2.5 Macroscopic Quantum State

→ quantization of circulation

→ Josephson effects

wave function of superfluid component

$$\psi(\boldsymbol{r}) = \psi_0\, \mathrm{e}^{\mathrm{i}\varphi(\boldsymbol{r})} \qquad (*) \qquad \text{with} \quad \psi^\star \psi = |\psi_0|^2 = \frac{\varrho_\mathrm{s}}{m_4}$$

mass of a ^{4}He atom

Schrödinger equation

$$-\mathrm{i}\hbar\nabla\psi = \boldsymbol{p}\,\psi$$

with $(*)$ $\longrightarrow$ $\boldsymbol{p} = \hbar\nabla\varphi(\boldsymbol{r}) = m_4 \boldsymbol{v}_\mathrm{s}$

$$\boxed{\boldsymbol{v}_\mathrm{s} = \frac{\hbar}{m_4}\nabla\varphi(\boldsymbol{r})}$$

comment:

only valid at sufficiently low velocity were ϱ_s is constant

➡ $\boldsymbol{v}_\mathrm{s}$ determines the phase shift of wave function

→ $\boldsymbol{v}_\mathrm{s} = 0$ $\longrightarrow$ phase is constant

→ $\boldsymbol{v}_\mathrm{s} =$ const. $\longrightarrow$ phase is changes uniformly

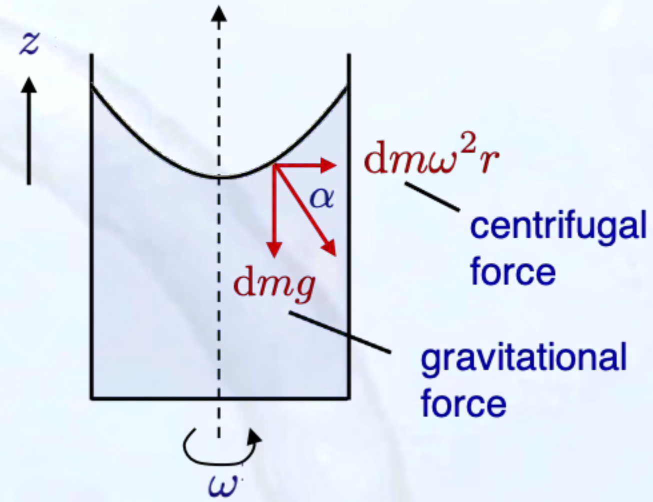
Interpretation

- phase is well-defined in entire liquid
- macroscopic wave function
- “rigid” coupling in momentum space

2.5 Macroscopic Quantum State

Proof of the concept: He-II under rotations

measurement of liquid meniscus



classical fluid $\triangleq$ normalfluid component ϱ_n

➡ solid body rotation $v_\mathrm{n} = \omega r$ — distance from axis of rotation

➡ profile of liquid surface ⟶ parabola

$$\tan\alpha = \frac{\mathrm{d}z}{\mathrm{d}r} = \frac{\omega^2 r}{g} \longrightarrow z = \frac{\omega^2}{2g} r^2$$

what about the superfluid component ?

two-fluid model $\mathrm{curl}\,\boldsymbol{v}_\mathrm{s} = 0$!

for a simply-connected region this means every loop can be contracted to a point

$$\int_A \underbrace{\mathrm{curl}\,\boldsymbol{v}_\mathrm{s}}_{=0} \cdot \mathrm{d}\boldsymbol{f} = \oint_L \boldsymbol{v}_\mathrm{s} \cdot \mathrm{d}\boldsymbol{l} = 0$$

