10.1 Phenomenological Description

London theory, Fritz und Heinz London 1935

idea: Maxwell equations and the properties of ideal conductor and ideal diamagnet

electrical conductivity: $m\dot{\boldsymbol{v}} = -e\boldsymbol{\mathcal{E}} - m\boldsymbol{v}/\tau$ (the friction term $m\boldsymbol{v}/\tau$ is struck through)

$\tau = \infty$ ideal conductor

with $\boldsymbol{j}_\mathrm{s} = -n_\mathrm{s} e_\mathrm{s} \boldsymbol{v}$

$$\longrightarrow \quad \boxed{\frac{\mathrm{d}\boldsymbol{j}_\mathrm{s}}{\mathrm{d}t} = \frac{n_\mathrm{s} e_\mathrm{s}^2}{m_\mathrm{s}} \boldsymbol{\mathcal{E}}}$$

1st London equation

insert in Maxwell equation $\mathrm{rot}\,\boldsymbol{\mathcal{E}} = -\partial \boldsymbol{B}/\partial t$

$$\longrightarrow \quad \frac{\partial}{\partial t} \underbrace{\left(\mathrm{rot}\,\boldsymbol{j}_\mathrm{s} + \frac{n_\mathrm{s} e_\mathrm{s}^2}{m_\mathrm{s}} \boldsymbol{B} \right)}_{=0} = 0$$

magnetic flux is constant for ideal conductors, but Meißner effect demands const. = 0, therefore not $\mathrm{rot}\,\partial \boldsymbol{B}/\partial t = 0$ but $B = 0$

$$\longrightarrow \quad \boxed{\mathrm{rot}\,\boldsymbol{j}_\mathrm{s} = -\frac{n_\mathrm{s} e_\mathrm{s}^2}{m_\mathrm{s}} \boldsymbol{B}}$$

2nd London equation

10.1 Phenomenological Description

application of London theory: penetration depth

fields enter a „little bit“, otherwise $j = \infty$

screening current

Maxwell equation $\operatorname{rot} \boldsymbol{B} = \mu_0 \boldsymbol{j}$

insert in 2nd London equation

$$\longrightarrow \quad \operatorname{rot}\operatorname{rot} \boldsymbol{B} = \mu_0 \operatorname{rot} \boldsymbol{j} = -\mu_0 \frac{n_\mathrm{s} e_\mathrm{s}^2}{m_\mathrm{s}} \boldsymbol{B}$$

special geometry:

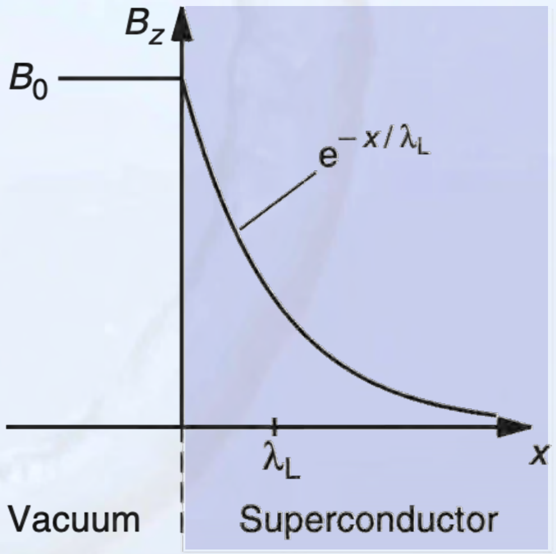
- superconductor fills half space $x > 0$
- magnetic field in z-direction

$$\frac{\mathrm{d}^2 B_z(x)}{\mathrm{d}x^2} - \frac{\mu_0 n_\mathrm{s} e_\mathrm{s}^2}{m_\mathrm{s}} B_z(x) = 0$$

ansatz $B_z(x) = B_0\, \mathrm{e}^{-x/\lambda_\mathrm{L}}$

$$\longrightarrow \quad \boxed{\lambda_\mathrm{L} = \sqrt{\frac{m_\mathrm{s}}{\mu_0 n_\mathrm{s} e_\mathrm{s}^2}}}$$

