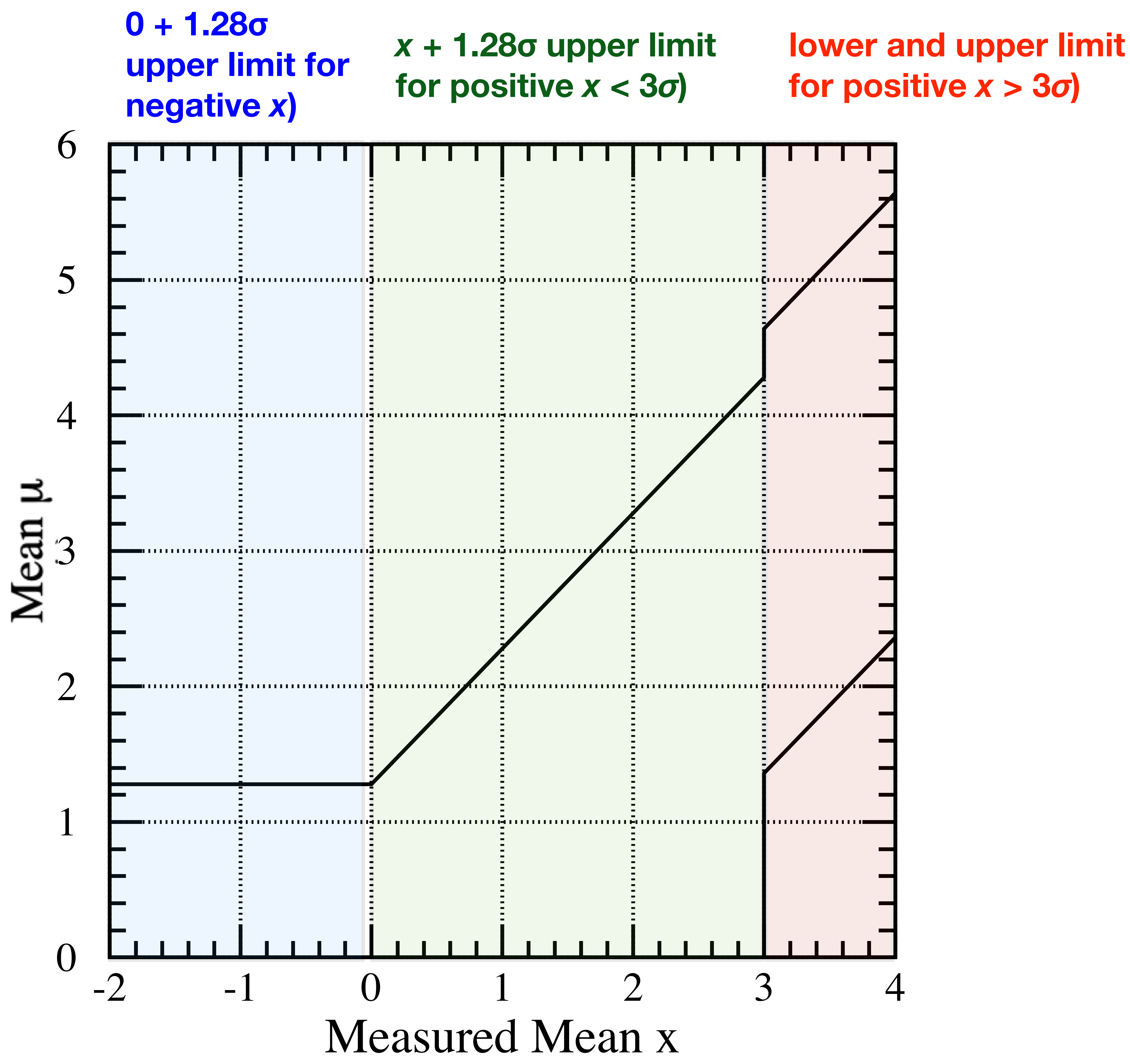
- If the result x is less then 3σ, I will state an upper limit
- If the result is greater than 3σ, I will state a central confidence interval from the standard tables

→ So what is reported in this case is decided *after* the measurement

Let's take a look at the confidence band

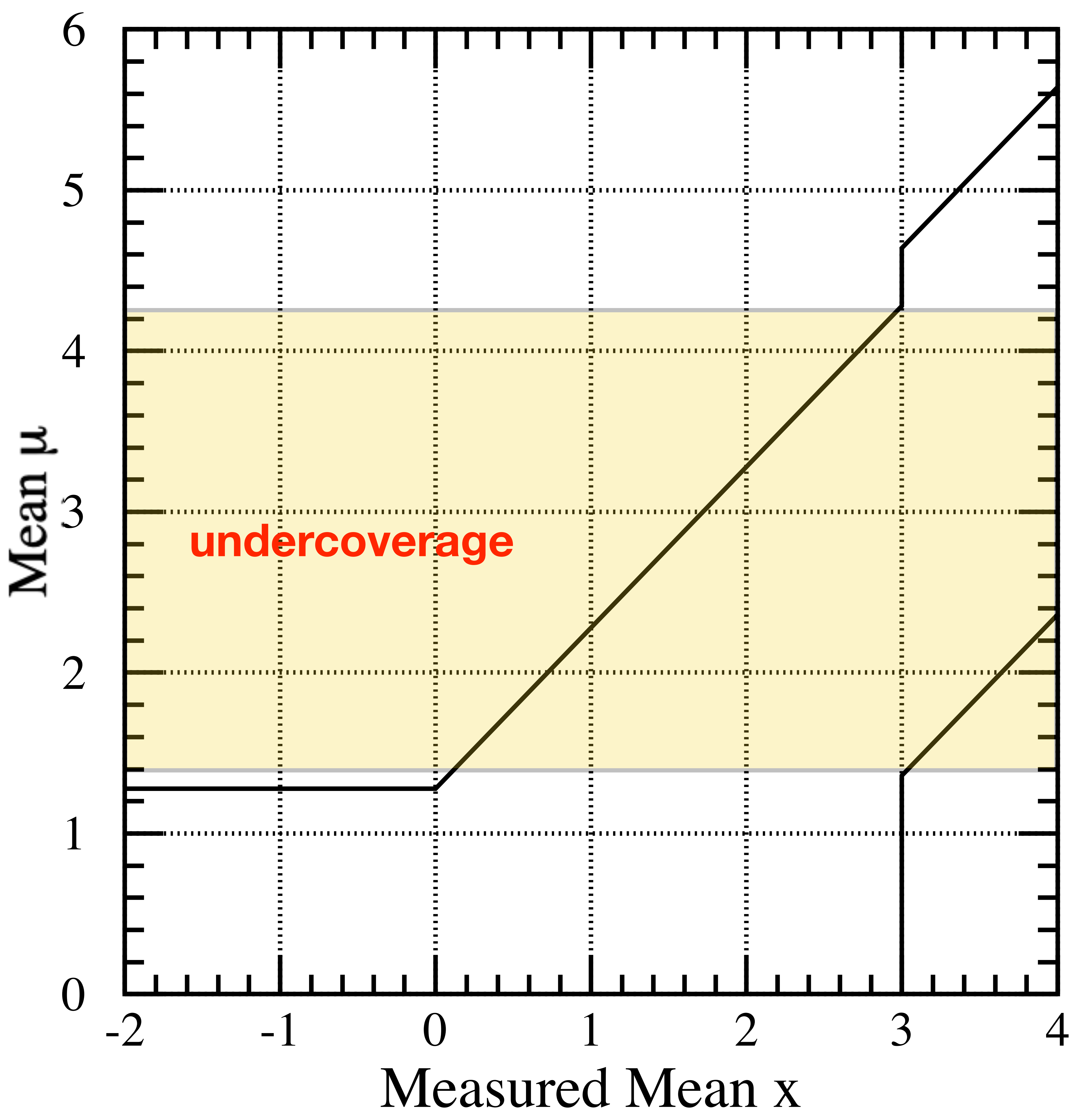
[Variables in the paper by Feldman and Cousins: $x \equiv \hat{\theta}, \;\; \mu \equiv \theta$. Confidence band for 90% CL. Otherwise same situation: Gaussian sampling distribution with $\sigma = 1$ and physical regime $\mu \geq 0$. In the following we'll use x and μ.]