area enclosed by contour L

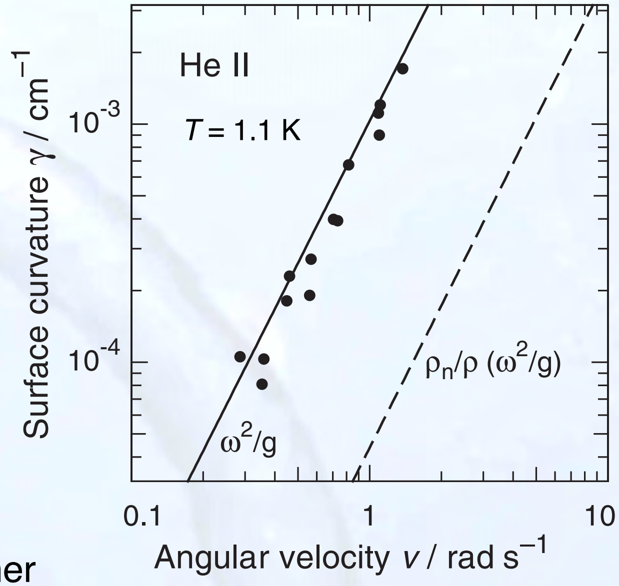
Stokes

- ϱ_s should not rotate (should be at rest)
- if so, centrifugal force is reduced

➡ $$\boxed{z = \frac{\varrho_\mathrm{n}}{\varrho} \frac{\omega^2}{2g} r}$$

2.5 Macroscopic Quantum State

Experimental results



surface curvature: $\gamma = \omega^2/g$ all liquid

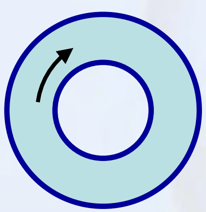
$\gamma = (\varrho_{\rm n}/\varrho)\omega^2/g$ only normalfluid

→ curvature for all liquid is observed in Osborn experiment

Why is this the case?

→ let's do a thought experiment with an annular-shaped container

multiply-connected region



circulation:

$$\kappa = \oint_L \boldsymbol{v}_{\rm s} \cdot {\rm d}\boldsymbol{l} \quad \Rightarrow \quad \kappa = \frac{\hbar}{m_4}\Delta\varphi_L$$

$$\boldsymbol{v}_{\rm s} = \frac{\hbar}{m_4}\nabla\varphi(\boldsymbol{r})$$

- since $\psi(\boldsymbol{r})$ is a uniquely-defined function
  → phase can only be changed by $2\pi\, n$ for full cycle
- $\Delta\varphi = 2\pi\, n \qquad n = 0,\ 1,\ 2,\ 3,\ \ldots\ .$

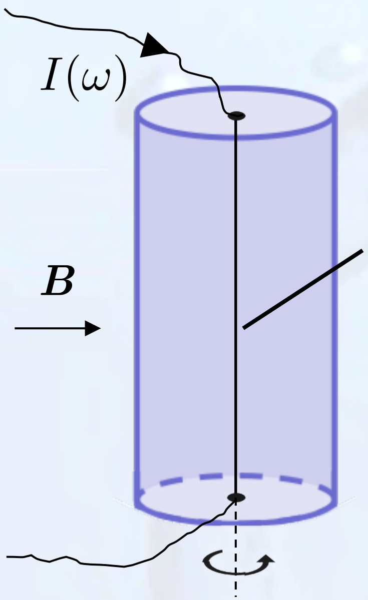
circulation is quantized !

$$\boxed{\kappa = \frac{h}{m_4} n}$$

2.5 Macroscopic Quantum State

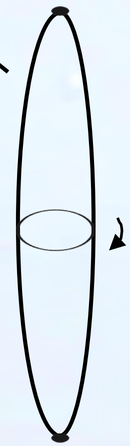
Experimental discovery of quantization of circulation

vibrating wire excited by current pules (Joe Vinen 1961)