London penetration depth



10.1 Phenomenological Description

insert in Maxwell equation $\mathrm{rot}\,\boldsymbol{B} = \mu_0 \boldsymbol{j}$ $\longrightarrow$ $j_{s,y}(x) = j_0\,\mathrm{e}^{-x/\lambda_L}$ screening current

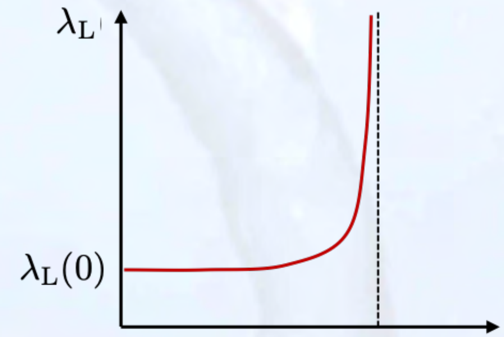
$$j_0 = B_0 / \mu_0\,\lambda_L$$

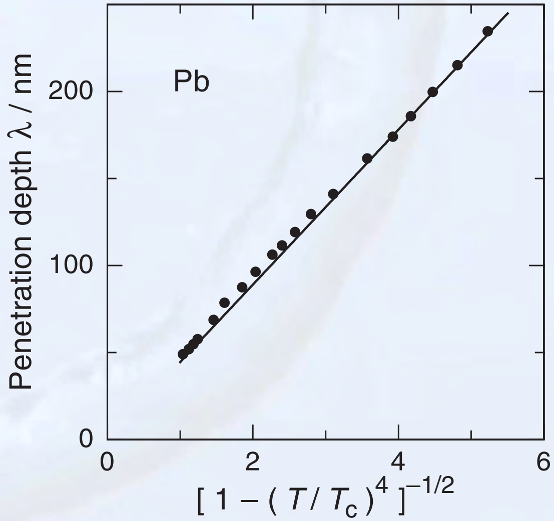
some numbers: $n_s \approx 10^{23}\ \mathrm{cm}^{-3}$ $\longrightarrow$ $\lambda_L = 30$ nm

experimental observation: susceptibility of thin lead cylinders

temperature dependence

$$n_s \propto 1 - (T/T_c)^4 \longrightarrow \lambda_L(T) = \frac{\lambda_L(0)}{\sqrt{1-(T/T_c)^4}}$$





- ► penetration depth of lead in the 100 nm range
- ► solid line: two-fluid model for superconductors

10.2 Microscopic Theory

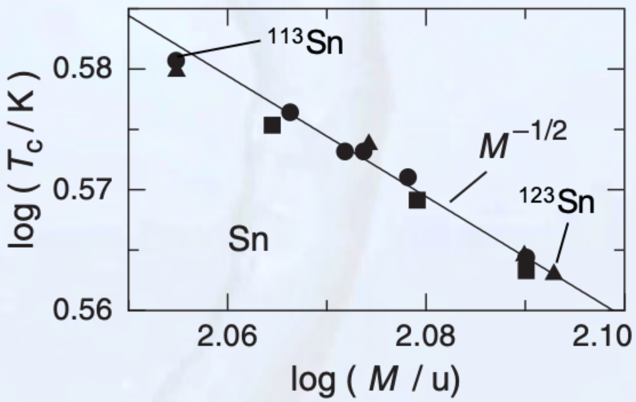
superconductivity occurs in many different materials

low transition temperatures ⟶ small energy differences matter ⟷ electrons have Fermi energy !