The "flip-flop" problem: Confidence band



Feldman, Cousins, physics/9711021v2

The "flip-flop" problem: Coverage



The coverage of the intervals is wrong

- Small μ: overcoverage
- Example: μ = 2
  acceptance region is
  $x \in [2 - 1.28, 2 + 1.64]$
  → coverage is only 85%
- More general:
  for $1.36 < \mu < 4.28$ the chance of finding a measured value x in acceptance region is only 85%, not the desired 90%

This is a serious problem of the flip-flopping approach

Feldman, Cousins, physics/9711021v2

Problems with classical confidence intervals

- in some situations the confidence interval can be an empty set
- they do not elegantly handle unphysical cases
- they do not continuously vary between

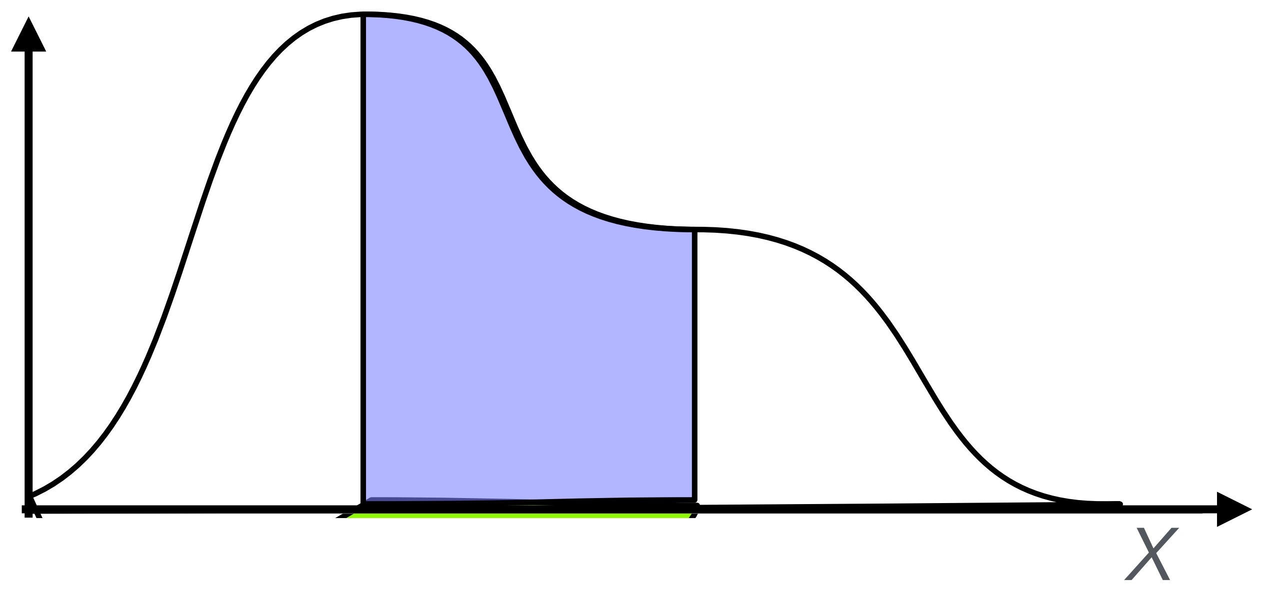
  a) giving upper limits in case of a very small signal and

  b) giving upper and lower limits in case of a more significant signal

Feldman & Cousins proposed a solution in their paper
→ Feldman-Cousins confidence intervals

Feldman-Cousins ordering principle for the construction of confidence intervals

The Neyman construction does not specify how, for a fixed true value μ, to define the interval that covers a fraction 1 – α (e.g. 95%) of the observed outcomes *x*.



Feldman & Cousins introduced an ordering principle based on the likelihood ratio:

$$R = \frac{P(x|\mu)}{P(x|\mu_{\text{best}})}$$

μ_{best} is the best fit obtained from data (maximum likelihood), taking the physically allowed region into account.

Order procedure for fixed μ: add values of *x* to the interval from highest *R* to lower *R* until the desired value 1 – α is reached.

Application of Feldman-Cousins to Gaussian upper limits with physical limit (1)

Sampling distribution in our example with physical limit $\mu \geq 0$ ($\sigma_x \equiv 1$):