thin wire
ø 2.5 μm,
5 cm long

transversal modes ≙ $\underbrace{\text{two circular polarized modes}}$



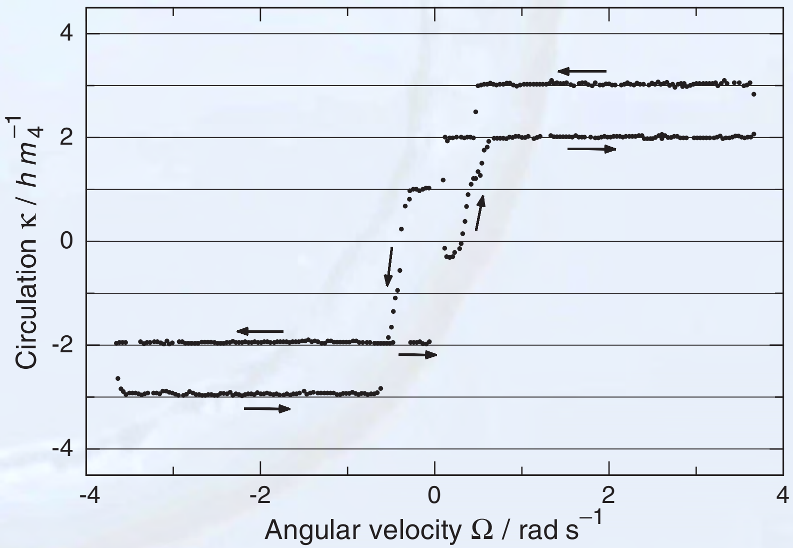
- without rotation: degenerate
- with rotation: lifting of degeneracy by Magnus force

frequency splitting: $\Delta\nu = \dfrac{\varrho_s}{2\pi\,\mathcal{M}}\,\kappa$

effective mass / length
(wire + ½ of expelled liquid)

experimental results

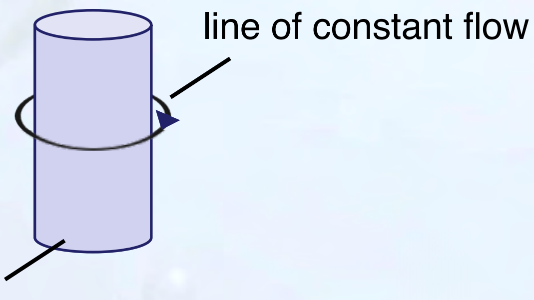
- quantization with expected value
- hysteresis effects are observed
- modern measurements up to $n = 4$

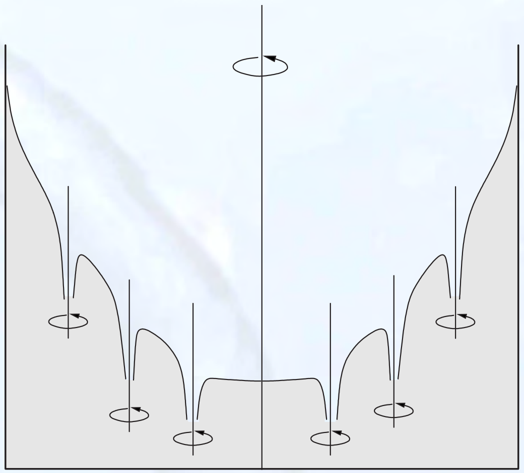


2.5 Macroscopic Quantum State

What has this to do with the rotation of bulk helium in a simply connected region?

- ➡ vortices may occur with normal fluid core
- ➡ resulting in a multiply connected region



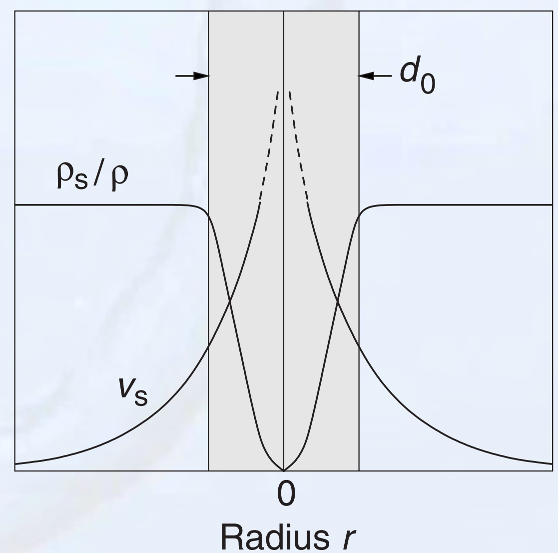


with $\kappa = \dfrac{h}{m_4} n$ and classical hydrodynamics one finds

$$v_\mathrm{s}(r) = \frac{\kappa}{2\pi\, r} = \frac{1.58 \times 10^{-8}}{r} n \qquad \left[\frac{\mathrm{m}}{\mathrm{s}}\right]$$

