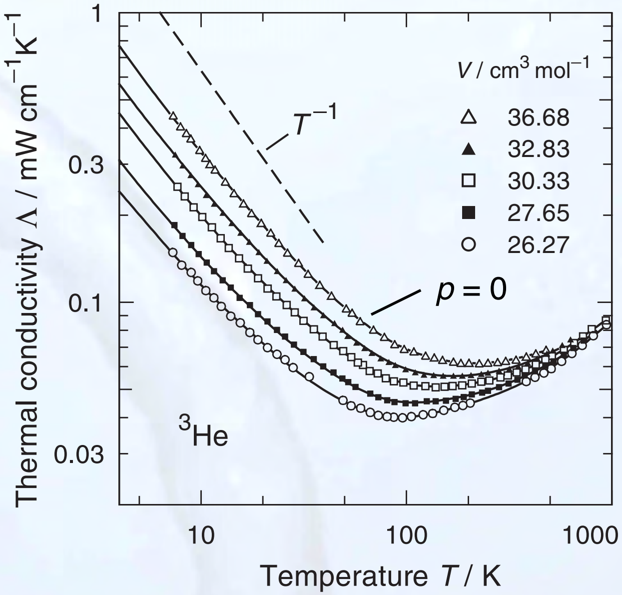
3.1 Ideal Fermi-Gas

(iii) Thermal conductivity

$$\Lambda = \frac{1}{3} C_V \, v \, \ell \quad \longrightarrow \quad \Lambda = \frac{1}{3} C_V \, \tau \, v_F^2$$

- low temperatures: $\left.\begin{array}{l} C \propto T \\ \tau \propto T^{-2} \end{array}\right\} \Lambda \propto T^{-1}$ and paramagnon contributions
- high temperatures: dense classical gas
- very small absolute value: $\Lambda \approx 10^{-4}\,\mathrm{W\,cm^{-1}K^{-1}}$ at 200 mK



Is ^{3}He a Fermi gas?

	^{3}He	Fermi Gas	Ratio
$C_V/\gamma T$	2.78	1.00	2.78
$v = v_F/\sqrt{3}\ (\mathrm{m\,s^{-1}})$	188	95	1.92
$\chi/\beta^2\ \ (\mathrm{J\,m^3})^{-1}$	3.3×10^{51}	3.6×10^{50}	9.1

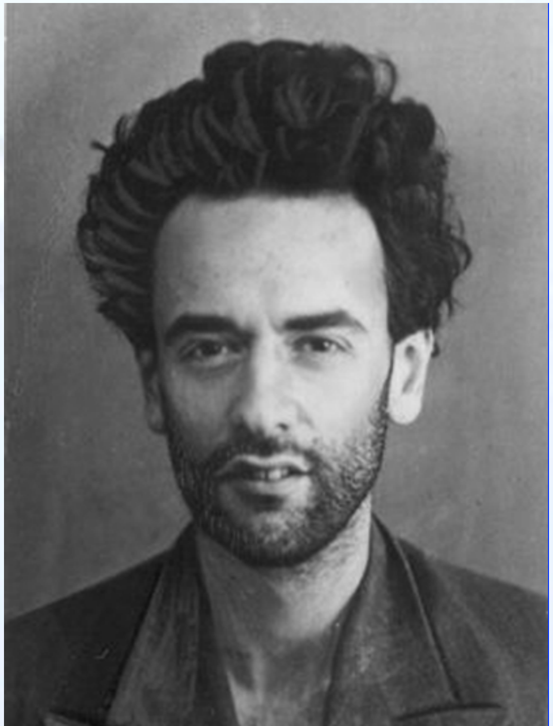
⟶ deviations are not too big, but still significant and in addition differently large for different properties

3.2 The Landau Fermi-Liquid Theory

free Fermi gas ⟶ strongly interacting ^{3}He atoms

↓

collective excitations $\widehat{=}$ quasi particles



Lev Davidovic Landau 1938

Landau theory of Fermi liquids 1956-1958

⟶ prediction of zero sound and collision-less spin waves

Basic idea

- interaction does change the energy of particles, but not momentum!
- plausible since momentum states are given by boundary conditions