$$g(x;\mu) = \frac{1}{\sqrt{2\pi}} \exp\left(-(x-\mu)^2/2\right)$$

In this case the best estimate is given by

$$\mu_{\text{best}} = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

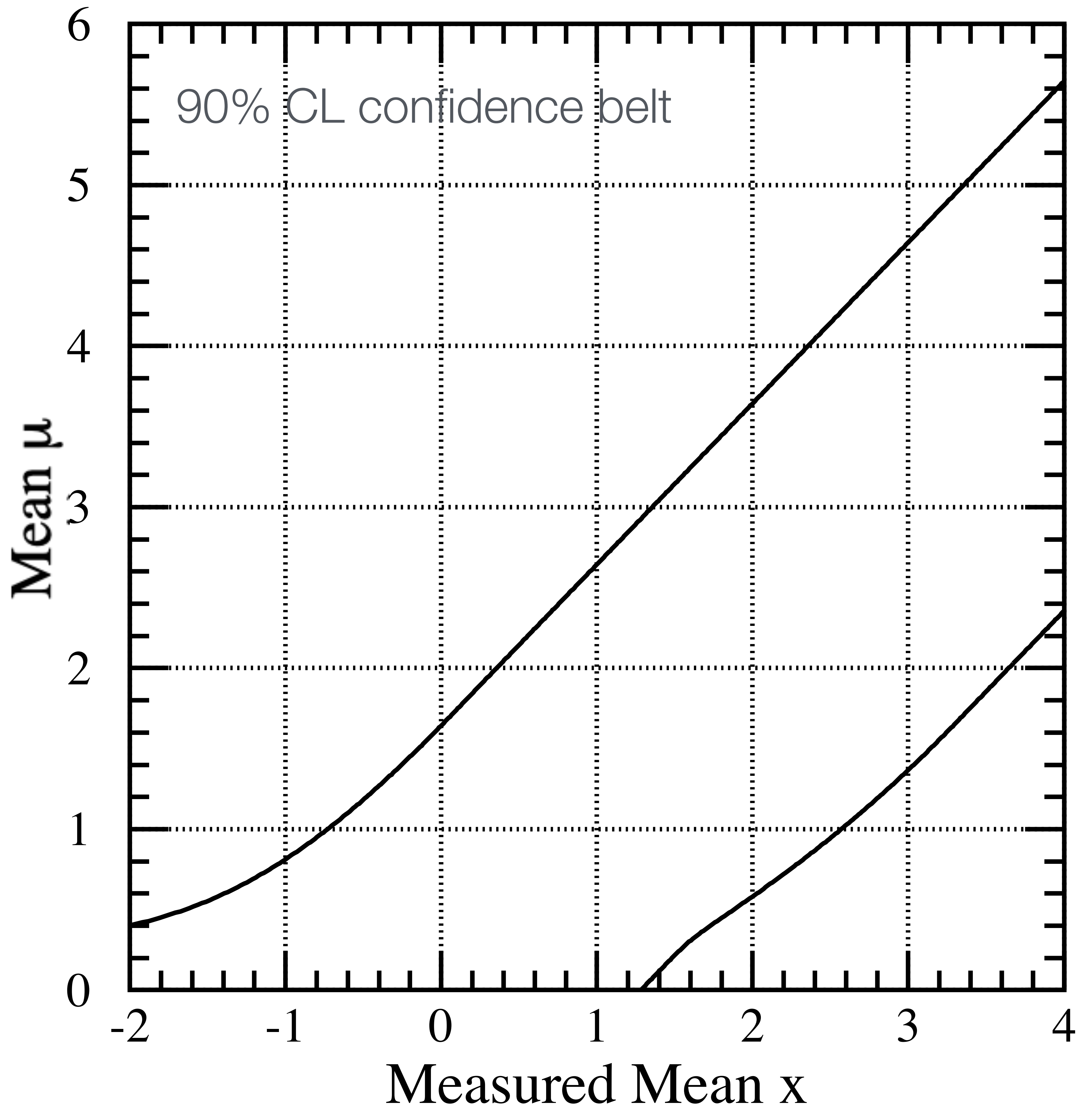
So R is given by

$$R = \frac{P(x|\mu)}{P(x|\mu_{\text{best}})} = \begin{cases} \dfrac{\exp\left(-\frac{1}{2}(x-\mu)^2\right)}{\exp\left(-\frac{1}{2}x^2\right)}, & x < 0 \\[2ex] \dfrac{\exp\left(-\frac{1}{2}(x-\mu)^2\right)}{1}, & x \geq 0 \end{cases}$$

In practice, for each μ find interval limits x_1 and x_2 by solving numerically:

$$R(x_1) = R(x_2) \qquad \text{and} \qquad \int_{x_1}^{x_2} g(x|\mu)\, \mathrm{d}x = 1 - \alpha$$

Application of Feldman-Cousins to Gaussian upper limits with physical limit (2)



Some nice features:

- Confidence interval is never empty
- Smooth transition from giving upper limit to two-sided interval
- Correct coverage
- No empty intervals

Feldman-Cousins confidence intervals for the mean of the Poisson Distribution (1)

Let's go back to the counting experiment with signal s and known average number of background counts b:

$$P(n|s) = \frac{(s+b)^n}{n!} e^{-(s+b)}$$

Classical method sometimes gives negative upper limit when $n_{\mathrm{obs}} < b$.

This problem is addressed by the Feldman-Cousins method.

The paper contains look-up tables for upper limits and confidence intervals.

Feldman-Cousins confidence intervals for the mean of the Poisson Distribution (2)

TABLE IV. 90% C.L. intervals for the Poisson signal mean μ, for total events observed n_0, for known mean background b ranging from 0 to 5.

$n_0 \backslash b$	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0
0	0.00, 2.44	0.00, 1.94	0.00, 1.61	0.00, 1.33	0.00, 1.26	0.00, 1.18	0.00, 1.08	0.00, 1.06	0.00, 1.01	*0.00, 0.98*
1	0.11, 4.36	0.00, 3.86	0.00, 3.36	0.00, 2.91	0.00, 2.53	0.00, 2.19	0.00, 1.88	0.00, 1.59	0.00, 1.39	0.00, 1.22
2	0.53, 5.91	0.03, 5.41	0.00, 4.91	0.00, 4.41	0.00, 3.91	0.00, 3.45	0.00, 3.04	0.00, 2.67	0.00, 2.33	0.00, 1.73
3	1.10, 7.42	0.60, 6.92	0.10, 6.42	0.00, 5.92	0.00, 5.42	0.00, 4.92	0.00, 4.42	0.00, 3.95	0.00, 3.53	0.00, 2.78
4	1.47, 8.60	1.17, 8.10	0.74, 7.60	0.24, 7.10	0.00, 6.60	0.00, 6.10	0.00, 5.60	0.00, 5.10	0.00, 4.60	0.00, 3.60
5	1.84, 9.99	1.53, 9.49	1.25, 8.99	0.93, 8.49	0.43, 7.99	0.00, 7.49	0.00, 6.99	0.00, 6.49	0.00, 5.99	0.00, 4.99
6	2.21,11.47	1.90,10.97	1.61,10.47	1.33, 9.97	1.08, 9.47	0.65, 8.97	0.15, 8.47	0.00, 7.97	0.00, 7.47	0.00, 6.47
7	3.56,12.53	3.06,12.03	2.56,11.53	2.09,11.03	1.59,10.53	1.18,10.03	0.89, 9.53	0.39, 9.03	0.00, 8.53	0.00, 7.53
8	3.96,13.99	3.46,13.49	2.96,12.99	2.51,12.49	2.14,11.99	1.81,11.49	1.51,10.99	1.06,10.49	0.66, 9.99	0.00, 8.99
9	4.36,15.30	3.86,14.80	3.36,14.30	2.91,13.80	2.53,13.30	2.19,12.80	1.88,12.30	1.59,11.80	1.33,11.30	0.43,10.30
10	5.50,16.50	5.00,16.00	4.50,15.50	4.00,15.00	3.50,14.50	3.04,14.00	2.63,13.50	2.27,13.00	1.94,12.50	1.19,11.50
11	5.91,17.81	5.41,17.31	4.91,16.81	4.41,16.31	3.91,15.81	3.45,15.31	3.04,14.81	2.67,14.31	2.33,13.81	1.73,12.81
12	7.01,19.00	6.51,18.50	6.01,18.00	5.51,17.50	5.01,17.00	4.51,16.50	4.01,16.00	3.54,15.50	3.12,15.00	2.38,14.00
13	7.42,20.05	6.92,19.55	6.42,19.05	5.92,18.55	5.42,18.05	4.92,17.55	4.42,17.05	3.95,16.55	3.53,16.05	2.78,15.05
14	8.50,21.50	8.00,21.00	7.50,20.50	7.00,20.00	6.50,19.50	6.00,19.00	5.50,18.50	5.00,18.00	4.50,17.50	3.59,16.50
15	9.48,22.52	8.98,22.02	8.48,21.52	7.98,21.02	7.48,20.52	6.98,20.02	6.48,19.52	5.98,19.02	5.48,18.52	4.48,17.52
16	9.99,23.99	9.49,23.49	8.99,22.99	8.49,22.49	7.99,21.99	7.49,21.49	6.99,20.99	6.49,20.49	5.99,19.99	4.99,18.99
17	11.04,25.02	10.54,24.52	10.04,24.02	9.54,23.52	9.04,23.02	8.54,22.52	8.04,22.02	7.54,21.52	7.04,21.02	6.04,20.02
18	11.47,26.16	10.97,25.66	10.47,25.16	9.97,24.66	9.47,24.16	8.97,23.66	8.47,23.16	7.97,22.66	7.47,22.16	6.47,21.16
19	12.51,27.51	12.01,27.01	11.51,26.51	11.01,26.01	10.51,25.51	10.01,25.01	9.51,24.51	9.01,24.01	8.51,23.51	7.51,22.51
20	13.55,28.52	13.05,28.02	12.55,27.52	12.05,27.02	11.55,26.52	11.05,26.02	10.55,25.52	10.05,25.02	9.55,24.52	8.55,23.52