➡ normal core: $v_\mathrm{s} \longrightarrow v_\mathrm{c}$

$d_0 \approx 2 - 3\,\text{Å}$ $\triangleq$ coherence length
healing length



2.5 Macroscopic Quantum State

Energy of a vortex

kinetic energy / volume

$$E_\mathrm{v} = \int_{a_0}^{b} \frac{\varrho_\mathrm{s} v_\mathrm{s}^2}{2} 2\pi r \, \mathrm{d}r$$ energy / length

a_0 : radius of vortex core

b : radius of vessel or ½ distance to next vortex

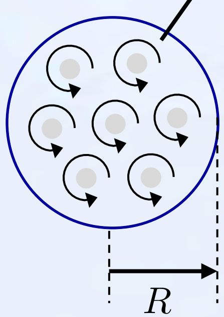
$$\kappa = v_\mathrm{s}\, 2\pi r \longrightarrow v_\mathrm{s}^2 = \frac{\kappa^2}{4\pi^2 r^2}$$

$$E_\mathrm{v} = \frac{\varrho_\mathrm{s}\kappa^2}{4\pi} \ln\left(\frac{b}{a_0}\right) \propto \kappa^2 \propto n^2$$

➡ vortex formation with *n* = 1 is preferred

Why is not a large vortex forming?

➡ splitting up in many small vortices prohibits large kinetic energy in core of vortex near the axis of rotation (velocity at the edge of vessel is given)



N vortices

$$v_\mathrm{s}(R) = N \frac{h}{m_4} \frac{1}{2\pi R}$$

$N \propto R^2$ if evenly distributed

2.5 Macroscopic Quantum State

At what velocity vortices are formed ?

critical angular velocity

$$\omega_\mathrm{c} = E_\mathrm{v}/L_\mathrm{v}$$

$$L_\mathrm{v} = \int_0^R \varrho_\mathrm{s} r\, v_\mathrm{s}\, 2\pi r\, \mathrm{d}r = \frac{1}{2}\varrho_\mathrm{s}\kappa R^2$$

angular momentum

$$\omega_\mathrm{c} = \frac{h}{2\pi m_4 R^2} \ln\left(\frac{R}{a_0}\right)$$

$R = 1\,\mathrm{cm} \longrightarrow \omega_\mathrm{c} \approx 10^{-3}\,\mathrm{s}^{-1}$

comment:

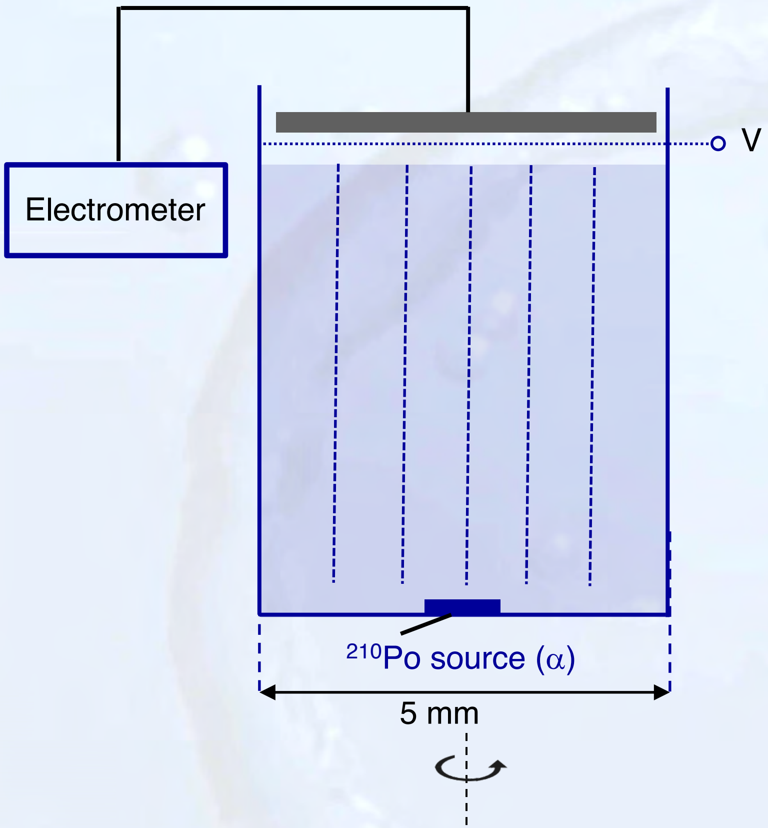
concept of critical velocity will be discussed in section 2.6