1950 Fröhlich (Bardeen) ⟶ interaction between electrons and lattice can mediate attraction between electrons

Isotope effect, discovered 1950

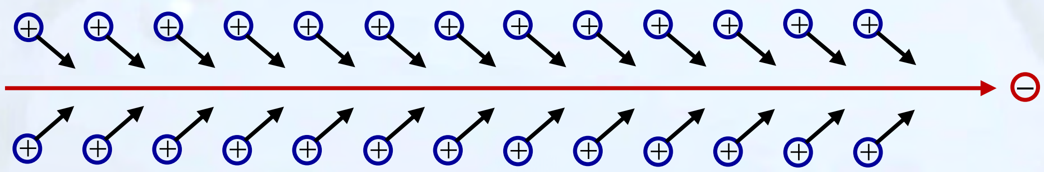
- T_c depends on atomic mass $T_c \propto 1/\sqrt{M}$
- for m = 113 u … 123 u T_c = 3.8 K … 3.66 K
- lattice properties are important for superconductivity



10.2 Microscopic Theory

schematic picture

- electron passes through lattice and attracts positive ions
- positive charge density maximum occurs long after electron has passed
- a second electron is attracted, but Coulomb repulsion is small since it is far away from first electron



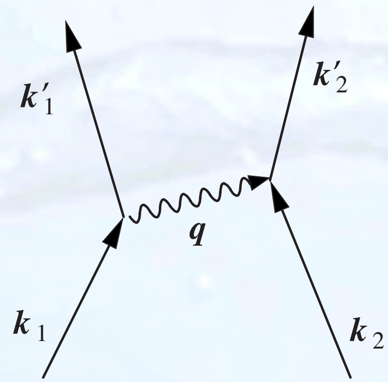
estimated distance between electron and positive charge density maximum

$$s = v_F t \approx 10^8 \times 10^{-13}\,\mathrm{cm} = 1000\,\text{Å}$$

time for ions to react $1/\omega_D$

10.2 Microscopic Theory

Cooper pairs



Leon N. Cooper

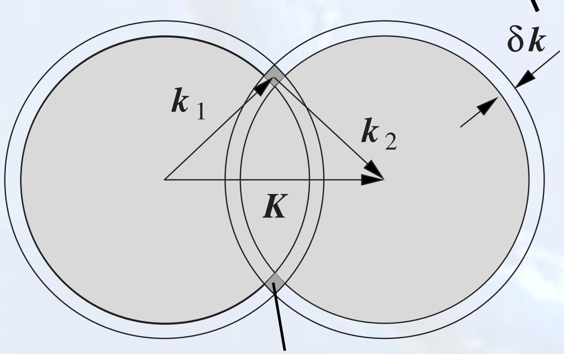
$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_1' + \boldsymbol{k}_2' = \boldsymbol{K}$$

center of mass motion

for $\hbar \boldsymbol{K} = 0$ phase space maximum ⟶ $\boldsymbol{k}_1 = -\boldsymbol{k}_2 = \boldsymbol{k},$

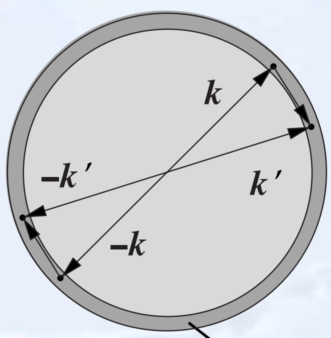
$\boldsymbol{K} \neq 0$

$\delta k = (m\omega_D / \hbar k_F)$



phase space

$\boldsymbol{K} = 0$



phase space

⟶ Cooper pair state $(\boldsymbol{k}, -\boldsymbol{k})$

in addition: $L = 0$

10.2 Microscopic Theory

stationary Schrödinger equation for two interacting particles

$$\left[-\frac{\hbar^2}{2m}(\Delta_1 + \Delta_2) + \mathcal{V}(\boldsymbol{r}_1, \boldsymbol{r}_2)\right]\psi(\boldsymbol{r}_1, \boldsymbol{r}_2) = E\psi(\boldsymbol{r}_1, \boldsymbol{r}_2)$$