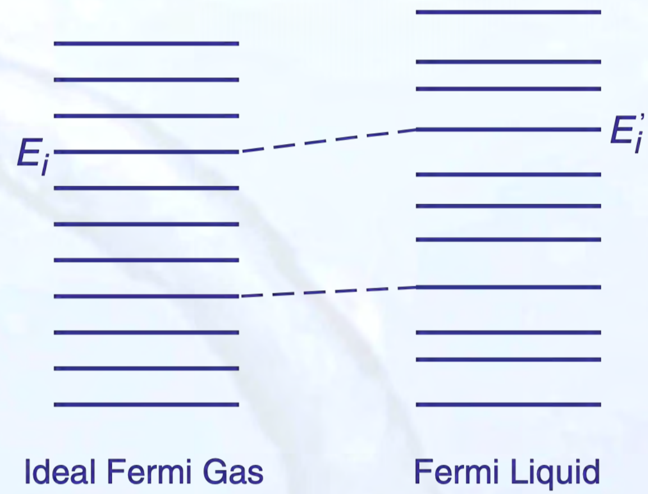
for each state in the Fermi gas there is a corresponding state in the liquid, but with modified energy

3.2 The Landau Fermi-Liquid Theory

Quasi-particle concept

important: total energy is not given by the sum of all individual states (isolated atoms)

$$U \neq \sum_i f_i E_i$$



Landau's Gedankenexperiment

consider that the interaction is switched on slowly

➡ number of states does not change

$$n = 2\varrho_k \int f \, \mathrm{d}^3 k = \int D(k) \, f \, \mathrm{d}k \longrightarrow$$

number of quasi particles per volume analog to Fermi gas

2 spin states

$4\pi k^2 \mathrm{d}k$

➡ energy of one quasi particle is defined by the change of energy of the complete system when a quasi particle is added:

$$\delta u = \int E \, \delta f \, \mathrm{d}^3 k$$

$\dfrac{\delta U}{V}$

small change in occupation when one quasi particle is added

3.2 The Landau Fermi-Liquid Theory

→ energy of one quasi particle is given by the energy of an isolated atom plus, the interaction with all other atoms

→ quasi particle states are not eigenstates

How does the distribution function look like? — does the Fermi distribution still hold?

Yes, as long as the energy levels (states) are well-defined!

but quasi particles aren't eigen states → transitions occur

→ broadening of levels $\delta E \approx \hbar/\tau$

collision time, lifetime

→ quasi particle states are well-defined as long as the uncertainty is small compared to the thermal broadening $\Delta E \approx k_\mathrm{B} T$

3.2 The Landau Fermi-Liquid Theory

→ this condition can always be fulfilled at sufficiently low temperatures, since

$$\tau \propto \frac{1}{T^2} \qquad \boxed{\delta E \propto T^2}$$

some numbers: $\tau \approx 5 \times 10^{-11} \frac{1}{T^2}\,[\,\mathrm{s}\,]$ Fermi gas

$\tau \approx 1 \times 10^{-12} \frac{1}{T^2}\,[\,\mathrm{s}\,]$ experimental result

$$\Delta E \approx k_\mathrm{B} T \longrightarrow T = 0.1\,\mathrm{K}$$

→ Fermi distribution holds $f(E,T) = \dfrac{1}{\mathrm{e}^{(E-\mu)/k_\mathrm{B}T} + 1}$

→ Landau theory is good for $T \ll 0.1\,\mathrm{K}$ in case of ^{3}He

3.2 The Landau Fermi-Liquid Theory

What is the dispersion relation ?

$T \to 0\,,$ states at E_{F} : $p_{\mathrm{F}} = \hbar\left(3\pi^2 n\right)^{1/3}$

general expression for states near Fermi level

$$E = E_{\mathrm{F}} + \left(\frac{\partial E}{\partial p}\right)_{\mathrm{F}} (p - p_{\mathrm{F}})$$

$$\left(\frac{\partial E}{\partial p}\right)_{\mathrm{F}} = \frac{p_{\mathrm{F}}}{m} = v_{\mathrm{F}}$$ Fermi gas

$$\left(\frac{\partial E}{\partial p}\right)_{\mathrm{F}} = \frac{p_{\mathrm{F}}}{m^*}$$ Fermi liquid

dispersion of quasi particles

$$E = E_{\mathrm{F}} + \frac{p_{\mathrm{F}}}{m^*}(p - p_{\mathrm{F}})$$

density of states at Fermi level

$$\boxed{D(E_{\mathrm{F}}) = \frac{m^* k_{\mathrm{F}}}{\pi^2 \hbar^2} = \frac{m^*}{\pi \hbar^2}\sqrt[3]{\frac{3n}{\pi}}}$$