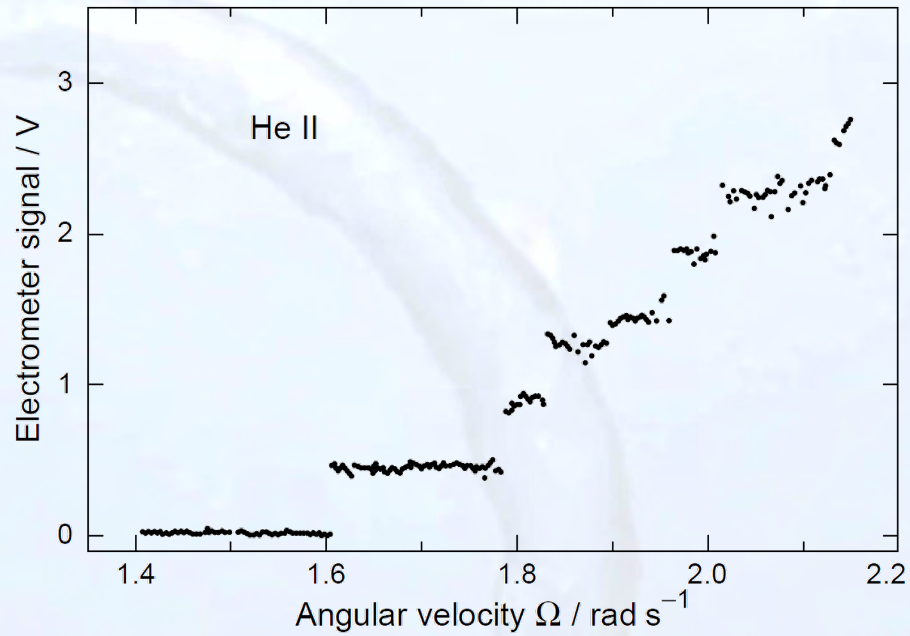
Experimental observation of vortices

- meniscus is rotating vessels
- damping of second sound
- electrometer experiments
- exploding electron bubbles
- decorating with hydrogen ice particles