electron-phonon interaction

two-particle wave function

$$\psi(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{V}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}_1}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}_2\cdot\boldsymbol{r}_2} = \frac{1}{V}\,\mathrm{e}^{\boldsymbol{k}\cdot\boldsymbol{r}} = \Psi(\boldsymbol{r})$$

$$\boldsymbol{r} = (\boldsymbol{r}_1 - \boldsymbol{r}_2)$$

electrons are scattered constantly into new pair states

$$\longrightarrow \quad \Psi(\boldsymbol{r}) = \sum_{\boldsymbol{k}} A_{\boldsymbol{k}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$A_{\boldsymbol{k}} \begin{cases} \neq 0 & \text{for} \quad k_\mathrm{F} < k < \sqrt{2m(E_\mathrm{F} + \hbar\omega_\mathrm{D})/\hbar^2} \\ = 0 & \text{otherwise}\,. \end{cases}$$

probability to find a particular pair state

insert $\Psi(\boldsymbol{r})$, multiplying with $\exp(-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{r})$ and integrate

$$\longrightarrow \quad \frac{\hbar^2 k^2}{m} A_{\boldsymbol{k}} + \frac{1}{V}\sum_{\boldsymbol{k}'} A_{\boldsymbol{k}'}\mathcal{V}_{\boldsymbol{k}\boldsymbol{k}'} = E A_{\boldsymbol{k}}$$

Fourier transform of electron-phonon interaction

10.2 Microscopic Theory

approximation for electron-phonon interaction

$$\mathcal{V}_{\boldsymbol{k}\boldsymbol{k}'} = \begin{cases} -\mathcal{V}_0 & \text{for} \quad E_\mathrm{F} < \epsilon_{\boldsymbol{k}}, \epsilon_{\boldsymbol{k}'} < E_\mathrm{F} + \hbar\omega_\mathrm{D} \\ 0 & \text{otherwise} \end{cases}$$

$$\longrightarrow \quad \left(\frac{\hbar^2 k^2}{m} - E \right) A_{\boldsymbol{k}} = \frac{\mathcal{V}_0}{V} \sum_{\boldsymbol{k}'} A_{\boldsymbol{k}'}$$

with $z = \hbar^2 k^2 / 2m$ $\longrightarrow$ $A_{\boldsymbol{k}} = \frac{\mathcal{V}_0}{V} \frac{1}{2z - E} \sum_{\boldsymbol{k}'} A_{\boldsymbol{k}'}$

with $\sum_{\boldsymbol{k}} A_{\boldsymbol{k}} = \sum_{\boldsymbol{k}'} A_{\boldsymbol{k}'}$ $\longrightarrow$ $1 = \frac{\mathcal{V}_0}{V} \sum_{\boldsymbol{k}} \frac{1}{2z - E}$

replacing the sum with an integral, and $D(E) \approx D(E_\mathrm{F})$ $\longrightarrow$ $1 = \mathcal{V}_0 \frac{D(E_\mathrm{F})}{2} \int_{E_\mathrm{F}}^{E_\mathrm{F} + \hbar\omega_\mathrm{D}} \frac{\mathrm{d}z}{2z - E}$

integration

$$\longrightarrow \quad \delta E = E - 2E_\mathrm{F} = \frac{2\,\hbar\omega_\mathrm{D}}{1 - \exp[4/\mathcal{V}_0 D(E_\mathrm{F})]} \approx -2\,\hbar\omega_\mathrm{D}\, \mathrm{e}^{-4/[\mathcal{V}_0 D(E_\mathrm{F})]}$$

energy reduction per Cooper pair

$\mathcal{V}_0\, D(E_\mathrm{F}) \ll 1$ weak coupling

- for Cu, Ag, K, … $\mathcal{V}_0$ is small, because they are good conductors $\longrightarrow$ no superconductor since small δE
- Al has small $\mathcal{V}_0$, but high density of states at Fermi energy $\longrightarrow$ superconductor with $T_c \approx 1$ K