Feldman, Cousins, physics/9711021v2

Feldman-Cousins method: Discussion

Nice features:

+ State-of-the art for frequentist confidence intervals
+ Avoids flip-flop problem, correct coverage
+ Handles interval estimates at physical boundaries

Drawbacks:

– Construction of F-C confidence intervals is complicated, usually has to be done numerically

– Systematic uncertainties not easily included

– Counter-intuitive result in case of counting experiments with different background (see next slide)

Feldman-Cousins method: The paradox of fewer than expected background events

Consider two counting experiments

- Experiment A: expects background $b = 0$ ("carefully designed experiment")
- Experiment B: expects background $b = 5$

Suppose now both experiments measure $n = 0$ counts.
Feldman-Cousins upper limits at 90% CL:

- Experiment A: $s_{up} = 2.44$
- Experiment B: $s_{up} = 0.98$

Weird: The FC method says that the experiment B in which a larger background is expected gives the better (more stringent) upper limit.

Experiment B must have observed a downward fluctuation of the background.
How can a fluctuation result in a better upper limit?

Suggestion in the Feldman-Cousins paper

"Our suggestion for doing this is that in cases in which the measurement is less than the estimated background, the experiment report both the upper limit and the “sensitivity” of the experiment, where the “sensitivity” is defined as the average upper limit that would be obtained by an ensemble of experiments with the expected background and no true signal. […]

Thus, an experiment that measures 2 events and has an expected background of 3.5 events would report a 90% C.L. upper limit of 2.7 events (from Tab. IV), but a sensitivity of 4.6 events (from Tab. XII)."

Feldman, Cousins, physics/9711021v2

CL_s method: Motivation

Consider an experiment with low sensitivity ("background dominated experiment").

- By construction, one rejects a true hypothesis with a certain probability (e.g. 5%)
- Problem: exclusion of parameter values to which one has no sensitivity
- Example Higgs search: m_H = 1000 TeV rejected with a chance of 5%
- "Spurious exclusion"

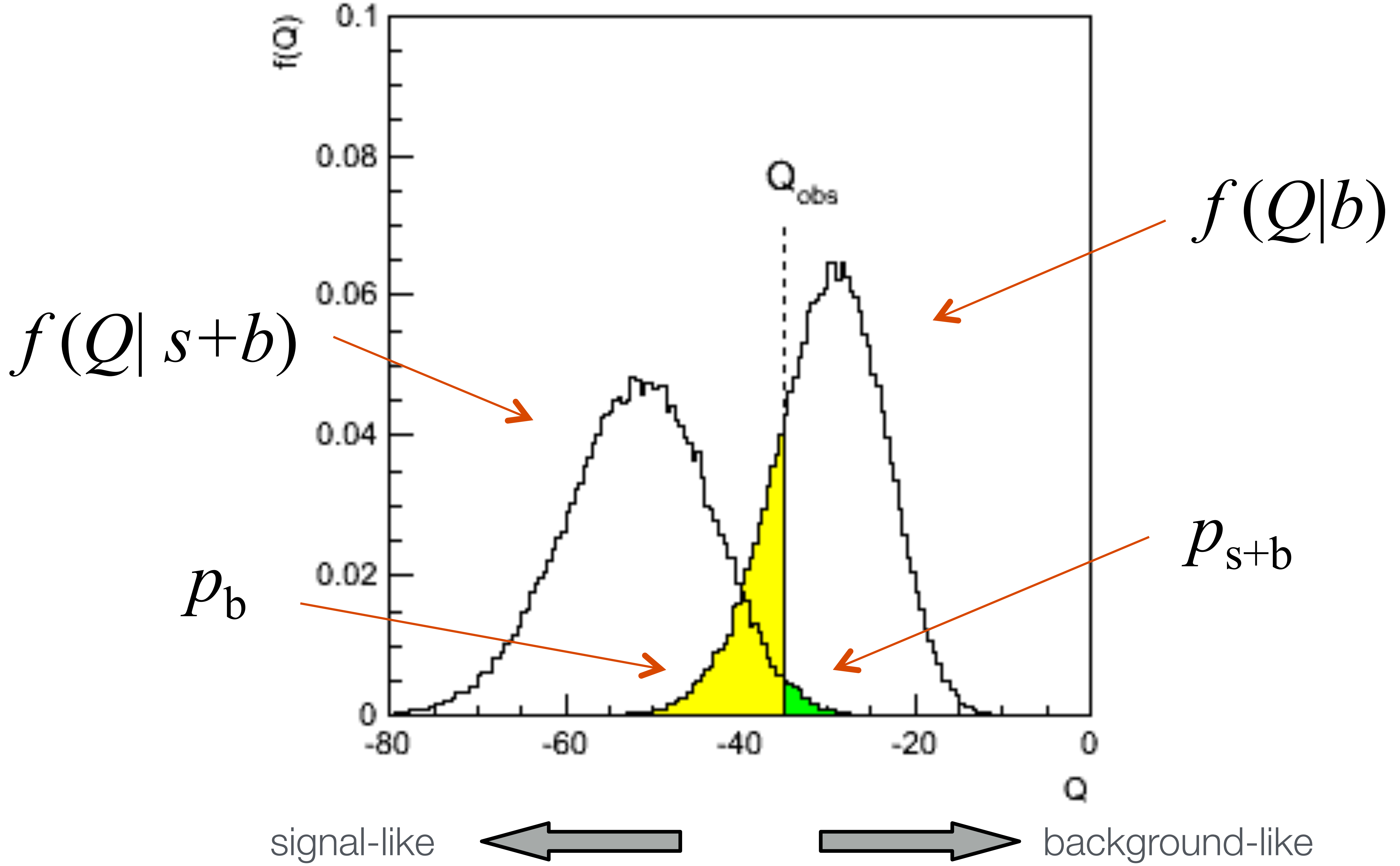
This problem was addressed for the LEP Higgs searches in the late 1990'ies and led to the CL_s method