3.2 The Landau Fermi-Liquid Theory

Central problem: Interaction term

- energy of quasi particles depends on the configuration of all quasi particles
- $E(\boldsymbol{p},T)$ changes when the occupation of states with $\boldsymbol{p}'$ differ by $\delta f(\boldsymbol{p}')$ from the one at $T=0$

Phenomenological ansatz (without spin term)

cannot be derived

$$E(\boldsymbol{p},T) = E(\boldsymbol{p},0) + 2\varrho_k \int h(\boldsymbol{p},\boldsymbol{p}')\,\delta f'\,\mathrm{d}^3p'$$

$$h(\boldsymbol{p},\boldsymbol{p}') = \frac{\partial^2 U}{\partial f(\boldsymbol{p})\,\partial f'(\boldsymbol{p}')}$$ interaction term

- $D(E_\mathrm{F})\;h(\boldsymbol{p},\boldsymbol{p}')$ corresponds to the scattering amplitude
- like for a Fermi gas only states at the Fermi surface are important $p \approx p' \approx p_\mathrm{F}$

⟶ $h(\boldsymbol{p},\boldsymbol{p}')$ depends only on the angle Θ between $\boldsymbol{p}'$ and $\boldsymbol{p}$

⟶ $\boxed{h(\boldsymbol{p},\boldsymbol{p}') = h(\Theta)}$

3.2 The Landau Fermi-Liquid Theory

Treatment of interaction term

consider new function: $F(\Theta) = D(E_\mathrm{F})\, h(\Theta)$

expansion in terms of Legendre polynomials

$$F(\Theta) = \sum_i F_i\, P_i(\cos\Theta) = F_0 + F_1\, \cos\Theta + F_2\, \frac{3\cos^2\Theta - 1}{2} + \ \ldots$$

these coefficients can (only) be determined experimentally

general expression with spin term: $\mathcal{F}(\boldsymbol{p}, \boldsymbol{s}, \boldsymbol{p}', \boldsymbol{s}') = h(\boldsymbol{p}, \boldsymbol{p}') + \underbrace{\xi(\boldsymbol{p}, \boldsymbol{p}')\ \boldsymbol{s}\cdot\boldsymbol{s}'}_{\text{spin term}}$

consider new function for spin term: $G(\Theta) = D(E_\mathrm{F})\, \xi(\Theta)$

expansion in terms of Legendre polynomials

$$G(\Theta) = D(E_\mathrm{F})\, \xi(\Theta) = \sum_i G_i\, P_i(\cos\Theta) = G_0 + G_1 \cos\Theta + \ \ldots$$

these coefficients can (only) be determined experimentally

3.2 The Landau Fermi-Liquid Theory

Application to liquid ^{3}He (not trivial)

(i) effective mass

$$\frac{1}{m} = \frac{1}{m^*} + \frac{p_F}{\hbar^3} \int F(\Theta) \cos\Theta \, d\Gamma$$

solid angle segment of Fermi surface

$$\boxed{\frac{m^*}{m} = \left(1 + \frac{1}{3} F_1\right)}$$

mean value of $F_1 \cos\Theta$

experimental results

pure ^{3}He: $\frac{m^*}{m} \approx 3$ normal pressure

$\frac{m^*}{m} \approx 6$ 30 bar

1% ^{3}He in ^{4}He: $\frac{m^*}{m} \approx 2.4$

Landau's Fermi liquid theory can be tested varying pressure and ^{3}He concentration

3.2 The Landau Fermi-Liquid Theory

(ii) specific heat

$$C = \frac{m^*}{m} C_{\rm FG} = \left(1 + \frac{1}{3} F_1\right) C_{\rm FG}$$

$\curvearrowright$ $C \propto T$ at $T \ll T_{\rm F}^*$ | $T_{\rm F}^* \approx 0.5\,{\rm K}$

(ii) sound velocity (first sound)

$$v_1^2 = \frac{p_{\rm F}^2}{3m^2} \frac{1 + F_0}{1 + \frac{1}{3} F_1} = \frac{1}{3} v_{\rm F}^2 \frac{1 + F_0}{1 + \frac{1}{3} F_1}$$

compare to:

$v = \frac{1}{3} v_{\rm F}$ Fermi gas

$$v_1^2 = \left(\frac{\partial p}{\partial \rho}\right)_S = N \frac{\partial \mu}{\partial N} \quad \Big| \quad \text{grad}\, \mu = \frac{1}{\rho}\, \text{grad}\, p$$

with $\mu = E_F = E(p_F)$

Since $p_F = \hbar \left(3\pi^2 \frac{N}{V}\right)^{1/3} \curvearrowright \frac{\partial \mu}{\partial N} \curvearrowright \frac{\partial \mu}{\partial p_F}$

$$\curvearrowright v_1^2 = \frac{p_F}{3m} \frac{\partial \mu}{\partial p_F}$$

$$= \frac{p_F}{3m} \left[\frac{p_F}{m} + \frac{2 p_F^2}{\hbar^3} \int F(\theta)(1 - \cos\theta)\, d\Gamma \right]$$

insert expansion

$$\curvearrowright v_1^2 = \frac{p_F^2}{3m^2} \left(\frac{1 + F_0}{1 + \frac{1}{3} F_1} \right)$$

(iii) magnetic susceptibility

$$\chi = \frac{m^*}{m} \left(\frac{1}{1 + \frac{1}{4} G_0} \right) \chi_{\rm FG}$$

– 2.8

➡ enhancement of susceptibility $\uparrow\uparrow$ against Fermi statistics $\downarrow\uparrow$

➡ if exchanged interaction larger by a factor 2 $\longrightarrow$ $1 + \frac{1}{4} G_0 < -1$ and ground state would be ferromagnetic

3.2 The Landau Fermi-Liquid Theory

Landau Fermi liquid parameters for ^{3}He

p (bar)	V_m (cm^3)	F_0	F_1	G_0	m^*/m
0	36.84	9.30	5.39	−2.78	2.80
3	33.95	15.99	6.49	−2.89	3.16
6	32.03	22.49	7.45	−2.93	3.48
9	30.71	29.00	8.31	−2.97	3.77
12	29.71	35.42	9.09	−2.99	4.03
15	28.89	41.73	9.85	−3.01	4.28
18	28.18	48.46	10.60	−3.03	4.53
21	27.55	55.20	11.34	−3.02	4.78
24	27.01	62.16	12.07	−3.02	5.02
27	26.56	69.43	12.79	−3.02	5.26
30	26.17	77.02	13.50	−3.02	5.50
33	25.75	84.79	14.21	−3.02	5.74

3.3 Sound Propagation in ^{3}He: Zero Sound

normal (first) sound: quasi particles reach local equilibrium by collisions $\omega\tau \ll 1$

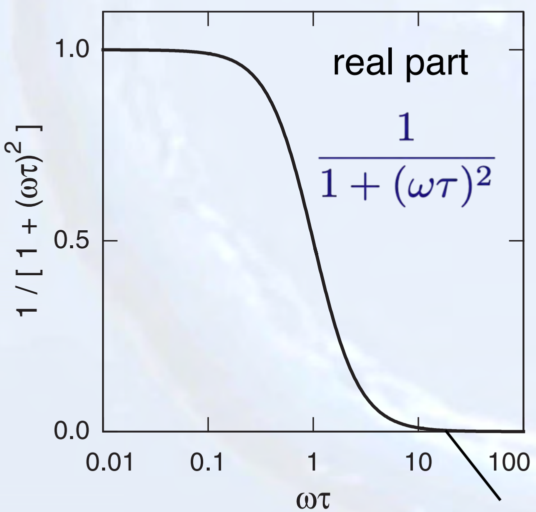
frequency of sound wave

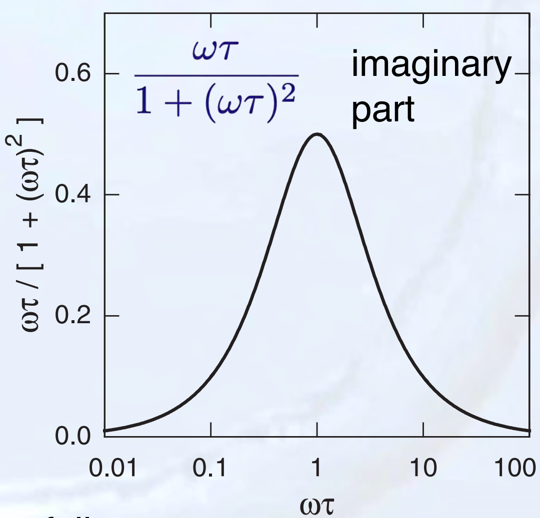
zero sound: collision-less propagation of sound $\omega\tau \gg 1$

^{3}He: particle density fluctuations in one region lead to density fluctuation in neighboring regions

→ propagation of sound-like modes → zero sound

Debye relaxation process (transition from hydrodynamic regime to collision-less regime)





systems cannot follow