2.5 Macroscopic Quantum State

Experimental observation of vortices

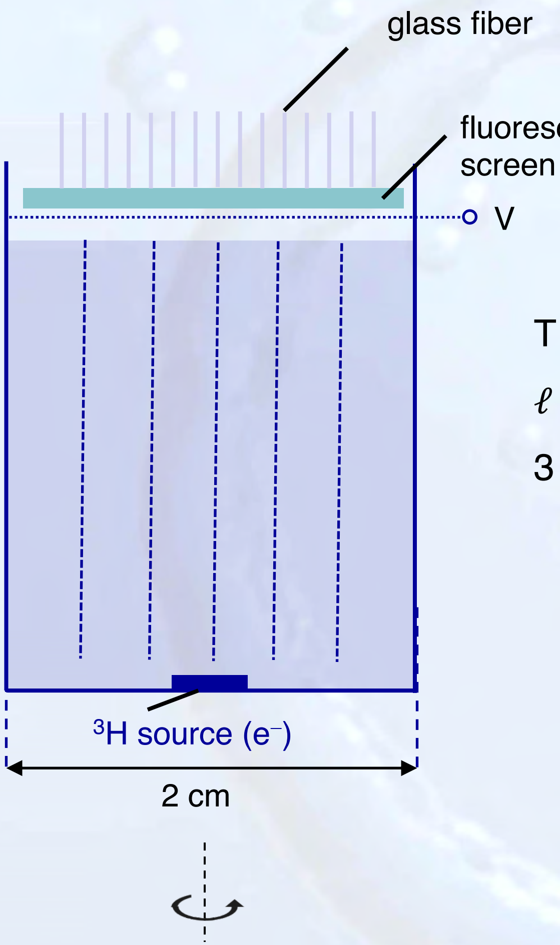




- α source ⟶ helium ionized ⟶ electrons form bubbles
- bubbles are captured by vortex lines via Magnus force
- *E* field is pulling bubbles alongside of vortex line to surface
- measurement of charge ⟶ is proportional to number vortex lines
- uniform acceleration over 10 h to 10 rot/min