A. Read, J. Phys. G 28, 2693 (2002), T. Junk, NIM A, 434, 435 (1999)

- Explicitly consider experimental sensitivity in limit setting
- Reduce spurious exclusion by a particular choice of the critical region
- Frequentist-motivated approach, but NOT frequentist ("modified frequentist method")
- Name a bit misleading, as the CL_s exclusion region is not a confidence interval
- Overcoverage by construction: conscious choice to give up frequentist coverage to take sensitivity into account
- "Despite its shaky foundations in statistical theory, it has been producing sensible results for over a decade" (http://cds.cern.ch/record/2203243)

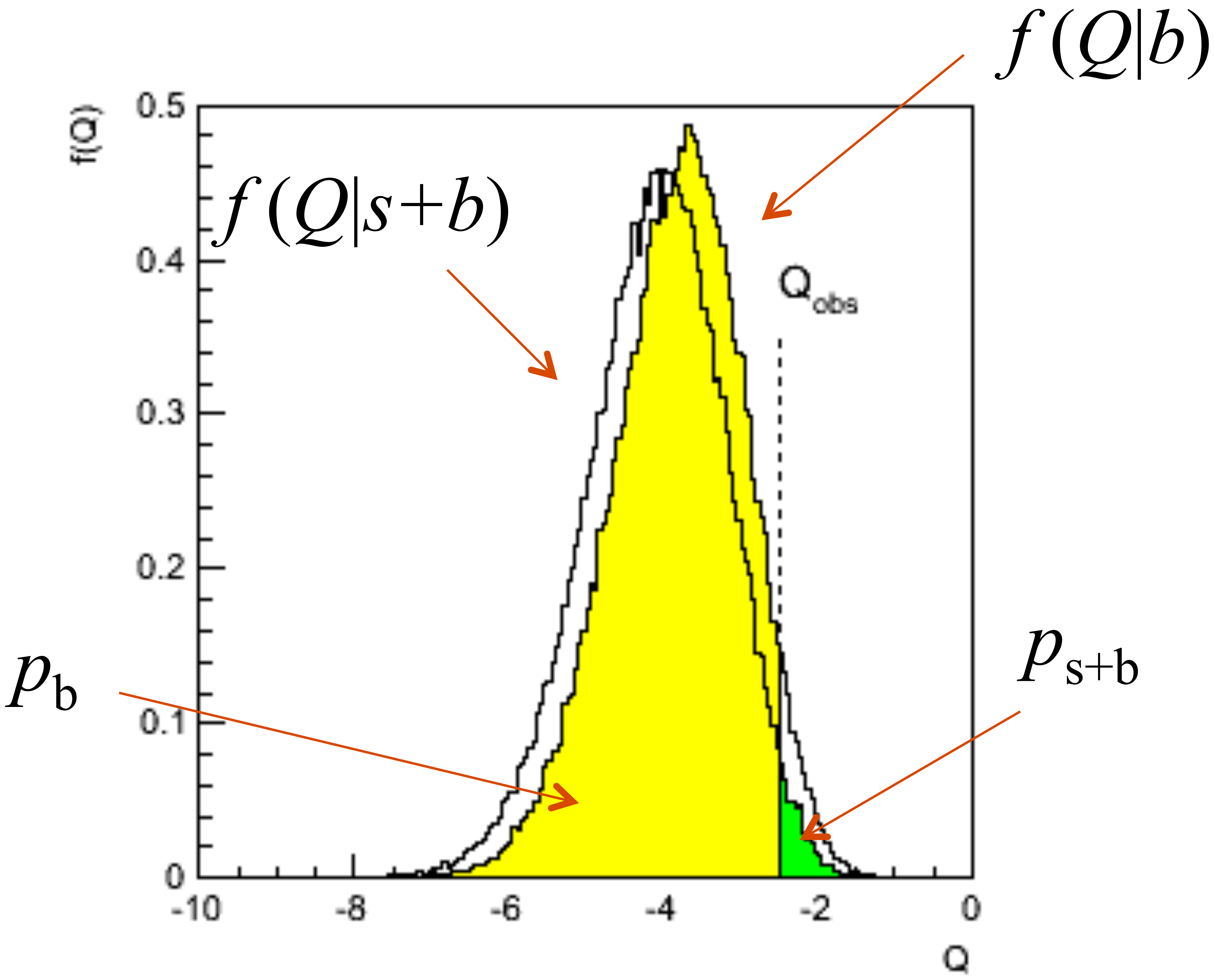
CL_s procedure (1)

Test statistic: $$Q = -2 \ln \frac{L(x|s+b)}{L(x|b)}$$



CL_s procedure (2)

Low sensitivity: the distributions under *s* and *s+b* are very close



$\mathrm{CL_s}$ Procedure (3)

Standard *p*-value test:

Reject s+b hypothesis if

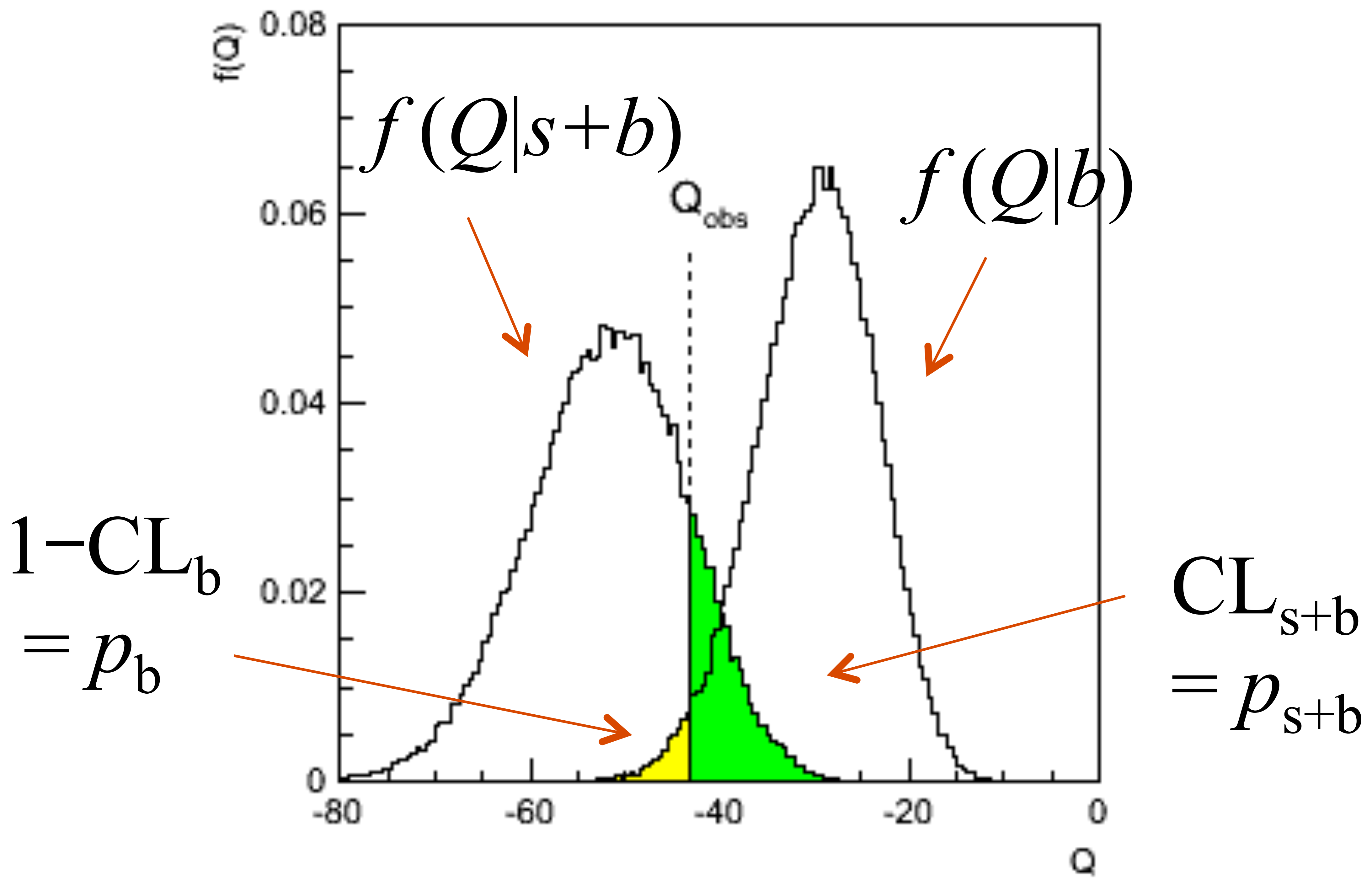
$$p_{s+b} \leq \alpha$$

$\mathrm{CL_s}$ method:

Reject s+b hypothesis if

$$\mathrm{CL}_s := \frac{p_{s+b}}{1 - p_b} \equiv \frac{\mathrm{CL}_{s+b}}{\mathrm{CL}_b} \leq \alpha$$

more stringent than standard *p*-value test as $1 - p_b \leq 1$



Increases “effective” *p*-value when the two distributions become close (prevents exclusion if sensitivity is low)

Upper Limits on $\mu = \sigma/\sigma_{SM}$ in Higgs searches

Signal for Higgs hypothesis: $s(m_H) = L_{\text{int}} \cdot \sigma_{\text{SM}}$

Signal strength μ: $n = \mu \cdot s(m_H) + b, \quad \mu = \dfrac{L_{\text{int}} \cdot \sigma(m_H)}{L_{\text{int}} \cdot \sigma_{\text{SM}}(m_H)} = \dfrac{\sigma(m_H)}{\sigma_{\text{SM}}(m_H)}$

μ = 1: SM w/ Higgs, μ = 0: SM w/o Higgs (background only model)

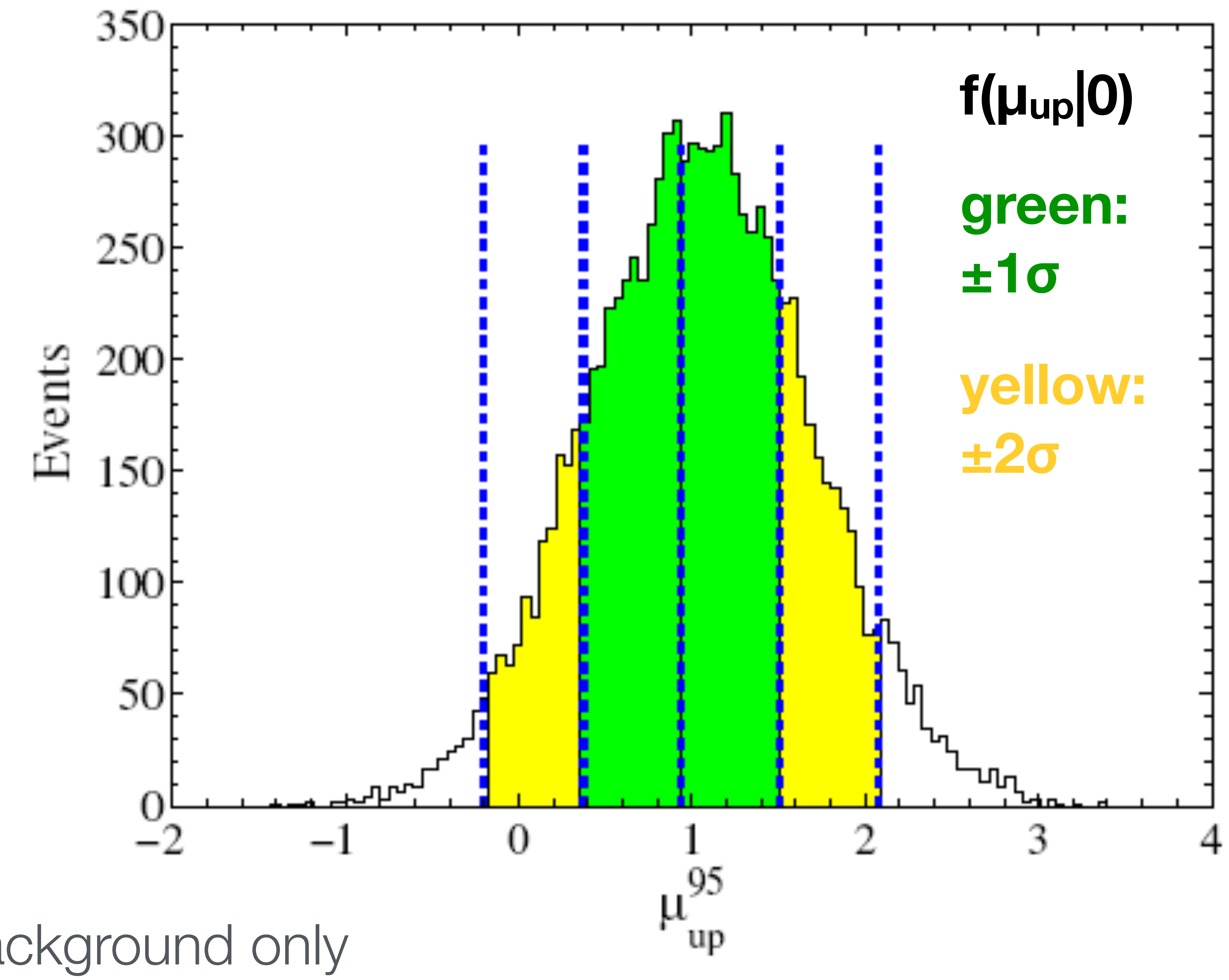
Carry out CL_s procedure for all values of $\mu = \sigma/\sigma_{SM}$. Reject μ if

$$\text{CL}_s := \frac{p_\mu}{1 - p_b} \leq 0.05$$

This defines upper limit μ_{up} at 95% CL (smallest value of μ that can be rejected by the CL_s criterion)