2.5 Macroscopic Quantum State

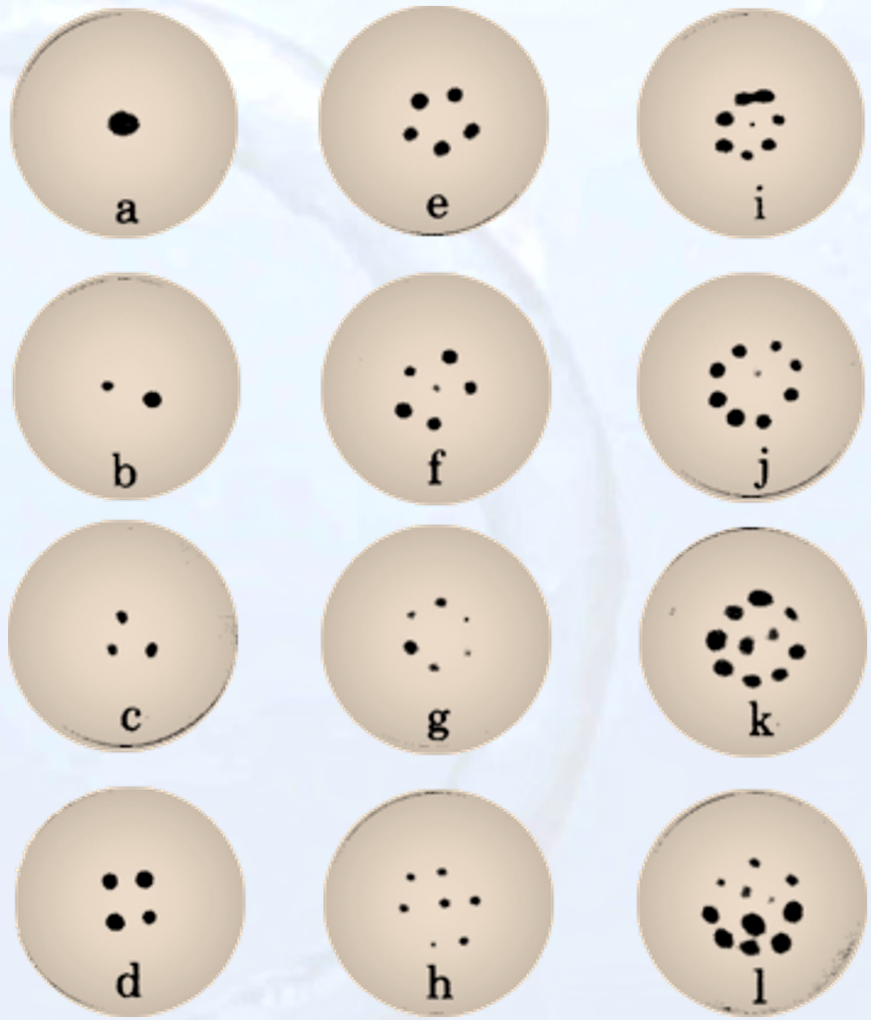
Experimental observation of vortices



T = 100 mK

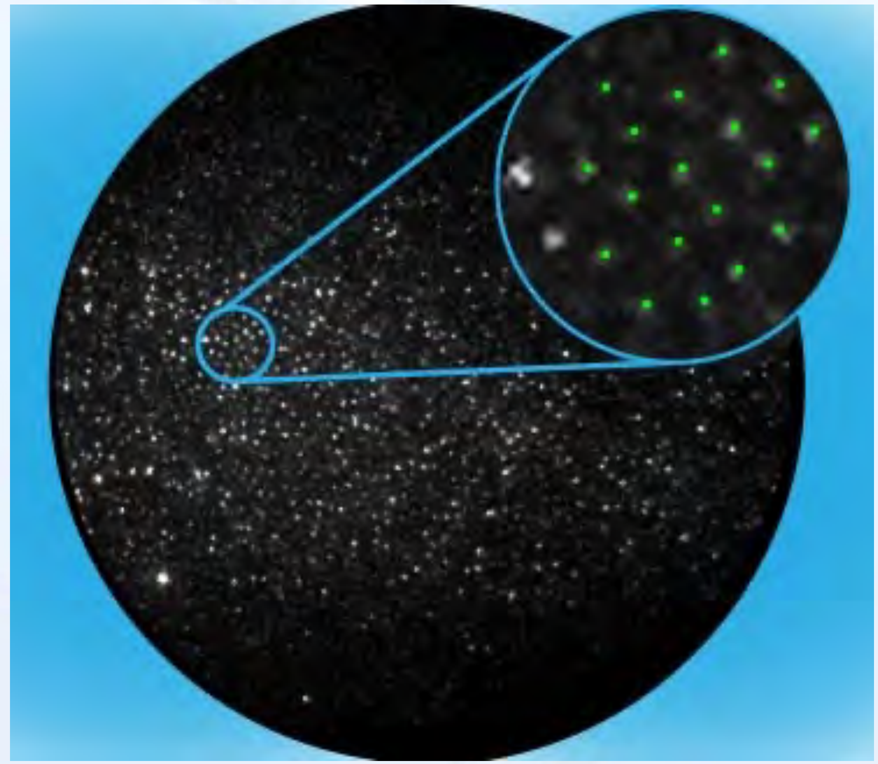
ℓ = 25 mm

3 … 8 rot/min



2.5 Macroscopic Quantum State

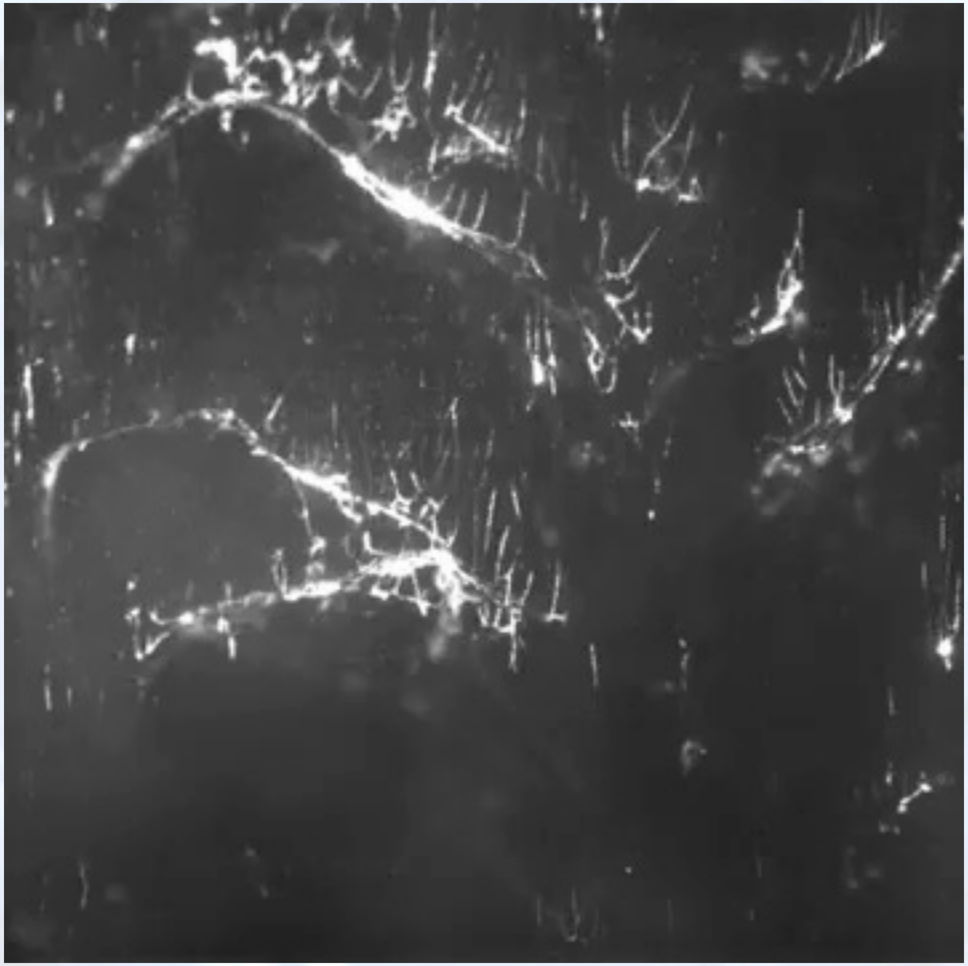
Experimental observation of vortices



Abrikosov lattice ⟶ Type 2 superconductor

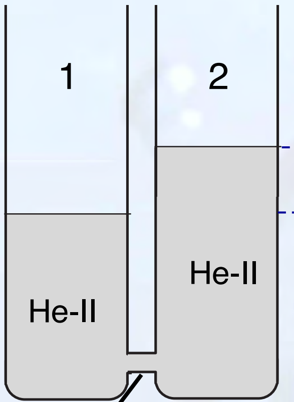
2.5 Macroscopic Quantum State

Experimental observation of vortices



2.5 Macroscopic Quantum State

Josephson Effects



Δz

$$\Delta\mu = \mu_2 - \mu_1 = m_4 g \Delta z$$

weak link

healing length

$$d \approx \xi = 1 \ldots 2\,\text{Å}$$

$$\xi_4 = \frac{0.3\,\text{nm}}{(1 - T/T_\lambda)^{2/3}}$$

diverges for $T \to T_\lambda$

Schrödinger Eq.

$$\mathrm{i}\hbar\dot{\Psi}_1 = \mu_1\Psi_1 + \mathcal{K}\Psi_2$$

$$\mathrm{i}\hbar\dot{\Psi}_2 = \mu_2\Psi_2 + \mathcal{K}\Psi_1$$