At a given value of m_H, we have an observed value of μ_{up}, and we can also find the distribution $f(\mu_{up}|0)$

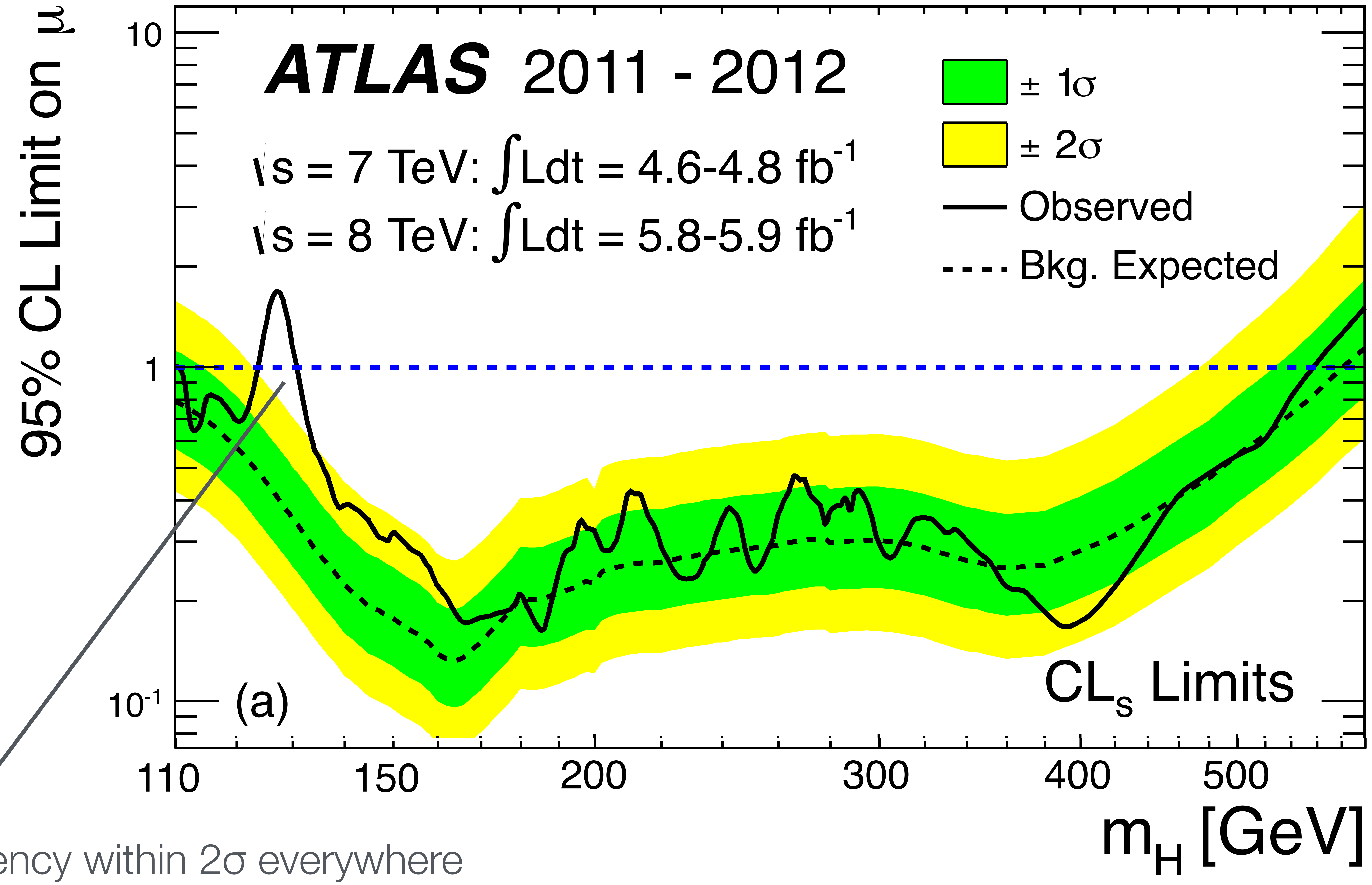
background only hypothesis



Upper limits on $\mu = \sigma/\sigma_{SM}$ in Higgs searches

95% CL limit on $\mu < 1 \Rightarrow$ Standard model with m_H rejected

ATLAS, Phys.Lett. B716 (2012) 1-29 (arXiv:1207.7214)