with $\Psi_1 = \sqrt{\varrho_\mathrm{s}}\mathrm{e}^{\mathrm{i}\varphi_1}$ and $\Psi_2 = \sqrt{\varrho_\mathrm{s}}\mathrm{e}^{\mathrm{i}\varphi_2}$

$$\boxed{\begin{aligned} \frac{\partial\varrho_\mathrm{s}}{\partial t} &= \frac{2\mathcal{K}}{\hbar}\,\varrho_\mathrm{s}\sin(\varphi_2 - \varphi_1) \\ \frac{\partial}{\partial t}(\varphi_2 - \varphi_1) &= -\frac{1}{\hbar}(\mu_2 - \mu_1) = -\frac{1}{\hbar}m_4 g\Delta z \end{aligned}}$$

$T = 0$

$\Delta\mu = 0$ phase difference constant $\longrightarrow$ Josephson dc effect

$\Delta\mu \neq 0$ phase difference changes $\longrightarrow$ Josephson ac effect

$$T = 0 \longrightarrow \quad \omega_\mathrm{J} = \frac{\Delta\mu}{\hbar} = \frac{m_4\Delta p}{\hbar\varrho} \quad \text{with } \Delta p = \varrho g\Delta z$$

$$T \neq 0 \longrightarrow \quad \omega_\mathrm{J} = \frac{\Delta\mu}{\hbar} = \frac{m_4\Delta p}{\hbar\varrho} - m_4\frac{S\Delta T}{\hbar}$$