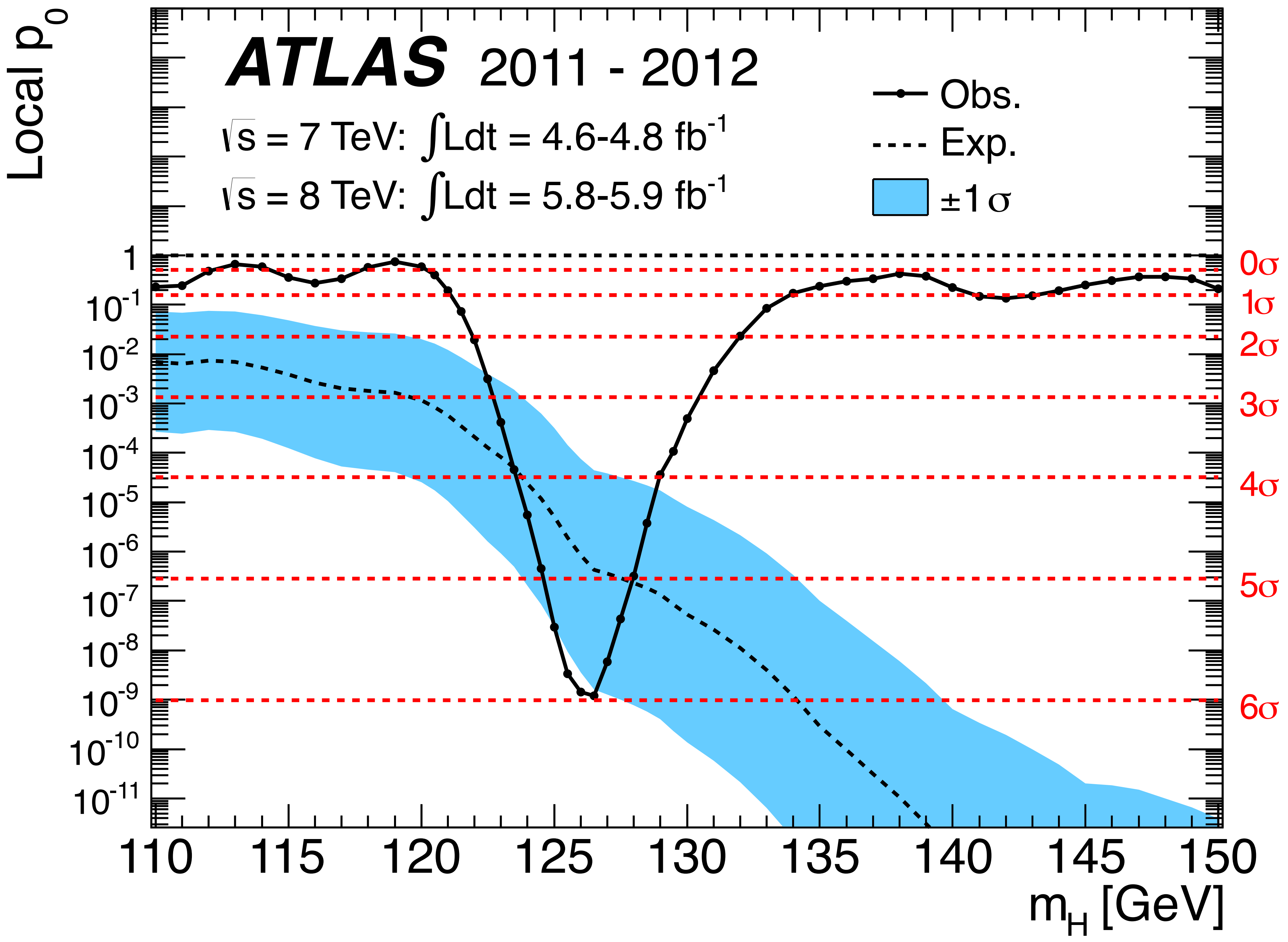


Consistency within 2σ everywhere except for m_H = 125 GeV

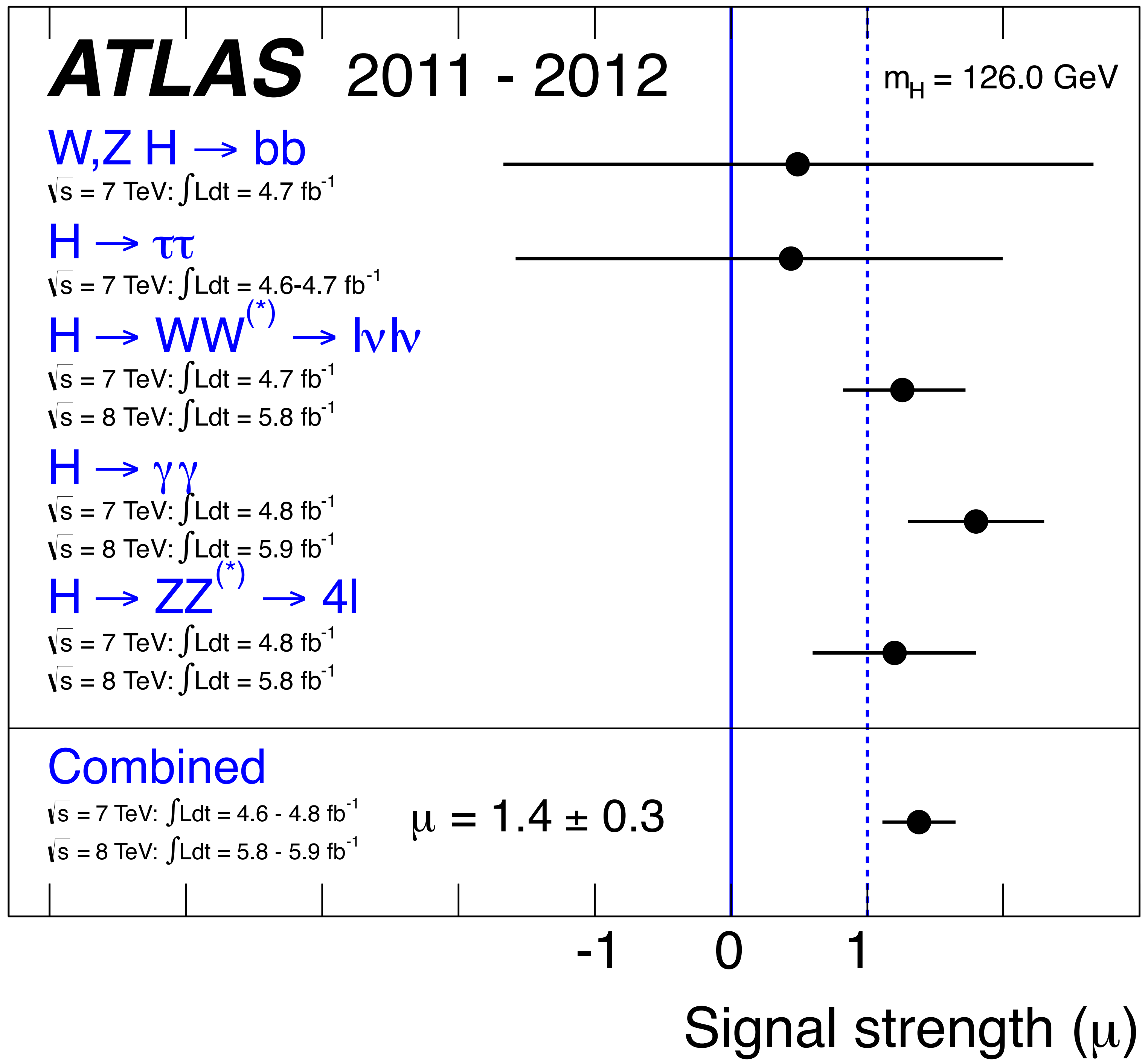
Higgs discovery (from ATLAS paper)

Reject also background only hypothesis at m_H = 125 GeV
and check consistency with μ = 1 → discovery!!!

local *p*-value



signal strength



ATLAS, Phys.Lett. B716 (2012) 1-29 (arXiv:1207.7214)

Higgs discovery

CERN Seminar on 4. July 2012

Higgs discovery

“I think we have it!”

(Rolf Heuer,
CERN director general in 2012

Summary

- Confidence intervals are based on the idea of *coverage*: a certain fraction of repeatedly measured intervals would cover the true value
- This leads to properties, that seem strange when interpreted as a region where the true parameter likely is (unphysical intervals, no good way to switch from interval to limit)
- The Feldman Cousins approach removes the strange properties, but does not change the basic definition → just because the interval is possible, it is not necessarily where you should expect the true parameter
- Compared to credible intervals a trade-off: Get objective results but lose the interpretation
- The CLs method is a compromise, giving intervals that are neither confidence intervals not credible intervals